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**Abstract**

We consider the Navier–Stokes equations with a pressure function satisfying a hard-sphere law. That means the pressure, as a function of the density, becomes infinite when the density approaches a finite critical value. Under some structural constraints imposed on the pressure law, we show a weak-strong uniqueness principle in periodic spatial domains. The method is based on a modified relative entropy inequality for the system. The main difficulty is that the pressure potential associated with the internal energy of the system is largely dominated by the pressure itself in the area close to the critical density. As a result, several terms appearing in the relative energy inequality cannot be controlled by the total energy.

**Keywords:** Navier–Stokes equations; hard-sphere pressure; weak-strong uniqueness.

## 1 Introduction

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a bounded domain. We consider the compressible Navier–Stokes equations in the time-space cylinder $(0, T) \times \Omega$:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) - \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = \varrho\, \mathbf{f}. \tag{1.2}$$

Here $\mathbb{S}(\nabla_x \mathbf{u})$ is the *Newtonian stress tensor* defined by

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu^S \left( \frac{\nabla_x \mathbf{u} + \nabla_x^{\mathrm{T}} \mathbf{u}}{2} - \frac{1}{d} (\operatorname{div}_x \mathbf{u}) \mathbb{I} \right) + \mu^B (\operatorname{div}_x \mathbf{u}) \mathbb{I}, \tag{1.3}$$

where $\mu^S > 0$ and $\mu^B \geq 0$ are the shear and bulk viscosity coefficients, respectively. Here, the velocity gradient matrix and the divergence on a matrix-valued function are defined as

$$(\nabla_x \mathbf{u})_{1 \leq i,j \leq d} = (\partial_{x_j} \mathbf{u}_i)_{1 \leq i,j \leq d}, \quad (\operatorname{div}_x \mathbb{S})_i = \sum_{j=1}^{d} \partial_{x_j} \mathbb{S}_{i,j}. \tag{1.4}$$

The external force $\mathbf{f}$ belongs to the class $L^\infty((0,T)\times\Omega;\mathbb{R}^d)$.

Unlike the commonly used isentropic setting, the pressure $p$ and the density $\varrho$ of the fluid are interrelated by a hard-sphere equation of state in the interval $[0,\bar\varrho)$:

$$p\in C^1[0,\bar\varrho),\ p(0)=0,\ p'>0 \text{ on } (0,\bar\varrho),\ \lim_{\varrho\to\bar\varrho-} p(\varrho)=+\infty. \tag{1.5}$$

## 2 Main results

In this section, we state our main results. Due to technical difficulties that will become clear in the course of the proofs, we consider solutions that are periodic in the spatial variable; that is we restrict ourselves to the periodic spatial domain (flat torus) $\Omega=\mathbb{T}^d:=\mathbb{R}^d/\mathbb{Z}^d$. The basic hypotheses imposed on the initial data are:

$$\begin{aligned} &\varrho(0,\cdot)=\varrho_0(\cdot) \text{ with } 0\le\varrho_0<\bar\varrho \text{ a.e. in } \Omega,\ \int_\Omega P(\varrho_0)\,\mathrm{d}x<\infty,\\ &\mathbf{u}(0,\cdot)=\mathbf{u}_0(\cdot),\ \int_\Omega \frac{|\mathbf{u}_0|^2}{\varrho_0}\mathrm{d}x<\infty. \end{aligned}\tag{2.1}$$

where $P\in C^1[0,\bar\varrho)$ is the pressure potential defined as

$$P(s)=s\int_{\frac{\bar\varrho}{2}}^{s}\frac{p(z)}{z^2}\,\mathrm{d}z. \tag{2.2}$$

Note that

$$P'(s)s-P(s)=p(s),\quad P''(s)=\frac{p'(s)}{s},\quad \text{for any } s\in[0,\bar\varrho). \tag{2.3}$$

Weak solutions are defined as follows:

**Definition 2.1.** *We say that $(\varrho,\mathbf{u})$ is a dissipative weak solution in $(0,T)\times\Omega$ to the system of equations (1.1)–(1.3), supplemented with initial data (2.1), if:*

- $0\le\varrho<\bar\varrho$ a.e. in $(0,T)\times\Omega$, $\varrho\in C_w([0,T];L^\gamma(\Omega))$ *for any* $\gamma>1$, $p(\varrho)\in L^1((0,T)\times\Omega)$, $\mathbf{u}\in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d))$, $\varrho\mathbf{u}\in C_w([0,T];L^2(\Omega;\mathbb{R}^d))$, $\varrho|\mathbf{u}|^2\in L^\infty(0,T;L^1(\Omega))$.
- *For any $\tau\in(0,T)$ and any test function $\phi\in C^\infty([0,T]\times\Omega)$, one has*

$$\int_0^\tau\int_\Omega\left[\varrho\partial_t\phi+\varrho\mathbf{u}\cdot\nabla_x\phi\right]\mathrm{d}x\,\mathrm{d}t=\int_\Omega\varrho(\tau,\cdot)\phi(\tau,\cdot)\,\mathrm{d}x-\int_\Omega\varrho_0\phi(0,\cdot)\,\mathrm{d}x. \tag{2.4}$$

- *For any $\tau\in(0,T)$ and any test function $\boldsymbol{\varphi}\in C^\infty([0,T]\times\Omega;\mathbb{R}^d)$, one has*

$$\begin{aligned}&\int_0^\tau\int_\Omega\left[\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}+(\varrho\mathbf{u}\otimes\mathbf{u}):\nabla_x\boldsymbol{\varphi}+p(\varrho)\,\mathrm{div}_x\boldsymbol{\varphi}-\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\boldsymbol{\varphi}\right]\mathrm{d}x\,\mathrm{d}t\\ &=-\int_0^\tau\int_\Omega\varrho\,\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\varrho\mathbf{u}(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot)\,\mathrm{d}x-\int_\Omega\varrho_0\mathbf{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x.\end{aligned}\tag{2.5}$$

- *The continuity equation holds in the sense of renormalized solutions:*

$$\partial_t b(\varrho)+\mathrm{div}_x(b(\varrho)\mathbf{u})+(b'(\varrho)\varrho-b(\varrho))\,\mathrm{div}_x\mathbf{u}=0\quad \textit{in } \mathcal{D}'((0,T)\times\Omega), \tag{2.6}$$

*for any $b\in C^1[0,\bar\varrho)$ satisfying*

$$|b'(s)|^2+|b(s)|^2\le C\,(1+p(s)) \textit{ for come constant } C \textit{ and any } s\in[0,\bar\varrho). \tag{2.7}$$

- *For a.e. $\tau \in (0,T)$, the* energy inequality *holds:*

$$
\begin{aligned}
&\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right](\tau,\cdot)\mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0)\right]\mathrm{d}x + \int_0^\tau \int_\Omega \varrho\,\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{2.8}
$$

In the sequel, we focus only on the 3D periodic domain: $d = 3$ and $\Omega = \mathbb{T}^3$. The two dimensional case can be done similarly. Concerning the existence of finite-energy weak solutions, we have the following remark:

**Remark 2.2.**
- *By employing the argument in [7] with the refined argument from [4], it can be shown that, under the following technical assumption on the pressure functional near the singular point $\bar{\varrho}$:*

$$
\lim_{\varrho\to\bar{\varrho}-} p(\varrho)(\bar{\varrho}-\varrho)^\beta > 0, \quad \textit{for some } \beta > 5/2,
\tag{2.9}
$$

*there exists a global-in-time weak solution in the sense of Definition 2.1.*

- *The constraint* (2.7) *on $b$ guarantees that $b(\varrho)$, $b'(\varrho) \in L^2((0,T)\times\Omega)$.*
- *It can be proven directly by using* (1.5) *that, for some $C > 0$, there holds $P(s) + C \ge 0$ for all $s \in [0,\bar{\varrho})$.*

Following [3, 6, 10], we define the relative entropy functional:

$$
\mathcal{E}(t) = \mathcal{E}\left(\varrho,\mathbf{u}\middle|r,\mathbf{U}\right)(t) := \int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + \big(P(\varrho) - P(r) - P'(r)(\varrho - r)\big)(t,\cdot)\,\mathrm{d}x,
\tag{2.10}
$$

We will show that any finite-energy weak solution satisfies a relative entropy inequality, a consequence of which is the weak-strong uniqueness.

Before stating our main results, we recall some notations. For a periodic function $f$ defined in $\Omega = \mathbb{T}^d$, one can write:

$$
f(x) = \sum_{k\in\mathbb{Z}^d} f_k e^{2\pi i k\cdot x},
$$

where $f_k$ are the Fourier coefficients. The mean value of $f$, denoted by $\langle f\rangle := \frac{1}{|\Omega|}\int_\Omega f\,\mathrm{d}x$, is the zero mode Fourier coefficient $f_0$. The inverse of the Laplacian coincides with the Fourier multiplier $\Delta_x^{-1}$ defined as

$$
\Delta_x^{-1} f(x) = \sum_{k\in\mathbb{Z}^d} \frac{1}{4\pi^2|k|^2} f_k e^{2\pi i k\cdot x}.
$$

Let $1 < q < \infty$ and set

$$
L_0^q(\Omega) := \{f \in L^q(\Omega) : \langle f\rangle = 0\}.
$$

Then by the classical elliptic theory, $\Delta_x^{-1}$ is a bounded linear mapping from $L_0^q(\Omega)$ to $W^{2,q}\cap L_0^q(\Omega)$ for any $1 < q < \infty$.

**Theorem 2.3.** *Suppose the pressure constraint* (2.9) *is satisfied. Let $(\varrho,\mathbf{u})$ be a finite-energy weak solution in $(0,T)\times\Omega$ in the sense of Definition 2.1. Let $(r,\mathbf{U}) \in C^1([0,T]\times\Omega)\times C^1([0,T];C^2(\Omega))$ such that*

$$
0 < r < \bar{\varrho}.
$$

*Let $b(s) \in C^1[0, \bar{\varrho})$ satisfy the condition*

$$|b'(s)|^{\frac{5}{2}} + |b(s)|^{\frac{5}{2}} \leq C\,(1 + p(s)) \text{ for come constant } C \text{ and any } s \in [0, \bar{\varrho}). \tag{2.11}$$

*Then the following relative entropy inequality holds for a.a. $\tau \in (0, T)$,*

$$\begin{aligned}
&\mathcal{E}\left(\varrho, \mathbf{u}\middle| r, \mathbf{U}\right)(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x(\mathbf{u} - \mathbf{U})) : \nabla_x(\mathbf{u} - \mathbf{U})\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega p(\varrho)b(\varrho)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \mathcal{E}\left(\varrho_0, \mathbf{u}_0\middle| r(0,\cdot), \mathbf{U}(0,\cdot)\right) + \int_0^\tau \mathcal{R}_1(t)\,\mathrm{d}t + \int_0^\tau \mathcal{R}_2(t)\,\mathrm{d}t + \mathcal{R}_3(\tau),
\end{aligned} \tag{2.12}$$

*where the remainder terms $\mathcal{R}_j,\ j = 1, 2, 3,$ are defined as*

$$\begin{aligned}
\mathcal{R}_1(t) :=& \int_\Omega \varrho(\partial_t \mathbf{U} + \mathbf{u}\cdot\nabla_x \mathbf{U})\cdot(\mathbf{U} - \mathbf{u})\,\mathrm{d}x \\
&+ \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x(\mathbf{U} - \mathbf{u})\,\mathrm{d}x + \int_\Omega \varrho\mathbf{f}\cdot(\mathbf{u} - \mathbf{U})\,\mathrm{d}x \\
&+ \int_\Omega (r - \varrho)\partial_t P'(r) + (r\mathbf{U} - \varrho\mathbf{u})\cdot\nabla_x P'(r)\,\mathrm{d}x + \int_\Omega \mathrm{div}_x \mathbf{U}(p(r) - p(\varrho))\,\mathrm{d}x, \\
\mathcal{R}_2(t) :=& \int_\Omega p(\varrho)\langle b(\varrho)\rangle\,\mathrm{d}x - \int_\Omega \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\nabla_x\Delta_x^{-1}\left(b(\varrho) - \langle b(\varrho)\rangle\right)\mathrm{d}x \\
&+ \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x\nabla_x\Delta_x^{-1}\left(b(\varrho) - \langle b(\varrho)\rangle\right)\mathrm{d}x - \int_\Omega \varrho\mathbf{f}\cdot\nabla_x\Delta_x^{-1}\left(b(\varrho) - \langle b(\varrho)\rangle\right)\mathrm{d}x \\
&+ \int_\Omega \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\left(\mathrm{div}_x(b(\varrho)\mathbf{u}) + (b'(\varrho)\varrho - b(\varrho))\,\mathrm{div}_x\mathbf{u} - \langle(b'(\varrho)\varrho - b(\varrho))\,\mathrm{div}_x\mathbf{u}\rangle\right)\mathrm{d}x, \\
\mathcal{R}_3(\tau) :=& \int_\Omega \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\left(b(\varrho) - \langle b(\varrho)\rangle\right)(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_0\mathbf{u}_0\cdot\nabla_x\Delta_x^{-1}\left(b(\varrho_0) - \langle b(\varrho_0)\rangle\right)\mathrm{d}x.
\end{aligned} \tag{2.13}$$

**Remark 2.4.**

- *In view of the conditions* (2.9) *and* (2.11)*, we will see later in the proof of Theorem 2.3 that all integrals appearing in* (2.12) *and* (2.13) *converge.*
- *The smoothness assumption on $r$, $\mathbf{U}$ can be relaxed accordingly, as long as all integrals appearing in* (2.12) *remain finite.*
- *Compared to the standard relative entropy inequality [3, 6, 10], the present form is augmented by the extra term $\int_0^\tau \int_\Omega p(\varrho)b(\varrho)\,\mathrm{d}x\,\mathrm{d}t$ on the left-hand side. To include this additional term, we have used the pressure identity*

  $$\int_0^\tau \int_\Omega p(\varrho)b(\varrho)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \mathcal{R}_2(t)\,\mathrm{d}t + \mathcal{R}_3(\tau)$$

  *that can be deduced by using the quantity $\nabla_x\Delta_x^{-1}\left(b(\varrho) - \langle b(\varrho)\rangle\right)$ as a test function in the momentum balance (2.5).*
- *In the case of more conventional no-slip boundary conditions, the pressure term $\int_0^\tau \int_\Omega p(\varrho)b(\varrho)\,\mathrm{d}x\,\mathrm{d}t$ is computed by means of*

  $$\mathcal{B}\left(b(\varrho) - \langle b(\varrho)\rangle\right)$$

  *as a test function, where $\mathcal{B}$ is the so–called Bogovskii operator (see[1] or Chapter III of Galdi's book [9]). While, the adjoint $B_j^*$ of the Bogovskii operator $B_j$ is considerably different compared to the Bogovskii operator. To handle the new terms in $\mathcal{R}_2$ and $\mathcal{R}_3$ in* (2.13)*, we would need employ the adjoint of $\mathcal{B}$. Unfortunately, the behavior of the adjoints of the operators $\nabla_x\Delta_x^{-1}$ and $\mathcal{B}$ is rather different and the use of $\mathcal{B}$ is connected with other technical difficulties. This is the main reason why we restrict ourselves to the purely periodic setting.*

Our next result is the weak-strong uniqueness principle:

**Theorem 2.5.** *Suppose the pressure constrain* (2.9) *is satisfied with* $\beta \geq 3$. *Let* $(\varrho, \mathbf{u})$ *be a finite-energy weak solution in* $(0,T) \times \Omega$ *in the sense of Definition 2.1. Let* $(r, \mathbf{U}) \in C^1([0,T] \times \Omega) \times C^1([0,T]; C^2(\Omega))$ *be a strong solution to* (1.1)–(1.3) *having the same initial data as* $(\varrho, \mathbf{u})$ *and such that*

$$0 < r < \bar{\varrho}.$$

*Then there holds*

$$(\varrho, \mathbf{u}) = (r, \mathbf{U}) \text{ in } (0,T) \times \Omega. \tag{2.14}$$

**Remark 2.6.** *The additional assumption* $\beta \geq 3$ *is needed only for estimating the term* $I_{19}$ *in Section 4.8. For all other terms,* $\beta > 5/2$ *is sufficient.*

The rest of the paper is devoted to the proof of Theorems 2.3 and 2.5. Throughout the paper, $C$ denotes some uniform constant of which the value may differ from line to line. In the sequel, to avoid notation complicity, we sometimes simply use $L^r(0,T; X(\Omega))$ to denote the scalar function space $L^r(0,T; X(\Omega))$, the vector valued function space $L^r(0,T; X(\Omega; \mathbb{R}^n))$ or the matrix valued function space $L^r(0,T; X(\Omega; \mathbb{R}^{n\times n}))$ if there is no confusion.

# 3 Relative entropy inequality

This section is devoted to the proof of Theorem 2.3. By employing the argument in [3], one can derive

$$\begin{aligned} &\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x(\mathbf{u} - \mathbf{U})) : \nabla_x(\mathbf{u} - \mathbf{U}) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad \leq \mathcal{E}\left(\varrho_0, \mathbf{u}_0 \middle| r(0,\cdot), \mathbf{U}(0,\cdot)\right) + \int_0^\tau \mathcal{R}_1(t) \,\mathrm{d}t. \end{aligned} \tag{3.1}$$

Now we include the terms related to $b(\varrho)$. By (2.2) and (2.9), we have

$$P(s) \leq C + p(s)(\bar{\varrho} - s) \leq 2C + P(s), \quad \text{for all } s \in [0, \bar{\varrho}). \tag{3.2}$$

Then, by (2.9) and (2.11), we have for any $s \in [0, \bar{\varrho})$ that

$$|b'(s)| + |b(s)| \leq C(1 + p(s))^{\frac{2}{5}} \leq C(1 + P(s)P(s)^{\frac{2}{3}})^{\frac{2}{5}} \leq C(1 + P(s))^{\frac{2}{3}}. \tag{3.3}$$

Hence, by the fact $p(\varrho) \in L^1((0,T) \times \Omega)$ in Definition 2.1, by the energy inequality (2.8) and Gronwall's inequality, we have $P(\varrho) \in L^\infty(0,T; L^1(\Omega))$ and furthermore,

$$b'(\varrho),\ b(\varrho) \in L^{\frac{5}{2}}((0,T) \times \Omega) \cap L^\infty(0,T; L^{\frac{3}{2}}(\Omega)). \tag{3.4}$$

Since the condition (2.11) is stronger than (2.7), the function $b(\varrho)$ satisfies the renormalized continuity equation (2.6). This gives us some information on the time derivative of $b(\varrho)$: by using (3.4), there holds

$$\partial_t b(\varrho) \in \left(L^2(0,T; W^{-1,\frac{6}{5}}(\Omega)) \cap L^{\frac{10}{9}}(0,T; W^{-1,\frac{30}{17}}(\Omega))\right) + L^{\frac{10}{9}}((0,T) \times \Omega). \tag{3.5}$$

By tedious, however direct, calculations, we can choose $\nabla_x \Delta_x^{-1}(b(\varrho) - \langle b(\varrho) \rangle)$ as a test function in the weak formulation (2.5) to deduce for any $\tau \in (0, T)$:

$$\int_0^\tau \int_\Omega p(\varrho) b(\varrho) \, \mathrm{d}x \, \mathrm{d}t = \int_0^\tau \mathcal{R}_2(t) \, \mathrm{d}t + \mathcal{R}_3(\tau), \tag{3.6}$$

where $\mathcal{R}_2(t)$ and $\mathcal{R}_3(\tau)$ are exactly as in (2.13). We briefly show that all the integrals appeared on the right-hand side of (3.6) are meaningful and uniformly bounded in $\tau \in (0, T)$.

By Definition 2.1 and (3.4), we have

$$\int_0^\tau \int_\Omega p(\varrho) \langle b(\varrho) \rangle \, \mathrm{d}x \, \mathrm{d}t \leq C \, \| p(\varrho) \|_{L^1((0,T)\times\Omega)} \| b(\varrho) \|_{L^\infty(0,T;L^1(\Omega))} \leq C. \tag{3.7}$$

By Definition 2.1 and Sobolev embedding $W^{1,2}(\Omega) \subset L^6(\Omega)$, we have

$$\varrho \mathbf{u} \otimes \mathbf{u} \in L^\infty(0, T; L^1(\Omega)) \cap L^1(0, T; L^3(\Omega)) \subset L^{\frac{5}{3}}((0, T) \times \Omega).$$

Together with (3.4) and the fact that $\Delta_x^{-1}$ is a linear continuous mapping from $L^q_0(\Omega)$ to $W^{2,q}(\Omega)$ for any $1 < q < \infty$, we deduce

$$\begin{aligned} & - \int_0^\tau \int_\Omega \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \nabla_x \Delta_x^{-1} \left( b(\varrho) - \langle b(\varrho) \rangle \right) \, \mathrm{d}x \, \mathrm{d}t \\ & \leq C \, \| \varrho \mathbf{u} \otimes \mathbf{u} \|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \| \left( b(\varrho) - \langle b(\varrho) \rangle \right) \|_{L^{\frac{5}{2}}((0,T)\times\Omega)} \leq C. \end{aligned} \tag{3.8}$$

Similarly, we have

$$\begin{aligned} & \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \nabla_x \Delta_x^{-1} \left( b(\varrho) - \langle b(\varrho) \rangle \right) \, \mathrm{d}x \, \mathrm{d}t \\ & \leq C \, \| \nabla_x \mathbf{u} \|_{L^2((0,T)\times\Omega)} \| \left( b(\varrho) - \langle b(\varrho) \rangle \right) \|_{L^2((0,T)\times\Omega)} \leq C, \end{aligned} \tag{3.9}$$

and

$$- \int_0^\tau \int_\Omega \varrho \mathbf{f} \cdot \nabla_x \Delta_x^{-1} \left( b(\varrho) - \langle b(\varrho) \rangle \right) \, \mathrm{d}x \, \mathrm{d}t \leq C \, \| \varrho \mathbf{f} \|_{L^\infty((0,T)\times\Omega)} \| \left( b(\varrho) - \langle b(\varrho) \rangle \right) \|_{L^2((0,T)\times\Omega)} \leq C.$$

By Definition 2.1, (3.4) and Sobolev embedding, we have:

$$\int_0^\tau \int_\Omega \varrho \mathbf{u} \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x (b(\varrho) \mathbf{u}) \, \mathrm{d}x \, \mathrm{d}t \leq C \, \| \varrho \mathbf{u} \|_{L^2(0,T;L^6(\Omega))} \| b(\varrho) \mathbf{u} \|_{L^2(0,T;L^{\frac{6}{5}}(\Omega))} \leq C. \tag{3.10}$$

Similarly,

$$\begin{aligned} & - \int_0^\tau \int_\Omega \varrho \mathbf{u} \cdot \nabla_x \Delta_x^{-1} \left( (b'(\varrho)\varrho - b(\varrho)) \, \mathrm{div}_x \mathbf{u} - \langle (b'(\varrho)\varrho - b(\varrho)) \, \mathrm{div}_x \mathbf{u} \rangle \right) \, \mathrm{d}x \, \mathrm{d}t \\ & \leq C \, \| \varrho \mathbf{u} \|_{L^{10}(0,T;L^{\frac{30}{13}}(\Omega))} \| (b'(\varrho)\varrho - b(\varrho)) \, \mathrm{div}_x \mathbf{u} \|_{L^{\frac{10}{9}}((0,T)\times\Omega)} \\ & \leq C \, \| \varrho \mathbf{u} \|_{L^2(0,T;L^6(\Omega)) \cap L^\infty(0,T;L^2(\Omega))} \| (b'(\varrho)\varrho - b(\varrho)) \|_{L^{\frac{5}{2}}((0,T)\times\Omega)} \| \mathrm{div}_x \mathbf{u} \|_{L^2((0,T)\times\Omega)} \\ & \leq C. \end{aligned} \tag{3.11}$$

For $\mathcal{R}_3(\tau)$, again by Definition 2.1, (3.4) and Sobolev embedding, we have

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla_x \Delta_x^{-1} \left( b(\varrho) - \langle b(\varrho) \rangle \right) (\tau, \cdot) \, \mathrm{d}x \leq C \, \| \varrho \mathbf{u} \|_{L^\infty(0,T;L^2(\Omega))} \| b(\varrho) \|_{L^\infty(0,T;L^{\frac{6}{5}}(\Omega))} \leq C, \tag{3.12}$$

and

$$
(3.13) \qquad -\int_\Omega \varrho_0\mathbf{u}_0\cdot\nabla_x\Delta_x^{-1}\left(b(\varrho_0)-\langle b(\varrho_0)\rangle\right)\,\mathrm{d}x \le C\,\|\varrho_0\mathbf{u}_0\|_{L^2(\Omega)}\|b(\varrho_0)\|_{L^{\frac{6}{5}}(\Omega)} \le C.
$$

Summarizing the estimates in (3.8)–(3.13) implies the right-hand side of (3.7) are uniformly bounded in $\tau\in(0,T)$. This shows the integral $\int_0^\tau\int_\Omega p(\varrho)b(\varrho)\,\mathrm{d}x\,\mathrm{d}t$ is meaningful and is uniformly bounded in $\tau\in(0,T)$.

Thus, summing up (3.1) and (3.6) implies our desired relative entropy inequality (2.12). We complete the proof of Theorem 2.3.

# 4 Weak-strong uniqueness

In this section, we prove Theorem 2.5 by using the relative entropy inequality. Let $(\varrho,\mathbf{u})$ and $(r,\mathbf{U})$ be the weak solution and the strong solution given in Theorem 2.5 issued from the same regular initial data. We choose $(r,\mathbf{U})$ as the test function in the relative entropy inequality (2.12) for the weak solution $(\varrho,\mathbf{u})$. The idea of proving Theorem 2.5 is the following: we analyze the corresponding right-hand side of (2.12) until some level that allows us to use Gronwall type inequalities to show the relative entropy is identically zero, which implies the weak solution and the strong one are equal. In this section, we let $\eta(t)$ be a universal $L^1(0,T)$ function.

## 4.1 New expression for remainder term $\mathcal{R}_1$

Since $r$ is strictly positive and strictly smaller than $\bar\varrho$, there exits $\alpha_0>0$ such that

$$
(4.1) \qquad 0<\alpha_0\le r\le\bar\varrho-\alpha_0<\bar\varrho.
$$

Thus, there holds

$$
(4.2) \qquad \partial_t\mathbf{U}+\mathbf{U}\cdot\nabla_x\mathbf{U}+r^{-1}\nabla_x p(r)-r^{-1}\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})=\mathbf{f}.
$$

By (2.3), we have

$$
(4.3) \qquad \nabla_x P'(r)=P''(r)\nabla_x r=r^{-1}p'(r)\nabla_x r=r^{-1}\nabla_x p(r).
$$

Plugging (4.2) into the expression of $\mathcal{R}_1$ in (2.13) and using (4.3) implies

$$
(4.4) \qquad \begin{aligned}
\mathcal{R}_1=&\int_\Omega\varrho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x+\int_\Omega\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})(r^{-1}\varrho-1)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\\
&-\int_\Omega\varrho\nabla_x P'(r)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x+\int_\Omega(r-\varrho)\partial_t P'(r)+(r\mathbf{U}-\varrho\mathbf{u})\cdot\nabla_x P'(r)\,\mathrm{d}x\\
&+\int_\Omega\mathrm{div}_x\mathbf{U}(p(r)-p(\varrho))\,\mathrm{d}x.
\end{aligned}
$$

By the continuity equation (1.1), and by (2.3), we have

$$
(4.5) \qquad \begin{aligned}
&-\varrho\nabla_x P'(r)\cdot(\mathbf{U}-\mathbf{u})+(r-\varrho)\partial_t P'(r)+(r\mathbf{U}-\varrho\mathbf{u})\cdot\nabla_x P'(r)\\
&=(r-\varrho)\left(\partial_t P'(r)+\mathbf{U}\cdot\nabla_x P'(r)\right)\\
&=(r-\varrho)\left[\partial_t P'(r)+\mathrm{div}_x(\mathbf{U}P'(r))+(P''(r)r-P'(r))\mathrm{div}_x\mathbf{U}\right]-(r-\varrho)P''(r)r\,\mathrm{div}_x\mathbf{U}\\
&=-(r-\varrho)p'(r)\mathrm{div}_x\mathbf{U}.
\end{aligned}
$$

Plugging (4.5) into (4.4) implies

$$\begin{aligned}\mathcal{R}_1 = &\int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x + \int_\Omega \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})r^{-1}(\varrho-r)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\\ &+\int_\Omega \mathrm{div}_x\mathbf{U}\big(p(r)-p(\varrho)-p'(r)(r-\varrho)\big)\,\mathrm{d}x.\end{aligned}\tag{4.6}$$

## 4.2 Property on the pressure potential

We give some properties concerning the quantity appearing in the relative entropy and related to the pressure potential:

**Lemma 4.1.** *Let $\varrho \geq 0$ and $0 < \alpha_0 \leq r \leq \bar{\varrho} - \alpha_0 < \bar{\varrho}$. There exits $\alpha_1 \in (0, \alpha_0)$ and a constant $c > 0$, such that*

$$P(\varrho) - P(r) - P'(r)(\varrho - r) \geq \begin{cases} c\,(\varrho - r)^2, & \text{if } \alpha_1 \leq \varrho \leq \bar{\varrho} - \alpha_1,\\ \dfrac{p(r)}{2}, & \text{if } 0 \leq \varrho \leq \alpha_1,\\ \dfrac{P(\varrho)}{2}, & \text{if } \bar{\varrho} - \alpha_1 \leq \varrho \leq \bar{\varrho}.\end{cases}\tag{4.7}$$

*Proof.* 1. We start by considering $\varrho$ near 0: $0 \leq \varrho \leq \alpha_1$. By (1.5), for $\alpha_1 < \alpha_0 < \bar{\varrho}$, we have

$$|p(\varrho)| \leq \sup_{0\leq s\leq\alpha_1} p'(s)\varrho \leq C\varrho, \text{ for any } 0 \leq \varrho \leq \alpha_1.$$

Then, for $\varrho \in [0, \alpha_1]$, there holds

$$|P(\varrho)| = \left|\varrho\int_{\frac{\bar{\varrho}}{2}}^{\varrho} \frac{p(z)}{z^2}\,\mathrm{d}z\right| \leq C\varrho\left|\log\frac{\bar{\varrho}}{2} - \log\varrho\right| \to 0, \text{ as } \varrho \to 0.\tag{4.8}$$

Since $r \in [\alpha_0, \bar{\varrho} - \alpha_0]$, by (1.5), (2.2) and (2.3), we have

$$|P'(r)| = \left|\frac{P(r) + p(r)}{r}\right| \leq C,\tag{4.9}$$

and

$$P'(r)r - P(r) = p(r) \geq C^{-1},\tag{4.10}$$

for some constant $C > 0$.

Thus, by (4.8)–(4.10), by choosing $\alpha_1 > 0$ small, we have for $0 \leq \varrho \leq \alpha_1,\ \alpha_0 \leq r \leq \bar{\varrho} - \alpha_0$ that

$$P(\varrho) - P(r) - P'(r)(\varrho - r) = \big[P(\varrho) - P'(r)\varrho\big] + \big[P'(r)r - P(r)\big] \geq \frac{p(r)}{2}.\tag{4.11}$$

2. We then consider $\varrho$ near $\bar{\varrho}$: $\bar{\varrho} - \alpha_1 \leq \varrho \leq \bar{\varrho}$. Since $r \in [\alpha_0, \bar{\varrho} - \alpha_0]$, by (1.5), (2.2) and (2.3), we have

$$|P(r) + P'(r)(\varrho - r)| \leq C.$$

By (2.2) and (2.9), we have

$$\lim_{\varrho\to\bar{\varrho}} P(\varrho) \to +\infty.$$

Thus, by choosing $\alpha_1 > 0$ small, there holds

$$P(\varrho) - P(r) - P'(r)(\varrho - r) \geq \frac{P(\varrho)}{2}. \tag{4.12}$$

3. For $\alpha_1 \leq \varrho \leq \bar{\varrho} - \alpha_1$, Taylor's formula gives

$$P(\varrho) - P(r) - P'(r)(\varrho - r) \geq \min_{\alpha_1 \leq s \leq \bar{\varrho} - \alpha_1} P''(s)(\varrho - r)^2 = \min_{\alpha_1 \leq s \leq \bar{\varrho} - \alpha_1} \frac{p'(s)}{s}(\varrho - r)^2 \geq c(\varrho - r)^2, \tag{4.13}$$

for some constant $c > 0$.

□

### 4.3 Estimate for the remainders: part 1

For the first term in the expression of $\mathcal{R}_1$ in (4.6), we have

$$\int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \leq \|\nabla_x \mathbf{U}(t)\|_{L^\infty(\Omega)} \int_\Omega \varrho |\mathbf{u} - \mathbf{U}|^2 \, \mathrm{d}x \leq \eta(t)\mathcal{E}(t). \tag{4.14}$$

Here in (4.14), we only need the assumption $\nabla_x \mathbf{U} \in L^1(0, T; L^\infty(\Omega))$. In the following, we will present, but will not emphasize, the lowest regularity assumption on $(r, \mathbf{U})$.

The estimate for the second term of (4.6) is more delicate. Let $\alpha_1 \in (0, \alpha_0)$ be as in Lemma 4.1, we write

$$\int_\Omega \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) r^{-1}(\varrho - r) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x = I_1 + I_2 + I_3, \tag{4.15}$$

with

$$\begin{aligned} I_1 &:= \int_{\alpha_1 \leq \varrho \leq \bar{\varrho} - \alpha_1} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) r^{-1}(\varrho - r) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x, \\ I_2 &:= \int_{\varrho \leq \alpha_1} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) r^{-1}(\varrho - r) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x, \\ I_3 &:= \int_{\varrho \geq \bar{\varrho} - \alpha_1} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) r^{-1}(\varrho - r) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x. \end{aligned} \tag{4.16}$$

For $I_1$, by Schwartz inequality, Poincaré inequality, we have for any $\sigma > 0$ that

$$|I_1| \leq \sigma^{-1} \|\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) r^{-1}(t)\|^2_{L^\infty(\Omega)} \int_{\alpha_1 \leq \varrho \leq \bar{\varrho} - \alpha_1} (\varrho - r)^2 \, \mathrm{d}x + \sigma \int_\Omega |\nabla_x (\mathbf{U} - \mathbf{u})|^2 \, \mathrm{d}x. \tag{4.17}$$

By Korn's inequality, there holds

$$\int_\Omega |\nabla_x (\mathbf{U} - \mathbf{u})|^2 \, \mathrm{d}x \leq C \int_\Omega \mathbb{S}(\nabla_x (\mathbf{u} - \mathbf{U})) : \nabla_x (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x.$$

By Lemma 4.1 and by choosing $\sigma > 0$ small, we deduce from (4.17) that

$$\begin{aligned} |I_1| &\leq C \|\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})(t)\|^2_{L^\infty(\Omega)} \int_{\alpha_1 \leq \varrho \leq \bar{\varrho} - \alpha_1} P(\varrho) - P(r) - P'(r)(\varrho - r) \, \mathrm{d}x \\ &\quad + \frac{1}{16} \int_\Omega \mathbb{S}(\nabla_x (\mathbf{u} - \mathbf{U})) : \nabla_x (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x \\ &\leq \eta(t)\mathcal{E}(t) + \frac{1}{16} \int_\Omega \mathbb{S}(\nabla_x (\mathbf{u} - \mathbf{U})) : \nabla_x (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x. \end{aligned} \tag{4.18}$$

For $I_2$, by Schwartz inequality, Poincaré inequality, Korn's inequality and Lemma 4.1, we can deduce

$$
\begin{aligned}
|I_2| &\le \sigma^{-1}\|\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})(t)\|^2_{L^\infty(\Omega)}\int_{\varrho\le\alpha_1}1\,\mathrm{d}x+\sigma\int_\Omega|\mathbf{U}-\mathbf{u}|^2\,\mathrm{d}x\\
&\le C\|\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})(t)\|^2_{L^\infty(\Omega)}\int_{\varrho\le\alpha_1}p(r)\,\mathrm{d}x+\frac{1}{16}\int_\Omega\mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})):\nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\\
&\le \eta(t)\mathcal{E}(t)+\frac{1}{16}\int_\Omega\mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})):\nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x.
\end{aligned}
\tag{4.19}
$$

Similarly, we have for $I_3$ that

$$
\begin{aligned}
|I_3| &\le C\sigma^{-1}\|\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})(t)\|^2_{L^\infty(\Omega)}\int_{\varrho\ge\bar\varrho-\alpha_1}1\,\mathrm{d}x+\sigma\int_\Omega|\mathbf{U}-\mathbf{u}|^2\,\mathrm{d}x\\
&\le C\|\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})(t)\|^2_{L^\infty(\Omega)}\int_{\varrho\ge\bar\varrho-\alpha_1}P(\varrho)\,\mathrm{d}x+\frac{1}{16}\int_\Omega\mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})):\nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\\
&\le \eta(t)\mathcal{E}(t)+\frac{1}{16}\int_\Omega\mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})):\nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x.
\end{aligned}
\tag{4.20}
$$

We offer another way to deal with $I_3$:

$$
\begin{aligned}
|I_3| &\le C\|\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})(t)\|^2_{L^\infty(\Omega)}\int_{\varrho\ge\bar\varrho-\alpha_1}1\,\mathrm{d}x+\int_{\varrho\ge\bar\varrho-\alpha_1}\varrho|\mathbf{U}-\mathbf{u}|^2\,\mathrm{d}x\\
&\le C\|\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})(t)\|^2_{L^\infty(\Omega)}\int_{\varrho\ge\bar\varrho-\alpha_1}P(\varrho)\,\mathrm{d}x+\int_\Omega\varrho|\mathbf{U}-\mathbf{u}|^2\,\mathrm{d}x\\
&\le \eta(t)\mathcal{E}(t).
\end{aligned}
\tag{4.21}
$$

The way to estimate the third term of $\mathcal{R}_1$ in (4.6) is more delicate. The main difficulty is that, as mentioned in the introduction, due to the hard-sphere pressure setting, we do not have a control of $p(\varrho)-p(r)-p'(r)(\varrho-r)$ by $P(\varrho)-P(r)-P'(r)(\varrho-r)$ which is the one appearing in the relative entropy. Indeed, by employing the proof of Lemma 4.1, we have for $\alpha_1$ small:

$$
p(\varrho)-p(r)-p'(r)(\varrho-r)\le\begin{cases}C\,(\varrho-r)^2, & \text{if } \alpha_1\le\varrho\le\bar\varrho-\alpha_1,\\ 1+p'(r)r-p(r), & \text{if } 0\le\varrho\le\alpha_1,\\ 2p(\varrho), & \text{if } \bar\varrho-\alpha_1\le\varrho\le\bar\varrho.\end{cases}
\tag{4.22}
$$

We see that we do have

$$
p(\varrho)-p(r)-p'(r)(\varrho-r)\le C\left[P(\varrho)-P(r)-P'(r)(\varrho-r)\right],\ \text{for } \varrho\le\bar\varrho-\alpha_1.
\tag{4.23}
$$

Thus, if we write

$$
\int_\Omega\mathrm{div}_x\mathbf{U}\big(p(r)-p(\varrho)-p'(r)(r-\varrho)\big)\,\mathrm{d}x=I_4+I_5,
\tag{4.24}
$$

with

$$
\begin{aligned}
I_4 &:= \int_{\varrho\le\bar\varrho-\alpha_1}\mathrm{div}_x\mathbf{U}\big(p(r)-p(\varrho)-p'(r)(r-\varrho)\big)\,\mathrm{d}x,\\
I_5 &:= \int_{\varrho\ge\bar\varrho-\alpha_1}\mathrm{div}_x\mathbf{U}\big(p(r)-p(\varrho)-p'(r)(r-\varrho)\big)\,\mathrm{d}x,
\end{aligned}
\tag{4.25}
$$

we have

$$|I_4| \le C\|\mathrm{div}_x \mathbf{U}(t)\|_{L^\infty(\Omega)} \int_{\varrho \ge \bar\varrho - \alpha_1} P(\varrho) - P(r) - P'(r)(\varrho - r)\,\mathrm{d}x \le \eta(t)\mathcal{E}(t). \tag{4.26}$$

Due to the fact $p(\varrho)/P(\varrho) \to +\infty$ as $\varrho \to \bar\varrho$, we can also see from (4.22) and Lemma 4.1 that we need to estimate $I_5$ differently. This the reason that we include the terms related to $b(\varrho)$.

## 4.4 Estimates for the remainder: part 2

In the sequel, we choose $b(s) \in C^\infty[0, \bar\varrho),\ b'(s) \ge 0$ as follows:

$$b(s) = \begin{cases} 0 & \text{if } s \le \bar\varrho - \alpha_1, \\ -\log(\bar\varrho - s), & \text{if } \bar\varrho - \alpha_2 \le s < \bar\varrho, \end{cases} \qquad b'(s) > 0 \text{ if } \bar\varrho - \alpha_1 < s < \bar\varrho - \alpha_2. \tag{4.27}$$

Here $\alpha_0$ and $\alpha_1$ are as in (4.1) and Lemma 4.1, respectively, and $0 < \alpha_2 < \alpha_1$. In the sequel, $\alpha_1$ and $\alpha_2$ will be chosen to be sufficiently small, while still be fixed. Such a choice of $b$ is admissible for Theorem 2.3. Indeed, by (2.9), the condition (2.11) is satisfied. Moreover, there holds for any $\gamma > 0$:

$$\lim_{s \to \bar\varrho -} \frac{p(s)}{(b(s))^\gamma} = \lim_{s \to \bar\varrho -} \frac{P(s)}{(b(s))^\gamma} = \lim_{s \to \bar\varrho -} \frac{p(s)}{(b'(s))^\beta} = \lim_{s \to \bar\varrho -} \frac{P(s)}{(b'(s))^{\beta-1}} = +\infty, \tag{4.28}$$

Thus, for any $\gamma \ge 1$ and any $2 \le \beta_0 \le \beta$, by Lemma 4.1 and (4.28), we have

$$\begin{aligned} \int_\Omega |b(\varrho)|^\gamma \,\mathrm{d}x &= \int_{\varrho \ge \bar\varrho - \alpha_1} |b(\varrho)|^\gamma \,\mathrm{d}x \\ &\le C \int_{\varrho \ge \bar\varrho - \alpha_1} P(\varrho)\,\mathrm{d}x \le C \int_{\varrho \ge \bar\varrho - \alpha_1} P(\varrho) - P(r) - P'(r)(\varrho - r)\,\mathrm{d}x, \\ \int_\Omega |b'(\varrho)|^{\beta_0 - 1}\,\mathrm{d}x &\le C \int_{\varrho \ge \bar\varrho - \alpha_1} P(\varrho)\,\mathrm{d}x \le C \int_{\varrho \ge \bar\varrho - \alpha_1} P(\varrho) - P(r) - P'(r)(\varrho - r)\,\mathrm{d}x, \\ \int_\Omega |b'(\varrho)|^{\beta_0}\,\mathrm{d}x &\le C \int_\Omega p(\varrho)\,\mathrm{d}x. \end{aligned} \tag{4.29}$$

By choosing $\alpha_2$ small such that

$$-\log(\bar\varrho - s) \ge 16\,\|\mathrm{div}_x \mathbf{U}\|_{L^\infty((0,T)\times\Omega)}, \text{ if } \bar\varrho - \alpha_2 \le s < \bar\varrho. \tag{4.30}$$

we deduce from (4.22) and Lemma 4.1 that

$$\begin{aligned} |I_5| &\le \int_{\bar\varrho - \alpha_1 \le \varrho \le \bar\varrho - \alpha_2} \mathrm{div}_x \mathbf{U}\big(p(r) - p(\varrho) - p'(r)(r - \varrho)\big)\,\mathrm{d}x \\ &\quad + \int_{\varrho \ge \bar\varrho - \alpha_2} \mathrm{div}_x \mathbf{U}\big(p(r) - p(\varrho) - p'(r)(r - \varrho)\big)\,\mathrm{d}x \\ &\le \int_{\bar\varrho - \alpha_1 \le \varrho \le \bar\varrho - \alpha_2} |\mathrm{div}_x \mathbf{U}| \max_{\bar\varrho - \alpha_1 \le s \le \bar\varrho - \alpha_2} p''(s)(r - \varrho)^2\,\mathrm{d}x + \frac{1}{8}\int_\Omega p(\varrho) b(\varrho)\,\mathrm{d}x \\ &\le C\|\mathrm{div}_x \mathbf{U}\|_{L^\infty(\Omega)} \int_{\bar\varrho - \alpha_1 \le \varrho \le \bar\varrho - \alpha_2} (r - \varrho)^2\,\mathrm{d}x + \frac{1}{8}\int_\Omega p(\varrho) b(\varrho)\,\mathrm{d}x \\ &\le C\|\mathrm{div}_x \mathbf{U}\|_{L^\infty(\Omega)} \int_{\bar\varrho - \alpha_1 \le \varrho \le \bar\varrho - \alpha_2} P(\varrho) - P(r) - P'(r)(\varrho - r)\,\mathrm{d}x + \frac{1}{8}\int_\Omega p(\varrho) b(\varrho)\,\mathrm{d}x \\ &\le \eta(t)\mathcal{E}(t) + \frac{1}{8}\int_\Omega p(\varrho) b(\varrho)\,\mathrm{d}x. \end{aligned} \tag{4.31}$$

Summarizing the estimates in (4.18), (4.19), (4.20), (4.26) and (4.31), we deduce

$$\mathcal{R}_1 \leq \eta(t)\mathcal{E}(t) + \frac{3}{16}\int_\Omega \mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})) : \nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x + \frac{1}{8}\int_\Omega p(\varrho)b(\varrho)\,\mathrm{d}x. \tag{4.32}$$

## 4.5 Estimates for the remainder: part 3

We start estimating the remainder term $\mathcal{R}_2$. By (4.29), we have

$$\langle b(\varrho)\rangle = \frac{1}{|\Omega|}\int_\Omega b(\varrho)\,\mathrm{d}x \leq C\mathcal{E}(t). \tag{4.33}$$

Then

$$\int_\Omega p(\varrho)\langle b(\varrho)\rangle\,\mathrm{d}x \leq C\int_\Omega p(\varrho)\,\mathrm{d}x\,\mathcal{E}(t) \leq \eta(t)\mathcal{E}(t). \tag{4.34}$$

We write

$$-\int_\Omega \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\nabla_x\Delta_x^{-1}\left(b(\varrho)-\langle b(\varrho)\rangle\right)\,\mathrm{d}x = \sum_{j=6}^{10} I_j \tag{4.35}$$

with

$$\begin{aligned}
I_6 &:= -\int_\Omega \varrho(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U}) : \nabla_x\nabla_x\Delta_x^{-1}\left(b(\varrho)-\langle b(\varrho)\rangle\right)\,\mathrm{d}x,\\
I_7 &:= -\int_\Omega \varrho\mathbf{U}\otimes(\mathbf{u}-\mathbf{U}) : \nabla_x\nabla_x\Delta_x^{-1}\left(b(\varrho)-\langle b(\varrho)\rangle\right)\,\mathrm{d}x,\\
I_8 &:= -\int_\Omega \varrho(\mathbf{u}-\mathbf{U})\otimes\mathbf{U} : \nabla_x\nabla_x\Delta_x^{-1}\left(b(\varrho)-\langle b(\varrho)\rangle\right)\,\mathrm{d}x,\\
I_9 &:= -\int_\Omega (\varrho-r)\mathbf{U}\otimes\mathbf{U} : \nabla_x\nabla_x\Delta_x^{-1}\left(b(\varrho)-\langle b(\varrho)\rangle\right)\,\mathrm{d}x,\\
I_{10} &:= -\int_\Omega r\mathbf{U}\otimes\mathbf{U} : \nabla_x\nabla_x\Delta_x^{-1}\left(b(\varrho)-\langle b(\varrho)\rangle\right)\,\mathrm{d}x.
\end{aligned} \tag{4.36}$$

For $I_6$, by Hölder's inequality, Young's inequality, Sobolev embedding and Korn's inequality, we have

$$\begin{aligned}
|I_6| &\leq \|\varrho(\mathbf{u}-\mathbf{U})\|_{L^2(\Omega)}\|\mathbf{u}-\mathbf{U}\|_{L^6(\Omega)}\|\nabla_x\nabla_x\Delta_x^{-1}\left(b(\varrho)-\langle b(\varrho)\rangle\right)\|_{L^3(\Omega)}\\
&\leq C\|\sqrt{\varrho}(\mathbf{u}-\mathbf{U})\|^2_{L^2(\Omega)}\|b(\varrho)\|^2_{L^3(\Omega)} + \frac{1}{16}\int_\Omega \mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})) : \nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x.
\end{aligned} \tag{4.37}$$

By (2.8) and (4.29), we have $P(\varrho)\in L^\infty(0,T;L^1)$ and $b(\varrho)\in L^\infty(0,T;L^\gamma(\Omega))$ for any $\gamma>1$. Thus, we deduce from (4.37) that

$$|I_6| \leq C\mathcal{E}(t) + \frac{1}{16}\int_\Omega \mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})) : \nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x. \tag{4.38}$$

For $I_7$ and $I_8$, by (4.29), there holds

$$\begin{aligned}
|I_7|+|I_8| &\leq \bar{\varrho}\|\mathbf{U}\|_{L^\infty(\Omega)}\int_\Omega \varrho|(\mathbf{u}-\mathbf{U})|^2\,\mathrm{d}x + \|\mathbf{U}\|_{L^\infty(\Omega)}\int_\Omega |\nabla_x\nabla_x\Delta_x^{-1}\left(b(\varrho)-\langle b(\varrho)\rangle\right)|^2\,\mathrm{d}x\\
&\leq \eta(t)\mathcal{E}(t) + \eta(t)\int_\Omega |b(\varrho)|^2\,\mathrm{d}x\\
&\leq \eta(t)\mathcal{E}(t) + \eta(t)\int_{\varrho\geq\bar{\varrho}-\alpha_1} P(\varrho)-P(r)-P'(r)(\varrho-r)\,\mathrm{d}x\\
&\leq \eta(t)\mathcal{E}(t).
\end{aligned} \tag{4.39}$$

For $I_9$, by Lemma 4.1 and (4.29), we have

$$
\begin{aligned}
|I_9| &\le \|\mathbf{U}\|^2_{L^\infty(\Omega)} \int_\Omega (\varrho - r)^2 \,\mathrm{d}x + \|\mathbf{U}\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla_x \nabla_x \Delta_x^{-1} \left(b(\varrho) - \langle b(\varrho)\rangle\right)|^2 \,\mathrm{d}x \\
&\le \eta(t) \int_\Omega P(\varrho) - P(r) - P'(r)(\varrho - r) \,\mathrm{d}x + \eta(t) \int_\Omega |b(\varrho)|^2 \,\mathrm{d}x \\
&\le \eta(t)\mathcal{E}(t).
\end{aligned}
\tag{4.40}
$$

For $I_{10}$, by (4.29), by Sobolev embedding and the smoothness of $r$, $\mathbf{U}$, we have

$$
\begin{aligned}
|I_{10}| &= \left| \int_\Omega \left(\Delta_x^{-1} \mathrm{div}_x \mathrm{div}_x (r\mathbf{U} \otimes \mathbf{U})\right) \left(b(\varrho) - \langle b(\varrho)\rangle\right) \,\mathrm{d}x \right| \\
&\le C\|(r, \mathbf{U})\|_{W^{1,\gamma}(\Omega)} \int_\Omega b(\varrho) \,\mathrm{d}x \quad (\text{with } \gamma > 3) \\
&\le \eta(t)\mathcal{E}(t),
\end{aligned}
\tag{4.41}
$$

## 4.6 Estimate for the remainder: part 4

We write

$$
\int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \nabla_x \Delta_x^{-1} \left(b(\varrho) - \langle b(\varrho)\rangle\right) \,\mathrm{d}x = I_{11} + I_{12}
\tag{4.42}
$$

with

$$
\begin{aligned}
I_{11} &:= \int_\Omega \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : \nabla_x \nabla_x \Delta_x^{-1} \left(b(\varrho) - \langle b(\varrho)\rangle\right) \,\mathrm{d}x, \\
I_{12} &:= \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x \nabla_x \Delta_x^{-1} \left(b(\varrho) - \langle b(\varrho)\rangle\right) \,\mathrm{d}x.
\end{aligned}
\tag{4.43}
$$

For $I_{11}$, by similar arguments as (4.39), we have

$$
\begin{aligned}
|I_{11}| &\le C \int_\Omega |b(\varrho)|^2 \,\mathrm{d}x + \frac{1}{16} \int_\Omega \mathbb{S}(\nabla_x (\mathbf{u} - \mathbf{U})) : \nabla_x (\mathbf{u} - \mathbf{U}) \,\mathrm{d}x \\
&\le C\mathcal{E}(t) + \frac{1}{16} \int_\Omega \mathbb{S}(\nabla_x (\mathbf{u} - \mathbf{U})) : \nabla_x (\mathbf{u} - \mathbf{U}) \,\mathrm{d}x.
\end{aligned}
\tag{4.44}
$$

For $I_{12}$, by similar arguments as (4.41), we have

$$
|I_{12}| \le C\|\mathbf{U}\|_{W^{2,\gamma}(\Omega)} \int_\Omega b(\varrho) \,\mathrm{d}x \le \eta(t)\mathcal{E}(t), \text{ with } \gamma > 3.
\tag{4.45}
$$

It is rather direct to deduce

$$
\begin{aligned}
-\int_\Omega \varrho \mathbf{f} \cdot \nabla_x \Delta_x^{-1} \left(b(\varrho) - \langle b(\varrho)\rangle\right) \,\mathrm{d}x &= \int_\Omega \left(\mathrm{div}_x \Delta_x^{-1} (\varrho \mathbf{f} - \langle \varrho \mathbf{f}\rangle)\right) b(\varrho) \,\mathrm{d}x \\
&\le C\|\varrho \mathbf{f}\|_{L^\infty(\Omega)} \int_\Omega b(\varrho) \,\mathrm{d}x \le C\mathcal{E}(t).
\end{aligned}
\tag{4.46}
$$

### 4.7 Estimate for the remainder: part 5

We write

$$\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\mathrm{div}_x(b(\varrho)\mathbf{u})\,\mathrm{d}x = \sum_{j=13}^{16} I_j, \tag{4.47}$$

where

$$\begin{aligned} I_{13} &:= \int_\Omega \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\mathrm{div}_x(b(\varrho)(\mathbf{u}-\mathbf{U}))\,\mathrm{d}x,\\ I_{14} &:= \int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\Delta_x^{-1}\mathrm{div}_x(b(\varrho)\mathbf{U})\,\mathrm{d}x,\\ I_{15} &:= \int_\Omega (\varrho-r)\mathbf{U}\cdot\nabla_x\Delta_x^{-1}\mathrm{div}_x(b(\varrho)\mathbf{U})\,\mathrm{d}x,\\ I_{16} &:= \int_\Omega r\mathbf{U}\cdot\nabla_x\Delta_x^{-1}\mathrm{div}_x(b(\varrho)\mathbf{U})\,\mathrm{d}x. \end{aligned} \tag{4.48}$$

Similarly as the derivation of the estimates for $I_j$ in Section 4.6, we can derive for some $\sigma > 0$ small that

$$\begin{aligned} |I_{13}| &\le C\|\mathbf{u}\|_{L^3(\Omega)}\|b(\varrho)(\mathbf{u}-\mathbf{U})\|_{L^{\frac{3}{2}}} \le C\sigma^{-1}\|\mathbf{u}\|^2_{L^6(\Omega)}\|b(\varrho)\|^2_{L^2(\Omega)} + \sigma\|(\mathbf{u}-\mathbf{U})\|^2_{L^6(\Omega)}\\ &\le \eta(t)\mathcal{E}(t) + \frac{1}{16}\int_\Omega \mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})) : \nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x, \end{aligned} \tag{4.49}$$

$$|I_{14}| \le \bar\varrho\int_\Omega \varrho|\mathbf{u}-\mathbf{U}|^2\,\mathrm{d}x + \int_\Omega |b(\varrho)\mathbf{U}|^2\,\mathrm{d}x \le \eta(t)\mathcal{E}(t) + \|\mathbf{U}\|^2_{L^\infty(\Omega)}\int_\Omega |b(\varrho)|^2\,\mathrm{d}x \le \eta(t)\mathcal{E}(t), \tag{4.50}$$

$$|I_{15}| \le \|\mathbf{U}\|^2_{L^\infty(\Omega)}\int_\Omega (\varrho-r)^2\,\mathrm{d}x + \int_\Omega |b(\varrho)\mathbf{U}|^2\,\mathrm{d}x \le \eta(t)\mathcal{E}(t), \tag{4.51}$$

and for $\gamma > 3$ :

$$|I_{16}| = \int_\Omega \nabla_x\Delta_x^{-1}\mathrm{div}_x(r\mathbf{U})\cdot(b(\varrho)\mathbf{U})\,\mathrm{d}x \le \|(r,\mathbf{U})\|_{W^{1,\gamma}(\Omega)}\int_\Omega b(\varrho\,\mathrm{d}x \le \eta(t)\mathcal{E}(t). \tag{4.52}$$

### 4.8 Estimate for the remainder: part 6

We write

$$\begin{aligned} &\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\Delta_x^{-1}\big((b'(\varrho)\varrho-b(\varrho))\,\mathrm{div}_x\mathbf{u} - \langle(b'(\varrho)\varrho-b(\varrho))\,\mathrm{div}_x\mathbf{u}\rangle\big)\,\mathrm{d}x\\ &= -\int_\Omega \big(\mathrm{div}_x\Delta_x^{-1}(\varrho\mathbf{u}-\langle\varrho\mathbf{u}\rangle)\big)\big((b'(\varrho)\varrho-b(\varrho))\,\mathrm{div}_x\mathbf{u}\big)\,\mathrm{d}x = \sum_{j=17}^{19} I_j, \end{aligned} \tag{4.53}$$

where

$$\begin{aligned} I_{17} &:= -\int_\Omega \big(\mathrm{div}_x\Delta_x^{-1}(\varrho\mathbf{u}-\langle\varrho\mathbf{u}\rangle)\big)\big((b'(\varrho)\varrho-b(\varrho))\,\mathrm{div}_x\mathbf{U}\big)\,\mathrm{d}x,\\ I_{18} &:= -\int_\Omega \big(\mathrm{div}_x\Delta_x^{-1}(\varrho\mathbf{U}-\langle\varrho\mathbf{U}\rangle)\big)\big((b'(\varrho)\varrho-b(\varrho))\,\mathrm{div}_x(\mathbf{u}-\mathbf{U})\big)\,\mathrm{d}x,\\ I_{19} &:= -\int_\Omega \big(\mathrm{div}_x\Delta_x^{-1}\left(\varrho(\mathbf{u}-\mathbf{U})-\langle\varrho(\mathbf{u}-\mathbf{U})\rangle\right)\big)\big((b'(\varrho)\varrho-b(\varrho))\,\mathrm{div}_x(\mathbf{u}-\mathbf{U})\big)\,\mathrm{d}x. \end{aligned} \tag{4.54}$$

For $I_{17}$, by (4.29), we have

$$
\begin{aligned}
|I_{17}| &\leq C\|\mathrm{div}_x\Delta_x^{-1}(\varrho\mathbf{u}-\langle\varrho\mathbf{u}\rangle)\|_{L^\infty(\Omega)}\|\mathrm{div}_x\mathbf{U}\|_{L^\infty(\Omega)}\int_\Omega|(b'(\varrho)\varrho-b(\varrho)|\,\mathrm{d}x \\
&\leq C\|\mathrm{div}_x\Delta_x^{-1}(\varrho\mathbf{u}-\langle\varrho\mathbf{u}\rangle)\|_{W^{1,6}(\Omega)}\|\mathrm{div}_x\mathbf{U}\|_{L^\infty(\Omega)}\int_{\varrho\geq\bar\varrho-\alpha_1}P(\varrho)\,\mathrm{d}x \\
&\leq C\|\varrho\mathbf{u}\|_{L^6(\Omega)}\|\mathrm{div}_x\mathbf{U}\|_{L^\infty(\Omega)}\mathcal{E}(t)\leq C\|\nabla_x\mathbf{u}\|_{L^2(\Omega)}\|\mathrm{div}_x\mathbf{U}\|_{L^\infty(\Omega)}\mathcal{E}(t)\leq\eta(t)\mathcal{E}(t).
\end{aligned}
\tag{4.55}
$$

For $I_{18}$, by Korn's inequality, we have for $\gamma>3$:

$$
|I_{18}|\leq C\|\mathbf{U}\|^2_{L^\gamma(\Omega)}\int_\Omega|(b'(\varrho)\varrho-b(\varrho)|^2\,\mathrm{d}x+\frac{1}{16}\int_\Omega\mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})):\nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x. \tag{4.56}
$$

By the fact $\mathbf{U}\in L^\infty(0,T;L^\gamma(\Omega))$ and (4.28), by choosing $\alpha_1$ be sufficiently small, we have

$$
|I_{18}|\leq\frac{1}{8}\int_\Omega p(\varrho)b(\varrho)\,\mathrm{d}x+\frac{1}{16}\int_\Omega\mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})):\nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x. \tag{4.57}
$$

By Sobolev embedding, by (4.29) and $\beta\geq3$, we have

$$
\begin{aligned}
|I_{19}| &\leq C\|\varrho(\mathbf{u}-\mathbf{U})\|_{L^2(\Omega)}\|(b'(\varrho)\varrho-b(\varrho)\|_{L^3(\Omega)}\|\mathrm{div}_x(\mathbf{u}-\mathbf{U})\|_{L^2(\Omega)} \\
&\leq C\|(b'(\varrho)\varrho-b(\varrho)\|^2_{L^3(\Omega)}\int_\Omega\varrho|\mathbf{u}-\mathbf{U}|^2\,\mathrm{d}x+\frac{1}{16}\int_\Omega\mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})):\nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x \\
&\leq\eta(t)\mathcal{E}(t)+\frac{1}{16}\int_\Omega\mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})):\nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x.
\end{aligned}
\tag{4.58}
$$

### 4.9 Estimate for the remainder: part 7

Since the initial data $\varrho_0=r_0\in[\alpha_0,\bar\varrho-\alpha_0]$, we have $b(\varrho_0)\equiv0$. We can write

$$
\mathcal{R}_3=I_{20}+I_{21} \tag{4.59}
$$

with

$$
\begin{aligned}
I_{20}:&=\int_\Omega\varrho\mathbf{U}\cdot\nabla_x\Delta_x^{-1}\left(b(\varrho)-\langle b(\varrho)\rangle\right)(\tau,\cdot)\,\mathrm{d}x, \\
I_{21}:&=\int_\Omega\varrho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\Delta_x^{-1}\left(b(\varrho)-\langle b(\varrho)\rangle\right)(\tau,\cdot)\,\mathrm{d}x.
\end{aligned}
\tag{4.60}
$$

For $I_{20}$, by (4.28) and choosing $\alpha_1$ be sufficiently small, we have

$$
\begin{aligned}
I_{20} &= -\int_\Omega\mathrm{div}_x\Delta_x^{-1}(\varrho\mathbf{U}-\langle\varrho\mathbf{U}\rangle)b(\varrho)(\tau,\cdot)\,\mathrm{d}x\leq C\int_{\varrho\geq\bar\varrho-\alpha_1}b(\varrho)(\tau,\cdot)\,\mathrm{d}x \\
&\leq\frac{1}{4}\int_{\varrho\geq\bar\varrho-\alpha_1}P(\varrho)-P(r)-P'(r)(\varrho-r)\,\mathrm{d}x,
\end{aligned}
\tag{4.61}
$$

which can be absorbed by the relative entropy on the left-hand side of the relative entropy inequality.

Similarly, for $I_{21}$ there holds by choosing $\alpha_1$ be sufficiently small:

$$
\begin{aligned}
I_{21} &\leq\frac{1}{4}\int_\Omega\varrho|\mathbf{u}-\mathbf{U}|^2(\tau,\cdot)\,\mathrm{d}x+C\int_\Omega|b(\varrho)|^2(\tau,\cdot)\,\mathrm{d}x \\
&\leq\frac{1}{4}\int_\Omega\varrho|\mathbf{u}-\mathbf{U}|^2(\tau,\cdot)\,\mathrm{d}x+\frac{1}{4}\int_{\varrho\geq\bar\varrho-\alpha_1}P(\varrho)-P(r)-P'(r)(\varrho-r)\,\mathrm{d}x,
\end{aligned}
\tag{4.62}
$$

which can also be absorbed by the relative entropy on the left-hand side of the relative entropy inequality.

### 4.10 End of the proof

Summarizing the estimates we obtained in Sections 4.3–(4.9) above, we derive

$$\mathcal{E}(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})) : \nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega p(\varrho)b(\varrho)\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^\tau \eta(t)\mathcal{E}(t)\,\mathrm{d}t, \tag{4.63}$$

for some $\eta(t) \in L^1((0,T))$. Then Grownwall's inequality implies $\mathcal{E}(t) \equiv 0$ in $(0,T)$. This implies our desired weak-strong uniqueness and the proof of Theorem 2.5 is completed.

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