Weak and measure–valued solutions to the complete Euler system

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Complete Euler system

Standard formulation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$
$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = 0$$
$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e + p \right) \mathbf{u} \right] = 0$$

Conservative variables

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$
$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0$$
$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$
$$p = (\gamma - 1)\varrho e, \ \ p = (\gamma - 1) \left(E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} \right)$$

Entropy

Gibbs' relation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy balance

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) \boxed{\geq} 0$$

Entropy in the polytropic case

$$s = S\left(\frac{p}{\varrho^\gamma}\right) = S\left((\gamma - 1)\frac{E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}}{\varrho^\gamma}\right)$$

Admissible weak solutions

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = 0$$

$$\partial_t \left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e \right) + \mathrm{div}_x \left[\left(\frac{1}{2}\varrho |\mathbf{u}|^2 + \varrho e + p \right) \mathbf{u} \right] = 0$$

Entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) \boxed{\geq} 0$$

Impermeability boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Infinitely many weak solutions

Initial data

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \vartheta(0,\cdot) = \vartheta_0.$$

Existence via convex integration

Let $N = 2, 3$. Let ϱ_0, ϑ_0 be piecewise constant (arbitrary).
Then the exists $\mathbf{u}_0 \in L^\infty$ such that the Euler system admits infinitely many admissible weak solution in $(0, T) \times \Omega$.

Measure–valued solutions

Parameterized measure and the dissipation defect

$$\mathcal{S} = \left\{ \varrho \geq 0,\ \mathbf{m} \in R^3,\ E \in [0, \infty) \right\}$$

$$\left\{ Y_{t,x} \right\}_{t \in (0,T), x \in \Omega},\ \ Y_{t,x} \in \mathcal{P}(\mathcal{S}),\ \ \mathcal{D} \in L^\infty(0,T),\ \ \mathcal{D} \geq 0$$

Field equations

$$\partial_t \left< Y_{t,x}; \varrho \right> + \mathrm{div}_x \left< Y_{t,x}; \mathbf{m} \right> = 0$$

$$\partial_t \left< Y_{t,x}; \mathbf{m} \right> + \mathrm{div}_x \left< Y_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right> + \nabla_x \left< Y_{t,x}; p \right> = D_x \mu_C$$

$$\partial_t \int_\Omega \left< Y_{t,x}; E \right> \ \mathrm{d}x + \mathcal{D} = 0,\ \ \partial_t \left< Y_{t,x}; \varrho s \right> + \mathrm{div}_x \left< Y_{t,x}; s\mathbf{m} \right> \geq 0$$

Compatibility

$$\int_0^\tau \int_\Omega |\mu_C| \ \mathrm{d}x\mathrm{d}t \leq C \int_0^\tau \mathcal{D} \mathrm{d}t$$

Relative energy

Relative energy in the standard variables

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle|\tilde{\varrho},\tilde{\vartheta},\tilde{\mathbf{u}}\right)$$

$$=\frac{1}{2}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2+\partial_\varrho H_{\tilde{\vartheta}}(\tilde{\varrho},\tilde{\vartheta})(\varrho-\tilde{\varrho})-H_{\tilde{\vartheta}}(\tilde{\varrho},\tilde{\vartheta})$$

$$H_{\tilde{\vartheta}}(\varrho,\vartheta)=\varrho\Big(e(\varrho,\vartheta)-\tilde{\vartheta}s(\varrho,\vartheta)\Big)$$

Relative energy in the conservative variables

$$\mathcal{E}\left(\varrho,\mathbf{m},E\middle|\tilde{\varrho},\tilde{\mathbf{m}},\tilde{E}\right)$$

$$=-\tilde{\vartheta}\Big[\varrho s-\partial_\varrho(\varrho s)(\varrho-\tilde{\varrho})-\nabla_{\mathbf{m}}(\varrho s)\cdot(\mathbf{m}-\tilde{\mathbf{m}})-\partial_E(\varrho s)(E-\tilde{E})$$

$$-\tilde{\varrho}\tilde{s}\Big]$$

Thermodynamic stability

Thermodynamic stability in the standard variables

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Thermodynamic stability in the conservative variables

$$(\varrho, \mathbf{m}, E) \mapsto \varrho s(\varrho, \mathbf{m}, E)$$

is a concave function

Thermodynamic stability in the polytropic case

$$\varrho s = \varrho S\left(\frac{p}{\varrho^{\gamma}}\right), \ \ p = (\gamma - 1)\varrho e$$

$$S'(Z) > 0, \ (1-\gamma)S'(Z) - \gamma S''(Z)Z > 0$$

Stability of strong solutions

Measure–valued strong uniqueness

Suppose the thermodynamic functions p, e, and s comply with the hypothesis of thermodynamic stability. Let $(\varrho, \mathbf{m}, E)$ be a smooth (C^1) solution of the Euler system and let $(Y_{t,x}; \mathcal{D})$ be a dissipative measure–valued solution of the same system with the same initial data, meaning

$$Y_{0,x} = \delta_{\varrho_0(x), \mathbf{m}_0(x), E_0(x)} \text{ for a.a. } x \in \Omega.$$

Then

$$\mathcal{D} \equiv 0, \ \ Y_{t,x} = \delta_{\varrho(t,x), \mathbf{m}(t,x), E(t,x)}$$

for a.a. $(t, x) \in (0, T) \times \Omega$.

Generating MV solutions, limits weak $\to$ MV

Navier–Stokes–Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$
$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x\left(\varrho \mathbf{u} \otimes \mathbf{u}\right) + \nabla_x(p + ap_R) = \nu \mathrm{div}_x \mathbb{S},$$
$$\partial_t\left(\varrho(e + ae_R)\right) + \mathrm{div}_x\left(\varrho(e + ae_R)\mathbf{u}\right) + \omega \nabla_x \mathbf{q}$$
$$= \nu \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u} - \lambda(\vartheta - \overline{\vartheta})^3.$$

Constitutive assumptions, radiative components

$$\mathbb{S}(\varrho, \nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right),$$
$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta$$
$$p_R = \frac{1}{3}\vartheta^4, \ e_R = \frac{\vartheta^4}{\varrho}, \ s_R = \frac{4}{3}\frac{\vartheta^3}{\varrho}$$

Limit (weak) $\to$ (MV)

Vanishing dissipation limit

Suppose that p and e are interrelated through the polytropic EOS with $\gamma = \frac{5}{3}$, and "other mostly technical conditions". Let

$$\nu = \omega = \varepsilon,\ a\varepsilon^{\alpha},\ \alpha > 1,\ \lambda = \varepsilon^{\beta},\ \beta < 1.$$

Let $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ be a family of weak solutions to the Navier–Stokes–Fourier system periodic in the space variable. Then $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ generates a Young measure Y and the energy defect measure a function $\mathcal{D}$ - a (DMV) solution of the Euler system.

Limits of Euler flows with strong stratification

Scaled Euler system

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \frac{1}{\varepsilon^2} \varrho \nabla_x \Phi,$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \varrho e(\varrho, \vartheta) \right) \mathbf{u} \right]$$

$$+ \mathrm{div}_x \left(\frac{1}{\varepsilon^2} p(\varrho, \vartheta) \mathbf{u} \right) = \frac{1}{\varepsilon^2} \varrho \nabla_x \Phi \cdot \mathbf{u}.$$

Geometry

$$\Omega = \mathcal{T}^2 \times (0,1), \ \mathcal{T}^2 = [0,1]|_{\{0,1\}} - \text{the two dimensional torus}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} - 0$$

Initial data

Stationary problem

$$p = \varrho\vartheta,\ \ \Phi = \Phi(z) = -z$$

$$\nabla_x(\varrho_s \overline{\Theta}) = -\varrho_s \nabla_x \Phi,\ \ \varrho_s = \exp\left(-\frac{z}{\overline{\Theta}}\right),\ \ \overline{\Theta} > 0$$

Well–prepared initial data

$$\varrho_{0,\varepsilon} = \varrho_s + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \ \vartheta_{0,\varepsilon} = \overline{\Theta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \ \mathbf{u}_{0,\varepsilon}$$

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)} + \|\vartheta^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^\infty(\Omega;R^N)} \leq c,$$

$$\varrho^{(1)}_{\varepsilon} \to 0,\ \ \vartheta^{(1)}_{0,\varepsilon} \to 0,\ \ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0,$$

$$\mathbf{U}_0 \in W^{k,2}(\Omega; R^3),\ \ k > 3,\ \ \mathbf{U}_0 = [U^1_0, U^2_0, 0],\ \ \mathrm{div}_h \mathbf{U}_0 = 0.$$

Target problem

Euler system

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_h \mathbf{U} + \nabla_x \Pi = 0,\ \operatorname{div}_h \mathbf{U} = 0,\ x_h \in \mathcal{T}^2,$$

Stratified initial data

$$\mathbf{U}(0,x) = \mathbf{U}_0(x_h, z) = [U_0^1(x_h,z), U_0^2(x_h,z), 0]$$

Singular limit (MV) $\to$ strong

Convergence to the target system

Let $\{Y^\varepsilon_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$, $\mathcal{D}^\varepsilon$ be a family of dissipative measure–valued solutions to the scaled system scaled Euler system, with the well prepared initial data

$$Y^\varepsilon_{0,x} = \delta_{\varrho_{0,\varepsilon},\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon},c_v\varrho_{0,\varepsilon}\vartheta_{0,\varepsilon}}.$$

Then

$$\mathcal{D}^\varepsilon \to 0 \text{ in } L^\infty(0,T),$$

and

$$Y^\varepsilon \to \delta_{\varrho_s,\varrho_s\mathbf{U},c_v\varrho_s\overline{\Theta}} \text{ in } L^\infty(0,T;\mathcal{M}^+(\mathcal{F})_{\mathrm{weak}-(*)}),$$

where $[\varrho_s,\overline{\Theta}]$ is the static state and $\mathbf{U}$ is the unique solution to the incompressible 2D Euler system