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Construction of Function Spaces Close to L^∞ with Associate Space Close to L^1

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Abstract The paper introduces a variable exponent space X which has in common with $L^\infty([0, 1])$ the property that the space $C([0, 1])$ of continuous functions on $[0, 1]$ is a closed linear subspace in it. The associate space of X contains both the Kolmogorov and the Marcinkiewicz examples of functions in L^1 with a.e. divergent Fourier series.

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1 Introduction

One of the fundamental facts about Fourier series is that $L^1(\mathbb{T})$ (where as usual $\mathbb{T}$ denotes the one-dimensional torus) is not especially pleasant in that, unlike $L^p(\mathbb{T})$ when $p > 1$, it contains a function with a Fourier series that is almost everywhere divergent. This was first shown by Kolmogorov, who with remarkable ingenuity constructed such a function. In fact his function belongs to the space $L \log \log L(\mathbb{T})$ that is slightly smaller than $L^1(\mathbb{T})$, and its partial Fourier series diverges unboundedly a.e. Some years later Marcinkiewicz gave an example of a function in $L^1(\mathbb{T})$ with a.e. divergent Fourier series even though its partial sums were bounded; various other examples have been given over the years. From this point of view the gulf between $L^1(\mathbb{T})$ and $\cup_{p>1} L^p(\mathbb{T})$ is wide. The situation is different if, instead of the Lebesgue spaces $L^p(\mathbb{T})$ with $p > 1$ we consider the so-called variable exponent Lebesgue spaces on $\mathbb{T}$. These have attracted considerable attention in recent years, principally because of the role they play in various applications, such as variational problems with integrands having non-standard growth. To explain briefly what they are, given a measurable $p : \mathbb{T} \to [1, \infty)$, the Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ with variable exponent p is the space of all measurable functions f on $\mathbb{T}$ such that for some $\lambda > 0$, $I(\lambda, f) := \int_{\mathbb{T}} (|f(x)|/\lambda)^{p(x)} dx < \infty$; it becomes a Banach space when endowed with the norm $\|f\|_{p(\cdot)} := \inf \{\lambda > 0 : I(\lambda, f) \le 1\}$; and it coincides with the classical L^p space when p is constant. It turns out that

$$L^1(\mathbb{T}) = \cup L^{p(\cdot)}(\mathbb{T}),$$

where the union is taken over all measurable p such that $p(x) > 1$ a.e. Thus any function with Fourier series that is divergent a.e. must belong to some variable exponent space $L^{p(\cdot)}(\mathbb{T})$, and just as it is interesting to know that the Kolmogorov function belongs to $L \log \log L(\mathbb{T})$, so it is natural to find out to which space $L^{p(\cdot)}(\mathbb{T})$ it belongs.

In this paper we show that there is a variable exponent space $L^{p(\cdot)}(\mathbb{T})$, with $1 < p(x) < \infty$ a.e., which has in common with $L^\infty(\mathbb{T})$ the property that the space $C(\mathbb{T})$ of continuous functions on $\mathbb{T}$ is a closed linear subspace in it. Moreover, both the Kolmogorov and the Marcinkiewicz functions belong to $L^{q(\cdot)}(\mathbb{T})$, where $1/q(x) = 1 - 1/p(x)$ for all x. As might be expected, some knowledge of the process of construction of the exceptional functions is necessary, and we give the crucial steps for the convenience of the reader.

2 Preliminaries

Let Ω be a non-empty open subset of $\mathbb{R}^n$; let $\mathcal{M}(\Omega)$ be the set of all measurable and almost everywhere finite real-valued functions on Ω; and given any measurable subset E of Ω, denote by $|E|$ the Lebesgue n-measure of E and by χ_E its characteristic function. The open ball in $\mathbb{R}^n$ with centre x and radius r will be denoted by $B(x, r)$. As usual, we say that a linear space $X = X(\Omega) \subset \mathcal{M}(\Omega)$, equipped with a norm

$\|\cdot\|_X$, is a *Banach function space* (BFS) on Ω if whenever $f, f_n, g \in \mathcal{M}(\Omega)$ $(n \in \mathbb{N})$, the following axioms hold:

(P1) $0 \le g \le f$ a.e. implies that $\|g\|_X \le \|f\|_X$;

(P2) $0 \le f_n \uparrow f$ a.e. implies that $\|f_n\|_X \uparrow \|f\|_X$;

(P3) $\|\chi_E\|_X < \infty$ if $E \subset \Omega$ and $|E| < \infty$;

(P4) given any $E \subset \Omega$ with $|E| < \infty$, there is a constant $C_E > 0$ such that for all $f \in X$,

$$\int_E f \, dx \le C_E \|f\|_X.$$

The associate space X' of a BFS X is the set of all $g \in \mathcal{M}(\Omega)$ such that $f.g \in L^1(\Omega)$; when endowed with the norm

$$\|g\|_{X'} := \sup \left\{ \|f.g\|_{L^1(\Omega)} : \|f\|_X \le 1 \right\}$$

it is a BFS on Ω. Moreover, X' is a closed, norm fundamental subspace of the dual X^* of X. We refer to [3] for basic properties of Banach function spaces.

Let f be a measurable, real-valued function on Ω. Its *non-increasing rearrangement* f^* is defined by

$$f^*(t) = \inf \left\{ \lambda \in (0, \infty) : |\{x \in \Omega : |f(x)| > \lambda\}| \le t \right\}, t \in [0, |\Omega|].$$

A BFS X is said to be *rearrangement-invariant* (r.i.) if

(P5) $\|f\|_X = \|g\|_X$ whenever $f^* = g^*$.

To every r.i. space $X(\Omega)$ there corresponds a unique r.i. space $\overline{X}((0, |\Omega|))$ such that $\|f\|_{X(\Omega)} = \|f^*\|_{\overline{X}((0,|\Omega|))}$ for all $f \in X(\Omega)$. This space, endowed with the norm

$$\|f\|_{\overline{X}((0,|\Omega|))} := \sup_{\|g\|_{X'(\Omega)} \le 1} \int_0^{|\Omega|} f^*(t) g^*(t) dt,$$

is called the *representation space* of $X(\Omega)$.

The *fundamental function* of an r.i. space $X(\Omega)$ is the map $\phi_X : [0, |\Omega|] \to [0, \infty)$ defined by

$$\phi_X(t) = \left\| \chi_{(0,t)} \right\|_{\overline{X}((0,|\Omega|))} \quad (t \in (0, |\Omega|)), \phi_X(0) = 0.$$

We now introduce various interesting subspaces of a BFS $X(\Omega)$. A function f in X is said to have *absolutely continuous norm* in X if $\left\| f \chi_{E_n} \right\|_X \to 0$ whenever $\{E_n\}$ is a sequence of measurable subsets of Ω such that $\chi_{E_n} \downarrow 0$ a.e. The set of all such functions is denoted by X_a. By X_b is meant the closure of the set of all bounded functions in X. Following Lai and Pick [15], a function $f \in X$ is said to

have *continuous norm* in X if for every $x \in \Omega$, $\lim_{\varepsilon \to 0} \left\| f \chi_{B(x,\varepsilon)} \right\|_X = 0$; the set of all these functions is written as X_c. The connection between this notion and the compactness of Hardy operators from a weighted BFS (X, w) to L^∞ is explored in [15]; for a connection with unconditional bases in BFSs see [13], [14]. In general, the relation between the subspaces X_a, X_b and X_c is complicated: for example (see [16]), there is a BFS X for which $\{0\} = X_a \subsetneq X_c = X$.

We now focus on the case in which Ω is a bounded interval in the real line, taken to be $(0, 1)$ for simplicity, although the arguments will work for any bounded interval (a, b). Let $I = [0, 1]$ and let $\mathcal{P}(I)$ be the family of all measurable functions $p : I \to [1, \infty)$. When $p \in \mathcal{P}(I)$ we denote by $L^{p(\cdot)}(I)$ the set of all measurable functions f on I such that for some $\lambda > 0$,

$$\int_0^1 \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

This set becomes a BFS when equipped with the norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \int_0^1 \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\};$$

it is often referred to as a (Lebesgue) space with variable exponent. When p is constant, the space coincides with the standard space $L^p(I)$. Spaces with variable exponent, and Sobolev spaces $W^{k,p(\cdot)}$ based upon them not only have intrinsic interest but also have applications to partial differential equations and the calculus of variations. More details will be found in [6] and [7]. For the particular BFS $X = L^{p(\cdot)}(I)$ the relation between it and its subspaces X_a, X_b and X_c was investigated in [8]: we give some of the results of that paper next.

Proposition 2.1 *Let $p \in \mathcal{P}(I)$ and set $X = L^{p(\cdot)}(I)$. Then*

  (i) $X_a = X_c$;
  (ii) $X_b = X$ *if and only if* $p(\cdot) \in L^\infty(I)$;
  (iii) $X_a = X_b$ *if and only if*

$$\int_0^1 A^{p^*(x)} dx < \infty \text{ for all } A > 1,$$

where p^* is the non-increasing rearrangement of p.

Further understanding of these relations is given by the following examples taken from [8].

Example 2.2 Let $n = 1$ and $\Omega = (0, 1/e)$. Then

  (i) if $p(x) = x^\alpha$ with $\alpha < 0$, then $X_a \subsetneq X_b$;
  (ii) if $p(x) = (\log x^{-1})^\alpha$, then $X_a = X_b$ if $\alpha \in (0, 1]$, and $X_a \subsetneq X_b$ if $\alpha > 1$.

3 A Variable Exponent Lebesgue Space Close to $L^\infty(I)$

For the remainder of the paper we shall denote by m the function defined, for every $x \in (0, 1]$ by

$$m(x) = \chi_{(0,1/200)}(x) \log(1/x);$$

$\{\delta_k\}$ will be a sequence of positive numbers with

$$\lim_{k\to\infty} \delta_k = 0 \text{ and } \sum_{k=1}^{\infty} \int_0^{\delta_k} \log(1/x)dx < \infty; \tag{3.1}$$

and $\{r_k\}$ will be an enumeration of the rationals in $I = [0, 1]$, (or some dense set in I). Moreover, $\widetilde{p}$ will be the function defined on I by

$$\widetilde{p}(x) = 2 + \sum_{k=1}^{\infty} m(x - r_k)\chi_{(r_k,r_k+\delta_k)}(x). \tag{3.2}$$

Using the elementary fact that

$$\int_a^{a+\delta} -\log x\, dx = \delta + \log\left\{\left(\frac{a}{a+\delta}\right)^a (a+\delta)^{-\delta}\right\} < \delta - \delta \log \delta,$$

it follows from the monotone convergence theorem and (3.1) that $\widetilde{p}(x)$ is finite a.e. on I. Moreover,

$$1 < \text{ ess inf } \widetilde{p}(x), \text{ ess sup } \widetilde{p}(x) = \infty,$$

and

$$L^\infty(I) \subset L^{\widetilde{p}(\cdot)}(I) \subset L^1(I).$$

To investigate further properties of $L^{\widetilde{p}(\cdot)}(I)$, we prove following theorem, (as we know it is new).

Theorem 3.1 *Let X be a BFS on I. The space $C(I)$ of continuous functions on I is a closed linear subspace of X if and only if there exists a positive constant c satisfying*

$$\|\chi_{(a,b)}\|_X \geq c \quad \text{whenever} \quad 0 \leq a < b \leq 1. \tag{3.3}$$

Proof For sufficiency part it is enough to show that there is a positive constant C such that for every $f \in C(I)$,

$$C\|f\|_{C(I)} \leq \|f\|_X \leq \|f\|_{C(I)}. \tag{3.4}$$

The second of these inequalities is clear. For the first, let $f \in C(I)$. There exists $x_0 \in I$ such that $\|f\|_{C(I)} = |f(x_0)|$; there exists $\varepsilon > 0$ such that $|f(x_0)| \leq 2|f(x)|$ if $x \in (x_0 - \varepsilon, x_0 + \varepsilon) \cap I := E$. Thus from (3.3) we see that

$$\|f\|_{C(I)} = |f(x_0)| \leq \frac{1}{c}|f(x_0)| \|\chi_E\|_X \leq \frac{2}{c}\|f\chi_E\|_X \leq \frac{2}{c}\|f\|_X.$$

Necessity. If $C(I)$ is a closed subset of X, then by the closed graph theorem, we have the estimate (3.4). Let given any interval $(a, b) \subset I$ be given, if we take a continuous function g on I such that $g \leq \chi_{(a,b)}$ and $\|g\|_{L^\infty} = 1$ we get (3.3). □

We now establish further properties of $L^{\widetilde{p}(\cdot)}(I)$.

Theorem 3.2 *For any $(a, b) \subset I$, we have*

$$\left\|\chi_{(a,b)}\right\|_{\widetilde{p}(\cdot)} > 1/e. \tag{3.5}$$

Proof Let $(a, b) \subset I$ and let $k \in \mathbb{N}$ be such that $a < s_k < s_k + \delta_k < b$ for some rational s_k. Then

$$\int_a^b \left(\frac{1}{1/e}\right)^{\widetilde{p}(x)} dx \geq \int_{s_k}^{s_k+\delta_k} \left(\frac{1}{1/e}\right)^{\widetilde{p}(x)} dx$$

$$\geq \int_{s_k}^{s_k+\delta_k} \exp\left\{\log\left(\frac{1}{x - s_k}\right)\right\} dx$$

$$= \int_0^{\delta_k} \frac{dx}{x} = \infty,$$

and so we have (3.5). □

Corollary 3.3 *The space $C(I)$ of continuous functions on I is a closed linear subspace of $L^{\widetilde{p}(\cdot)}(I)$.*

Proof Using Theorem 3.2 from Theorem 3.1 we obtain that there is a positive constant C such that for every $f \in C(I)$,

$$C\|f\|_{C(I)} \leq \|f\|_{\widetilde{p}(\cdot)} \leq \|f\|_{C(I)}.$$

From these estimates the proof of corollary follows. □

Remark 3.4 With the conjugate exponent p' defined by $1/p(x) + 1/p'(x) = 1$ $(x \in I, p \in \mathcal{P}(I))$, it is known that $L^{p'(\cdot)}(I)$ is isomorphic to the dual $\left(L^{p(\cdot)}(I)\right)^*$ of $L^{p(\cdot)}(I)$ if and only if $p(\cdot) \in L^\infty(I)$; when ess sup $p(x) = \infty$, $L^{p'(\cdot)}(I)$ is isomorphic to a proper closed subspace of $\left(L^{p(\cdot)}(I)\right)^*$. The space $L^{\widetilde{p}(\cdot)}(I)$ constructed above has some properties similar to those of $L^\infty(I)$. For example, the dual of $L^\infty(I)$ is the set of finitely additive measures that are absolutely continuous with respect to Lebesgue measure. These functionals are extensions to $L^\infty(I)$ of continuous linear functionals on $C(I)$. Since $C(I)$ is a closed linear subspace of $L^{\widetilde{p}(\cdot)}(I)$, it follows from the Hahn-Banach theorem that any bounded linear functional on $C(I)$ can be extended to $L^{\widetilde{p}(\cdot)}(I)$.

Remark 3.5 Given any interval $(a, b) \subset I$, there is a no r.i. space $X((a, b))$ different from $L^\infty((a, b))$ such that $L^\infty((a, b)) \subset X((a, b)) \subset L^{\widetilde{p}(\cdot)}((a, b))$. For by (3.5), there exists $C > 0$ such that $\phi_X(t) \geq C$ for all $t \in (0, b - a)$, and thus $X((a, b)) = L^\infty((a, b))$.

It is clear that

$$L_a^{\widetilde{p}(\cdot)}(I) \neq \{0\}.$$

For each $n \in \mathbb{N}_0$ let $E_n = \{x \in I : n \leq \widetilde{p}(x) < n + 1\}$ and let $\{G_n\}$ be a sequence of disjoint sets such that $G_n \subset E_n$ and $|G_n| < \exp(-e^n)$; define a function g by

$$g(x) = \sum_{n=0}^{\infty} n \chi_{G_n}(x).$$

This function does not belong to $L^\infty(I)$, but since $\int_0^1 (Ag(x))^{\widetilde{p}(x)} dx < \infty$ for every $A > 1$, it is in $L_a^{\widetilde{p}(\cdot)}(I)$.

Therefore

$$L^\infty \subsetneq L_a^{\widetilde{p}(\cdot)}(I).$$

4 A Variable Exponent Lebesgue Space Close to $L^1(I)$

Let $\widetilde{q}(\cdot)$ be the conjugate exponent of the function $\widetilde{p}(\cdot)$ defined by (3.2), i.e.

$$\frac{1}{\widetilde{q}(\cdot)} + \frac{1}{\widetilde{p}(\cdot)} = 1, \quad x \in I,$$

with the convention that $1/\infty = 0$. Note that $\widetilde{q}(x) > 1$ for a.e. $x \in I$, and that the essential infimum of $\widetilde{q}(x)$ on every interval $(a, b) \subset I$ is 1; moreover, $L^{\widetilde{q}(\cdot)}(I)$ can be identified with the associate space $\left(L^{p(\cdot)}(I)\right)'$.

The conjugate of the function $x \longmapsto 1 + m(x - r_k)\chi_{(r_k, r_k + \delta_k)}(x)$ on the interval $(r_k, r_k + \delta_k)$ is $\widetilde{q}_k$, where $\widetilde{q}_k(x) = 1 + 1/\log\left(\frac{1}{x - r_k}\right)$: thus $\widetilde{q}_k(\cdot)$ satisfies the estimates

$$1 - \frac{1}{\widetilde{q}_k(x)} \leq \frac{1}{\log\frac{1}{\varepsilon}}, \quad x \in (r_k, r_k + \varepsilon) \quad \text{and} \quad 0 < \varepsilon \leq \delta_k.$$

Therefore,

$$\varepsilon^{\frac{1}{(\widetilde{q}_k)_+}} \leq e\varepsilon, \tag{4.1}$$

where $(\widetilde{q}_k)_+$ is essential supremum of $\widetilde{q}_k(x)$ on the interval $(r_k, r_k + \delta_k)$. As

$$0 < \|\chi_{(a,b)}\|_{\widetilde{p}(\cdot)} \leq 1,$$

by Corollary 2.23 from [6]

$$|(a, b)|^{1/(\widetilde{q}_k)_-} \leq \|\chi_{(a,b)}\|_{\widetilde{q}_k(\cdot)} \leq |(a, b)|^{1/(\widetilde{q}_k)_+}$$

and using (4.1), we have

$$\|\chi_{(r_k, r_k+\varepsilon)}\|\|\widetilde{q}_k(\cdot) \approx \varepsilon, \quad \text{for every} \quad 0 < \varepsilon \le \delta_k.$$

Since $\widetilde{q}(x) \le q_k(x)$ on $(r_k, r_k + \varepsilon)$ when $0 < \varepsilon \le \delta_k$, we thus have

$$\varepsilon = \|\chi_{(r_k,r_k+\varepsilon)}\|_{L^1} \lesssim \|\chi_{(r_k,r_k+\varepsilon)}\|_{\widetilde{q}(\cdot)} \lesssim \|\chi_{(r_k,r_k+\varepsilon)}\|_{q_k(\cdot)} \lesssim \varepsilon. \qquad (4.2)$$

Let f be a non-negative decreasing step function on $(r_k, r_k + \delta_k)$: this can be written as

$$f(x) = \sum_{i=1}^{\infty} a_i \chi_{(r_k, x_i)}(x)$$

where $r_k + \delta_k = x_1 > x_2 > \cdots > x_i > \ldots$ and each $a_i \ge 0$. Using (4.2) we obtain

$$\begin{aligned}
\|f\|_{L^1} \lesssim \|f\|_{\widetilde{q}(\cdot)} &= \left\|\sum_{i=1}^{\infty} a_i \chi_{(r_k,x_i)}\right\|_{\widetilde{q}(\cdot)} \\
&\lesssim \sum_{i=1}^{\infty} a_i \left\|\chi_{(r_k,x_i)}\right\|_{\widetilde{q}(\cdot)} \approx \sum_{i=1}^{\infty} a_i (x_i - r_k) \\
&= \left\|\sum_{i=1}^{\infty} a_i \chi_{(r_k,x_i)}\right\|_{L^1} = \|f\|_{L^1}.
\end{aligned}$$

It follows that for every non-negative decreasing function f on $(r_k, r_k + \delta_k)$,

$$\left\|f \chi_{(r_k,r_k+\delta_k)}\right\|_{\widetilde{q}(\cdot)} \approx \left\|f \chi_{(r_k,r_k+\delta_k)}\right\|_{L^1}. \qquad (4.3)$$

Lemma 4.1 *Let* $f = \sum_{k=1}^{\infty} f_k$, *where each* f_k *is a non-negative, non-increasing function on* $(r_k, r_k + \delta_k)$ *that is zero outside* $(r_k, r_k + \delta_k)$. *Then*

$$\|f\|_{\widetilde{q}(\cdot)} \approx \|f\|_{L^1},$$

where $\approx$ *means that the left-hand side is bounded above and below by positive, constant multiples of the right-hand side that are independent of the particular* f.

Proof As in the proof of (4.3),

$$\begin{aligned}
\|f\|_{L^1} \lesssim \|f\|_{\widetilde{q}(\cdot)} &= \left\|\sum_{k=1}^{\infty} f_k\right\|_{\widetilde{q}(\cdot)} \le \sum_{k=1}^{\infty} \|f_k\|_{\widetilde{q}(\cdot)} \\
&\lesssim \sum_{k=1}^{\infty} \|f_k\|_{L^1} = \|f\|_{L^1}. \qquad \square
\end{aligned}$$

The space $L^{\widetilde{q}(\cdot)}(I)$ is subspace of $L^1(I)$ since $\widetilde{q}(x) > 1$ a.e. on I; although it has many bad properties it is quite like $L^1(I)$ in various respects. A simple illustration of this is given in the next theorem, in which it is shown that there is a function

$f \in L^{\tilde{q}(\cdot)}(I)$ such that the Hardy–Littlewood maximal function Mf is not integrable over any interval, no matter how small, contained in I. [We recall that the Hardy–Littlewood operator M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over all intervals $Q \subset I$ that contain x.] This stems from the bad oscillatory behaviour of $\tilde{q}$ and the fact that $\tilde{q}$ is not continuous or strictly greater than 1 in any small interval.

Theorem 4.2 *There exists $f \in L^{\tilde{q}(\cdot)}(I)$ such that the Hardy–Littlewood maximal function Mf is not integrable in any interval $(a, b) \subset I$.*

Proof Let f be defined by

$$f(x) = \frac{d}{dx}\left(\frac{1}{\log(1/x)}\right) \quad (x \in (0, 1/e)).$$

Then f is non-negative, decreasing and integrable on $(0, 1/e)$; for $x \in (0, 1/e)$ we have

$$Mf(x) \geq \frac{1}{x} \int_0^x \frac{d}{dt}\left(\frac{1}{\log(1/t)}\right) dt = \frac{1}{x \log(1/x)}.$$

The function $x \longmapsto \frac{1}{x \log(1/x)}$ is non-negative and decreasing, but not integrable on $(0, 1/e)$.

Now consider the function g defined on I by

$$g(x) = \sum_{k=1}^{\infty} a_k f(x - r_k) \chi_{(r_k, r_k + \delta_k)}(x),$$

where each $a_k > 0$ and $\sum_{k=1}^{\infty} a_k < \infty$. Use of Lemma 4.1 shows that $f \in L^{\tilde{q}(\cdot)}(I)$, but $\left\| \chi_{(a,b)} Mf \right\|_{L^1} = \infty$, no matter what interval $(a, b) \subset I$ we choose. $\qquad \square$

Note that conditions on the exponent function p sufficient for the validity of an inequality of the form

$$\|Mf\|_1 \leq C \|f\|_{p(\cdot)}$$

are considered in [5, 9, 10, 18], in case when $p(\cdot)$ satisfies a decay condition and when $p(\cdot)$ is close to 1 in value.

To provide further information about the properties of $L^{\tilde{q}(\cdot)}(I)$ we recall that given a Banach space X, a sequence $\{(f_n, g_n)\}_{n \in \mathbb{N}} \subset X \times X^*$ is said to be a biorthogonal system if, for all $m, n \in \mathbb{N}$,

$$\langle g_n, f_n \rangle = 1 \text{ and } \langle g_m, f_n \rangle = 0 \text{ for } m \neq n,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in X. Given a biorthogonal system $\{(f_n, g_n)\}_{n\in\mathbb{N}}$, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is called fundamental in X if the closure of span $\{f_n : n \in \mathbb{N}\}$ is X; it is a basis of X if for every $f \in X$,

$$f = \sum_{n=1}^{\infty} \langle g_n, f \rangle f_n,$$

with convergence in the norm of X. If $\{f_n\}_{n\in\mathbb{N}}$ is a basis of X, then $\{g_n\}_{n\in\mathbb{N}}$ is a basis in the closed linear span $\overline{\text{span}}\,\{g_n : n \in \mathbb{N}\}$.

Theorem 4.3 *Let $\{(f_n, g_n)\}_{n\in\mathbb{N}}$ be a biorthogonal system in $L^{\tilde{q}(\cdot)}(I) \times \left(L^{\tilde{q}(\cdot)}(I)\right)^*$ and suppose $\{g_n\}_{n\in\mathbb{N}}$ is fundamental in $C(I)$. If $\{f_n\}_{n\in\mathbb{N}}$ is a basis in $L^{\tilde{q}(\cdot)}(I)$, then $\{g_n\}_{n\in\mathbb{N}}$ is a basis in $C(I)$.*

Proof This follows immediately from Theorem 3.1. □

Further results of this kind are given in [13] and [14].

5 Almost Everywhere Divergence of Fourier Series in $L^{p(\cdot)}(\mathbb{T})$

We conclude the paper by exhibiting the role played by the spaces of variable exponent that we have been considering in connection with functions with almost everywhere divergent Fourier series. To fix the notation, we denote as usual $\mathbb{R}/(2\pi\mathbb{Z})$ by $\mathbb{T}$, and associate with any function $f \in L^1(\mathbb{T})$ its Fourier series

$$f(x) \sim \sum_{-\infty}^{\infty} \widehat{f}(k)e^{ikx},$$

where

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\exp(-ikx)dx.$$

The nth partial sum of the trigonometric Fourier series of f is

$$S_n(x, f) := \sum_{k=-n}^{n} \widehat{f}(k)e^{ikx}.$$

In [11], Kolmogorov constructed his famous example of a function $f \in L^1(\mathbb{T})$ such that its partial sums $S_n(x, f)$ diverge unboundedly almost everywhere. Later, Marcinkiewicz [17] produced a function in which the Fourier series diverged a.e. even though the partial sums were bounded. Kolmogorov's function belongs to $L \log \log L$; Chen [4] gave examples of functions in $L(\log\log L)^{1-\varepsilon}$, $(0 < \varepsilon < 1)$, that have a.e. divergent Fourier series; and Konyagin [12] produced functions, with similar bad properties, in the space $L\phi(L)$, where $\phi(t) = o\left(\sqrt{\log t / \log\log t}\right)$.

The function spaces between $L^1(\mathbb{T})$ and $\cup_{p>1}L^p(\mathbb{T})$ ($p =$ constant) play an important role in the problem of the a.e. convergence of Fourier series, since every $f \in \cup_{p>1}L^p(\mathbb{T})$ has an a.e. convergent Fourier series, while as shown by Kolmogorov

there is a function $f \in L^1(\mathbb{T})$ with a.e. divergent Fourier series. Further discussion of this point is given in [1]. Turning now to spaces with variable exponent, we remark that in contrast to the situation for classical Lebesgue spaces,

$$L^1(\mathbb{T}) = \cup L^{p(\cdot)}(\mathbb{T}), \tag{5.1}$$

where the union is over all $p(\cdot) \in \mathcal{P}(\mathbb{T})$ (defined just as $\mathcal{P}(I)$ was defined in Sect. 2) that are greater than 1 a.e. To establish this claim, let $f \in L^1(\mathbb{T})$ and for each $n \in \mathbb{N}$ define

$$E_n = \{x \in \mathbb{T} : n - 1 \le |f(x)| < n\};$$

plainly $\sum_n n|E_n| < \infty$. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that

$$\sum_n n^{1+\varepsilon_n} |E_n| < \infty;$$

for example, we could take $\varepsilon_n = 1/n$. Now define $p(t) = 1 + \varepsilon_n \ (t \in E_n)$. It is apparent that $f \in L^{p(\cdot)}(\mathbb{T})$. In view of (5.1) it is natural to seek to characterise those spaces $L^{p(\cdot)}(\mathbb{T})$ that contain functions with a.e. divergent Fourier series. To prepare for a discussion of this question we give some details of the procedure used in the construction of the Kolmogorov and the Marcinkiewicz functions. The following lemma (see [2]) is crucial for the construction of both examples: condition (iii) of the lemma is necessary only for the Marcinkiewicz example; for the Kolmogorov example it is sufficient for conditions (i), (ii) and (iv) to be satisfied.

Lemma 5.1 *There is a sequence of functions ϕ_n satisfying the following conditions:*

(i) *For all $n \in \mathbb{N}$,*

$$\phi_n \ge 0 \ and \ \int_0^{2\pi} \phi_n(x)dx = 2;$$

(ii) *each ϕ_n has bounded variation;*

(iii) *there is a sequence of subsets H_n of $[0, 2\pi]$, with*

$$\lim_{n \to \infty} |H_n| = 2\pi,$$

such that there exists A with the property that for all $n, r \in \mathbb{N}$ and all $x \in H_n$,

$$|S_r(x, \phi_n)| \le A \log n;$$

(iv) *if $\varepsilon > 0$, there exist $\alpha > 0$ and $N \in \mathbb{N}$ such that given any $n > N$ there is a set $E_n \subset [0, 2\pi]$ for which*

(a) *$|E_n| > 2\pi - \varepsilon$,*

(b) *for any $x \in E_n$, there exists $r_x \in \mathbb{N}$ such that $\left| S_{r_x}(x, \phi_n) \right| > \alpha \log n$,*

(c) *$n \le r_x \le m_n$, where m_n depends only on n but not on ε.*

The proof is based on the following constructions of the functions ϕ_n. Let

$$A_k = \frac{4\pi k}{2n+1} \quad (k = 1, 2, \cdots, n)$$

and suppose that

$$\lambda_1 = 1, \lambda_2, \cdots, \lambda_n,$$

is an increasing sequence of odd numbers, chosen as detailed below. Let

$$m_1 = n, \, 2m_k + 1 = \lambda_k(2n+1) \text{ if } k = 2, \ldots, n,$$

and define non-overlapping intervals

$$\Delta_k = \left(A_k - \frac{1}{m_k^2}, A_k + \frac{1}{m_k^2} \right) \quad (k = 1, 2, \ldots, n).$$

Let

$$\phi_n(x) = \begin{cases} m_k^2/n, & x \in \Delta_k \ (k = 1, 2, \ldots, n), \\ 0, & x \in [0, 2\pi] \setminus \cup_{k=1}^n \Delta_k, \\ \phi_n(x + 2\pi) = \phi_n(x), & x \in \mathbb{R}. \end{cases}$$

For $n \geq 2$ let

$$D_k = \left[A_k + \frac{1}{n \log n}, A_{k+1} - \frac{1}{n \log n} \right] \quad (k = 1, 2, \ldots, n-1)$$

and set

$$H_n = \cup_{k=1}^{n-1} D_k.$$

The m_k are defined inductively as follows: suppose we have $\lambda_1 < \cdots < \lambda_{k-1}(k \geq 2)$ and correspondingly $m_1 < \cdots < m_{k-1}$. We may then choose m_k so large that

$$\left| \frac{1}{\pi} \int_{\cup_{j=1}^{k-1} \Delta_j} \phi_n(t) L_{m_k}(t - x) dt \right| < 1 \tag{5.2}$$

for all $x \in D_{k-1}$, where L_{m_k} is the Dirichlet kernel of order m_k:

$$L_{m_k}(t) = \frac{\sin\left(m_k + \frac{1}{2}\right)t}{2\sin(t/2)}.$$

The choice of m_k may be made as follows. For $x \in D_{k-1}$ and $t \in \Delta_j$, $(j = 1, 2, \ldots, k-1)$,

$$|t - x| > \frac{1}{n \log n} - \frac{1}{n^2} > \frac{1}{2n \log n};$$

and the function

$$t \longmapsto \frac{\phi_n(t)}{2 \sin \{(t-x)/2\}}$$

is bounded on every Δ_j $(j = 1, \ldots, k-1)$. Thus the integral

$$\int_{\Delta_j} \frac{\phi_n(t)}{2 \sin \{(t-x)/2\}} \sin \left(m_k + \frac{1}{2} \right) (t-x) dt$$

can be made as small as desired if m_k is sufficiently large.

The examples of Kolmogorov and Marcinkiewicz are of the following form:

$$K(x) = \sum_{k=1}^{\infty} \frac{\phi_{n_k}(x)}{\sqrt{\log n_k}}$$

and

$$M(x) = \sum_{k=1}^{\infty} \frac{\phi_{n_k}(x)}{\log n_k},$$

respectively, where the sequence of integers n_k is strictly increasing and satisifes a number of conditions (see [2, pp. 437–439]). For K we have

$$\overline{\lim}_{n \to \infty} |S_n(x, K)| = \infty \text{ a.e.,}$$

while

$$\overline{\lim}_{n \to \infty} |S_n(x, M)| < \infty \text{ a.e.}$$

Note that in the construction involved in the proof of Lemma 5.1 the symmetric intervals $\left(A_k - \frac{1}{m_k^2}, A_k + \frac{1}{m_k^2} \right)$ may be replaced by the non-symmetric intervals $\left(A_k, A_k + \frac{2}{m_k^2} \right)$.

In view of (5.1) it is clear that the Kolmogorov example belongs to some Lebesgue space with variable exponent p with $p(x) > 1$ a.e.; the same holds for the Marcinkiewicz example. More information is provided by the following theorem.

Theorem 5.2 *There exists $p \in \mathcal{P}(\mathbb{T})$, with $1 < p(x) < \infty$ a.e., such that the space of continuous functions $C(\mathbb{T})$ is a closed subspace of $L^{p(\cdot)}(\mathbb{T})$ and both the Kolmogorov and the Marcinkiewicz example belong to $L^{q(\cdot)}(\mathbb{T})$, where q is the conjugate of p.*

Proof Let

$$t_n^k = \frac{4\pi k}{2n+1} \quad (n \in \mathbb{N}, k = 1, \ldots, n) \, ;$$

the set $\left\{ t_n^k : n \in \mathbb{N}, k = 1, \ldots, n \right\}$ is a dense subset of $[0, 2\pi]$. Noting that

$$\sum_{n=1}^{\infty} \int_0^{2/n^2} \log(1/x)dx < \infty,$$

define δ_n^k $(n \in \mathbb{N}, k = 1, \ldots, n)$ in such a way that $\delta_n^1 = 2/n^2$ and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \int_{t_n^k}^{t_n^k + \delta_n^k} m \left(x - t_n^k \right) dx < \infty.$$

Now define the exponent p by

$$p(x) = 2 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} m \left(x - t_n^k \right) \chi_{\left(t_n^k, t_n^k + \delta_n^k\right)}(x).$$

Given any fixed $n \in \mathbb{N}$, choose the numbers m_k in the estimate (5.2) so that

$$t_n^k + \frac{2}{m_k^2} < \delta_n^k \quad (k = 2, \ldots, n) \, .$$

Finally, observe that by Lemma 4.1 both K and M belong to $L^{q(\cdot)}(\mathbb{T})$, where q is the conjugate of p. □

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