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EMBEDDINGS OF LORENTZ-TYPE SPACES INVOLVING WEIGHTED INTEGRAL MEANS

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Abstract. We characterize embeddings between Lorentz-type spaces defined with respect to two different weighted means. In particular, we establish two-sided estimates of the optimal constant C in the inequality

$$\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_2} u_2(s)\,ds\right)^{\frac{m_2}{p_2}} w_2(t)\,dt\right)^{\frac{1}{m_2}} \le C \left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1} u_1(s)\,ds\right)^{\frac{m_1}{p_1}} w_1(t)\,dt\right)^{\frac{1}{m_1}},$$

where $m_1, m_2, p_1, p_2 \in (0,\infty)$, u_1, u_2, w_1, w_2 are weights on $(0,\infty)$ and $m_2 > p_2$. The most innovative part consists of the fact that possibly different general inner weights u_1 and u_2 are allowed. Proofs are based on a combination of duality techniques with weighted inequalities for iterated operators involving integrals and suprema.

1. Introduction and the main result

In this paper we study weighted inequalities of the form

$$\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_2} u_2(s)\,ds\right)^{\frac{m_2}{p_2}} w_2(t)\,dt\right)^{\frac{1}{m_2}} \le C \left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1} u_1(s)\,ds\right)^{\frac{m_1}{p_1}} w_1(t)\,dt\right)^{\frac{1}{m_1}}, \tag{1.1}$$

where m_1, m_2, p_1, p_2 are positive real numbers and u_1, u_2, w_1, w_2 are *weights*, that is, measurable non-negative functions on $(0,\infty)$ and $m_2 > p_2$. The inequality is required to hold with some positive constant C for all scalar measurable functions f defined on a σ-finite measure space $(\mathcal{R},\mu)$. By f^* we denote the *non-increasing rearrangement* of f, given by

$$f^*(t) = \inf\{\lambda \in \mathbb{R} \colon \mu(\{x \in \mathcal{R} \colon |f(x)| > \lambda\}) \le t\} \quad \text{for } t \in (0,\infty).$$

Our main goal is to establish easily verifiable necessary and sufficient conditions on the parameters $m_1, m_2, p_1, p_2 \in (0,\infty)$ and the weights u_1, u_2, w_1, w_2 for which (1.1) holds and to give two-sided estimates of the optimal constant C.

We denote by $\mathfrak{M}(\mathcal{R},\mu)$ the set of all μ-measurable functions on $\mathcal{R}$ whose values belong to $[-\infty,\infty]$. We also define $\mathfrak{M}_+(\mathcal{R},\mu) = \{g \in \mathfrak{M}(\mathcal{R},\mu) \colon g \ge 0\}$.

The inequality (1.1) can be viewed as a continuous embedding between appropriate function spaces. As usual, we say that a (quasi-)normed space X is *embedded* into another such space, Y, if $X \subset Y$ and the identity operator is continuous from X to Y. We denote by $\mathrm{G}\Gamma^{m,p}_{u,w}$ the collection of all functions $f \in \mathfrak{M}(\mathcal{R},\mu)$ such that

$$\|f\|_{\mathrm{G}\Gamma^{m,p}_{u,w}} := \left(\int_0^\infty \left(\int_0^t f^*(s)^p u(s)\,ds\right)^{\frac{m}{p}} w(t)\,dt\right)^{\frac{1}{m}} < \infty,$$

where $m, p \in (0,\infty)$ and w, u are weights (on $(0,\infty)$). Under this notation, (1.1) is equivalent to the continuous embedding

$$\mathrm{G}\Gamma^{m_1,p_1}_{u_1,w_1} \hookrightarrow \mathrm{G}\Gamma^{m_2,p_2}_{u_2,w_2}. \tag{1.2}$$

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Moreover, the norm of the embedding (1.2) coincides with the optimal (smallest) constant C that renders (1.1) true.

The study of function spaces involving weights and rearrangements goes back to early 1950's, when the fundamental paper of Lorentz [41] appeared, followed later by [42]. In [41], the space $\Lambda^p(v)$ was defined as the set of all $f \in \mathfrak{M}(\mathcal{R},\mu)$ for which the functional

$$\|f\|_{\Lambda^p(v)} := \left(\int_0^\infty f^*(t)^p v(t)\,dt\right)^{\frac{1}{p}}$$

is finite, where $p \in (0,\infty)$ and v is a weight on $(0,\infty)$. These spaces proved to be indispensable in a wide range of disciplines of mathematical analysis, in particular in theory of interpolation, theory of operators of harmonic analysis and theory of partial differential equations. A major breakthrough in the theory was seen in 1990, when Ariño and Muckenhoupt in [2] characterized those parameters $p \in (1,\infty)$ and weights v for which the Hardy–Littlewood maximal operator is bounded on $\Lambda^p(v)$, and Sawyer in [47] developed a duality concept for spaces $\Lambda^p(v)$. Among other results, Sawyer obtained a generalization of the theorem of Ariño and Muckenhoupt to the situation in which two possibly different exponents and two possibly different weights are allowed. He also reformulated the action of the maximal operator on weighted Lebesgue spaces restricted to the cone of non-decreasing functions in terms of embeddings between function spaces by introducing the space $\Gamma^p(v)$ as the family of all $f \in \mathfrak{M}(\mathcal{R},\mu)$ for which the functional

$$\|f\|_{\Gamma^p(v)} := \left(\int_0^\infty f^{**}(t)^p v(t)\,dt\right)^{\frac{1}{p}}$$

is finite, where f^{**} is the *maximal non-increasing rearrangement* of f, defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds \quad \text{for } t \in (0,\infty). \tag{1.3}$$

For every $f \in \mathfrak{M}(\mathcal{R},\mu)$ and every $t \in (0,\infty)$, the estimate $f^*(t) \le f^{**}(t)$ holds. As a consequence, one trivially has $\Gamma^p(v) \hookrightarrow \Lambda^p(v)$ for any p and v.

During the 1990's, the spaces $\Lambda^p(v)$ and $\Gamma^p(v)$ were put under a serious scrutiny under the common label *classical Lorentz spaces.* Their basic functional properties as well as embedding relations between them were characterized. It would be next to impossible to give a complete account of the literature which is available to this subject nowadays. Let us quote at least the efforts of M. Carro, A. García del Amo, M. Gol'dman, H. Heinig, L. Maligranda, J. Martín, C. Neugebauer, R. Oinarov, J. Soria, G. Sinnamon, V.D. Stepanov that resulted in a long series of papers, see [4, 7, 8, 9, 10, 24, 31, 32, 33, 34, 40, 44, 45, 48, 51, 52, 53, 54]. The first attempt to survey the situation in the field was given in [7] where the contemporary state of the art was described. Since then, however, important new results have been obtained and things have changed essentially again.

A significant progress in the study of classical Lorentz spaces was made in the early 2000's due to the efforts of Sinnamon [49, 50] and to the development of a new approach based on discretization and anti-discretization techniques in [25]. Using these new techniques, embeddings of classical Lorentz spaces in cases that had resisted for years were finally characterized, the notable last missing case being added in [6]. This rounded off one particular level of results.

As a consequence of these advances, the field could have been explored deeper (see e.g. [5, 6, 26, 27]). One of the most important innovations was the involvement of function spaces involving *inner weighted means.* In order to describe such function spaces, let us first consider the weighted version of (1.3), namely

$$f_u^{**}(t) = \frac{1}{U(t)}\int_0^t f^*(s)u(s)\,ds \quad \text{for } t \in (0,\infty), \tag{1.4}$$

where u is a given weight on $(0,\infty)$ and

$$U(t) := \int_0^t u(s)\,ds \quad \text{for } t \in (0,\infty).$$

Given $p \in (0,\infty)$ and another weight, v, on $(0,\infty)$, we define the space $\Gamma^p_u(v)$ as the collection of all functions $f \in \mathfrak{M}(\mathcal{R},\mu)$ such that

$$\|f\|_{\Gamma^p_u(v)} := \left(\int_0^\infty f^{**}_u(t)^p v(t)\,dt\right)^{\frac{1}{p}} < \infty.$$

Some effort was spent in order to recover general embedding results for classical Lorentz spaces by methods that would avoid the powerful but technically complicated discretization-antidiscretization scheme, but only with a partial success (see e.g. [29, 30, 19]). A recent overview of the field of embeddings of classical Lorentz spaces can be found in [46, Chapter 10].

There exists plenty of motivation for studying relations between classical Lorentz spaces in great detail. For example, in the recent work [1], information about classical Lorentz spaces is used in order to investigate the continuity properties of local solutions to the n-Laplace equation

$$-\operatorname{div}(|\nabla u|^{n-2}\nabla u) = f(x) \quad \text{in } \Omega,$$

where Ω is a bounded open subset of $\mathbb{R}^n$.

Recently, new spaces came into play, for a good reason. Given two parameters $m, p \in (0,\infty)$ and a weight v, on $(0,\infty)$, the space $\mathrm{G}\Gamma(p,m,v)$ is defined as the the collection of all functions $f \in \mathfrak{M}(\mathcal{R},\mu)$ such that

$$\|f\|_{\mathrm{G}\Gamma(p,m,v)} := \left(\int_0^b \left(\int_0^t f^*(s)^p\,ds\right)^{\frac{m}{p}} v(t)\,dt\right)^{\frac{1}{m}} < \infty.$$

These spaces turn out to be important among other reasons because of their intimate connection to the so-called *grand Lebesgue spaces* and their slightly younger relatives called *small Lebesgue spaces*. The grand Lebesgue space was introduced by Iwaniec and Sbordone in [35] in connection with integrability properties of Jacobians. Since it is a relatively complicated structure, it took some time before its dual was characterized. This was done by Fiorenza in [14]. In that paper also the small Lebesgue spaces were introduced. It was shown later by Fiorenza and Karadzhov in [15] that the norm in the small Lebesgue space can be equivalently expressed in terms of the functional governing the $\mathrm{G}\Gamma(p,m,w)$ space with appropriate parameters and weights. Further results in this direction were obtained e.g. in [16, 17, 18]. The associate space of $\mathrm{G}\Gamma(p,m,w)$ was then completely characterized in [28].

The techniques in the background of many of the results mentioned inevitably involve weighted inequalities involving Hardy-type integral operators. However, we also witness a still growing importance of weighted inequalities involving *supremum operators*. These operators have been studied recently (see e.g. [11], [23] or [21]) in connection with several problems in analysis including action of fractional maximal operators, optimality of function spaces in Sobolev embeddings, or the interpolation theory, but the available results are far from being complete.

In [25], the characterization of the embeddings of the form

$$\Gamma^q_u(w) \hookrightarrow \Gamma^p_u(v), \tag{1.5}$$

where $p, q \in (0,\infty)$ and u, v, w are weights on $(0,\infty)$, was completed. It was an important step ahead and applications followed instantly, but it still suffered from the principal restriction that the inner weight u had to be the same on both sides of the embedding.

On the side of applications, there exists a significant desire for two-sided estimates of optimal constants in embeddings of the type (1.5) with *two possibly different inner weights*. The motivation arises usually in tasks that involve, in a way, two possibly different integral mean operators. To give at least one example, let us recall the long-time extensive research of the optimality of function spaces in Sobolev-type embeddings, carried out e.g. in [13, 36, 37, 38, 12]. For instance, the considerations in [38, Theorem 3.1], where the explicit formula for the optimal rearrangement-invariant function norm in a Sobolev inequality is sought and the known implicit one is reduced to a formula involving an integral mean with respect to another weight function, show that characterizations of embeddings of the form (1.1) are useful.

Most of the functions which we shall deal with will be defined on $(0,\infty)$. If this is the case, then $(\mathcal{R},\mu)$ is the interval $(0,\infty)$ endowed with the one-dimensional Lebesgue measure λ_1, and we shall write just $\mathfrak{M}$ and $\mathfrak{M}_+$ instead of $\mathfrak{M}((0,\infty),\lambda_1)$ and $\mathfrak{M}_+((0,\infty),\lambda_1)$ respectively.

Let u_1, u_2, w_1 and w_2 be weights on $(0,\infty)$ and $t\in(0,\infty)$. We will use the following notation:

$$U_1(t)=\int_0^t u_1(s)\,ds,\quad U_2(t)=\int_0^t u_2(s)\,ds,\quad W_1(t)=\int_0^t w_1(s)\,ds,\quad W_2(t)=\int_0^t w_2(s)\,ds.$$

Further, let $p_1,p_2,m_1,m_2\in(1,\infty)$. We define

$$\varphi(t)=\int_0^t U_1(s)^{\frac{m_1}{p_1}}w_1(s)\,ds+U_1(t)^{\frac{m_1}{p_1}}\int_t^\infty w_1(s)\,ds\quad\text{for } t\in(0,\infty).$$

Note that, for every $t\in(0,\infty)$, one has $\varphi(t)=\|\chi_{(0,t)}\|_{\mathrm{G}\Gamma^{m_1,p_1}_{u_1,w_1}(0,\infty)}$. We also set

$$\sigma(t)=\frac{U_1(t)^{\frac{m_1^2}{p_1(m_1-p_2)}-1}u_1(t)\int_0^t U_1(s)^{\frac{m_1}{p_1}}w_1(s)\,ds\int_t^\infty w_1(s)\,ds}{\varphi(t)^{\frac{m_1}{m_1-p_2}+1}},\quad t\in(0,\infty).$$

Throughout the paper, the expressions of the form $0\cdot\infty$ or $\frac{0}{0}$ are taken as zero. For $p\in(1,\infty)$, we define $p'=\frac{p}{p-1}$. We write $A\approx B$ when the ratio A/B is bounded from below and from above by positive constants independent of appropriate quantities appearing in expressions A and B.

We shall now state the principal result of the paper.

Theorem 1.1. *Let $p_1,p_2,m_1,m_2\in(1,\infty)$. Assume that $m_2>p_2$. Let u_1, u_2, w_1 and w_2 be weights. Assume that*

- *u_1 is strictly positive, $\int_0^t u_1(s)\,ds<\infty$ for all $t\in(0,\infty)$, $\int_0^\infty u_1(t)\,dt=\infty$,*
- *$\int_0^t w_1(s)U_1(s)^{\frac{m_1}{p_1}}\,ds<\infty$, $\int_t^\infty w_1(s)U_1(s)^{\frac{m_1}{p_1}}\,ds=\infty$ for all $t\in(0,\infty)$,*
- *$\int_0^t w_1(s)\,ds=\infty$, $\int_t^\infty w_1(s)\,ds<\infty$ for all $t\in(0,\infty)$.*

Let

$$C=\sup_{f\in\mathfrak{M}}\frac{\left(\int_0^\infty\left(\int_0^t f^*(s)^{p_2}u_2(s)\,ds\right)^{\frac{m_2}{p_2}}w_2(t)\,dt\right)^{\frac{1}{m_2}}}{\left(\int_0^\infty\left(\int_0^t f^*(s)^{p_1}u_1(s)\,ds\right)^{\frac{m_1}{p_1}}w_1(t)\,dt\right)^{\frac{1}{m_1}}}.\tag{1.6}$$

(a) *Let $p_1\le p_2$ and $m_1\le p_2$. Then*

$$C\approx B_1,$$

where

$$B_1=\sup_{t\in(0,\infty)}\frac{\left(\int_0^t U_2(s)^{\frac{m_2}{p_2}}w_2(s)\,ds+U_2(t)^{\frac{m_2}{p_2}}\int_t^\infty w_2(s)\,ds\right)^{\frac{1}{m_2}}}{\varphi(t)^{\frac{1}{m_1}}}.$$

(b-i) *Let $p_1\le p_2$, $m_1>p_2$ and $m_1\le m_2$. Then*

$$C\approx B_2+B_3,$$

where

$$B_2=\sup_{t\in(0,\infty)}\left(U_1(t)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}}\int_0^t\sigma(s)\,ds+\int_t^\infty U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}}\sigma(s)\,ds\right)^{\frac{m_1-p_2}{m_1p_2}}\left(\int_0^t U_2(s)^{\frac{m_2}{p_2}}w_2(s)\,ds\right)^{\frac{1}{m_2}}$$

and

$$B_3=\sup_{t\in(0,\infty)}\left(\int_0^t\sup_{y\in(s,t)}U_2(y)^{\frac{m_1}{m_1-p_2}}U_1(y)^{-\frac{p_2m_1}{p_1(m_1-p_2)}}\sigma(s)\,ds\right)^{\frac{m_1-p_2}{m_1p_2}}\left(\int_t^\infty w_2(s)\,ds\right)^{\frac{1}{m_2}}.$$

(b-ii) *Let $p_1\le p_2$, $m_1>p_2$ and $m_1>m_2$. Then*

$$C\approx B_4+B_5+B_6+B_7,$$

where

$$B_4 = \left(\int_0^\infty \left(\int_t^\infty U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \sigma(s)\,ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} U_1(t)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \right.$$
$$\left. \times \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds \right)^{\frac{m_1}{m_1-p_2}} \sigma(t)\,dt \right)^{\frac{m_1-m_2}{m_1 m_2}},$$

$$B_5 = \left(\int_0^\infty \sup_{s\in(t,\infty)} U_1(s)^{-\frac{m_1 m_2}{p_1(m_1-m_2)}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy \right)^{\frac{m_1}{m_1-m_2}} \left(\int_0^t \sigma(s)\,ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} \sigma(t)\,dt \right)^{\frac{m_1-m_2}{m_1 m_2}},$$

$$B_6 = \left(\int_0^\infty \sup_{s\in(t,\infty)} U_2(s)^{\frac{m_1 m_2}{p_2(m_1-m_2)}} U_1(s)^{-\frac{m_1 m_2}{p_1(m_1-m_2)}} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{m_1}{m_1-m_2}} \right.$$
$$\left. \times \left(\int_0^t \sigma(s)\,ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} \sigma(t)\,dt \right)^{\frac{m_1-m_2}{m_1 m_2}}$$

and

$$B_7 = \left(\int_0^\infty \sup_{s\in(t,\infty)} U_2(s)^{\frac{m_1}{m_1-p_2}} U_1(s)^{-\frac{m_1 p_2}{p_1(m_1-p_2)}} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{m_1}{m_1-m_2}} \right.$$
$$\left. \times \left(\int_0^t \sup_{y\in(s,t)} U_2(y)^{\frac{m_1}{m_1-p_2}} U_1(y)^{-\frac{m_1 p_2}{p_1(m_1-p_2)}} \sigma(s)\,ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} \sigma(t)\,dt \right)^{\frac{m_1-m_2}{m_1 m_2}}.$$

(c-i) *Let $p_1 > p_2$, $m_1 \le p_2$ and $p_1 \le m_2$. Then*

$$C \approx B_8 + B_9,$$

where

$$B_8 = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{1}{p_1}}}{\varphi(t)^{\frac{1}{m_1}}} \sup_{s\in(t,\infty)} U_1(s)^{-\frac{1}{p_1}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy \right)^{\frac{1}{m_2}}$$

and

$$B_9 = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{1}{p_1}}}{\varphi(t)^{\frac{1}{m_1}}} \sup_{s\in(t,\infty)} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{1}{m_2}} \left(\int_t^s U_2(y)^{\frac{p_1}{p_1-p_2}} U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y)\,dy \right)^{\frac{p_1-p_2}{p_1 p_2}}.$$

(c-ii) *Let $p_1 > p_2$, $m_1 \le p_2$ and $p_1 > m_2$. Then*

$$C \approx B_{10} + B_{11} + B_{12},$$

where

$$B_{10} = \sup_{t\in(0,\infty)} \frac{\left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds \right)^{\frac{1}{m_2}}}{\varphi(t)^{\frac{1}{m_1}}},$$

$$B_{11} = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{1}{p_1}} \left(\int_t^\infty \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy \right)^{\frac{m_2}{p_1-m_2}} U_2(s)^{\frac{m_2}{p_2}} w_2(s) U_1(s)^{-\frac{m_2}{p_1-m_2}}\,ds \right)^{\frac{p_1-m_2}{p_1 m_2}}}{\varphi(t)^{\frac{1}{m_1}}}$$

and

$$B_{12} = \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{1}{p_1}} \left(\int_t^\infty \left(\int_t^s U_2(y)^{\frac{p_1}{p_1-p_2}} U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y)\,dy \right)^{\frac{m_2(p_1-p_2)}{p_2(p_1-m_2)}} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{m_2}{p_1-m_2}} w_2(t)\,dt \right)^{\frac{p_1-m_2}{p_1 m_2}}}{\varphi(t)^{\frac{1}{m_1}}}.$$

(d-i) *Let* $p_2 < m_1 < p_1 \le m_2$. *Then*

$$C \approx B_{13} + B_{14} + B_{15},$$

where

$$B_{13} = \sup_{t\in(0,\infty)} \left(\int_0^t \sigma(s)\,ds\right)^{\frac{m_1-p_2}{m_1p_2}} U_1(t)^{-\frac{1}{p_1}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds\right)^{\frac{1}{m_2}},$$

$$B_{14} = \sup_{t\in(0,\infty)} \left(\int_t^\infty U_1(s)^{-\frac{m_1p_2}{p_1(m_1-p_2)}} \sigma(s)\,ds\right)^{\frac{m_1-p_2}{m_1p_2}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds\right)^{\frac{1}{m_2}}$$

and

$$B_{15} = \sup_{t\in(0,\infty)} \left(\int_t^\infty w_2(s)\,ds\right)^{\frac{1}{m_2}} \left(\int_0^t \left(\int_s^t U_1(y)^{-\frac{p_1}{p_1-p_2}} U_2(y)^{\frac{p_1}{p_1-p_2}} u_1(y)\,dy\right)^{\frac{m_1(p_1-p_2)}{p_1(m_1-p_2)}} \sigma(s)\,ds\right)^{\frac{m_1-p_2}{m_1p_2}}.$$

(d-ii) *Let* $p_2 < m_1 \le m_2 < p_1$. *Then*

$$C \approx B_{14} + B_{15} + B_{16},$$

where

$$\begin{aligned} B_{16} = &\sup_{t\in(0,\infty)} \left(\int_0^t \sigma(s)\,ds\right)^{\frac{m_1-p_2}{m_1p_2}} \left(\int_t^\infty U_1(s)^{-\frac{m_2}{p_1-m_2}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy\right)^{\frac{m_2}{p_1-m_2}} U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds\right)^{\frac{p_1-m_2}{p_1m_2}} \\ &+ \sup_{t\in(0,\infty)} \left(\int_0^t \sigma(s)\,ds\right)^{\frac{m_1-p_2}{m_1p_2}} \left(\int_t^\infty \left(\int_s^\infty U_1(y)^{-\frac{p_1}{p_1-p_2}} U_2(y)^{\frac{p_1}{p_1-p_2}} u_1(y)\,dy\right)^{\frac{m_2(p_1-p_2)}{p_2(p_1-m_2)}}\right. \\ &\qquad \left. \times \left(\int_s^\infty w_2(y)\,dy\right)^{\frac{m_2}{p_1-m_2}} w_2(s)\,ds\right)^{\frac{p_1-m_2}{p_1m_2}}. \end{aligned}$$

(d-iii) *Let* $p_2 < m_2 < m_1 < p_1$. *Then*

$$C \approx B_{17},$$

where

$$
\begin{aligned}
B_{17} = {} & \left(\int_0^\infty \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds \right)^{\frac{m_1}{m_1-m_2}} \left(\int_t^\infty U_1(s)^{-\frac{m_1 p_2}{p_1(m_1-p_2)}} \sigma(s)\,ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} U_1(t)^{-\frac{m_1 p_2}{p_1(m_1-p_2)}} \sigma(t)\,dt \right)^{\frac{m_1-m_2}{m_1 m_2}} \\
& + \left(\int_0^\infty \left(\int_t^\infty U_1(s)^{-\frac{m_2}{p_1-m_2}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy \right)^{\frac{m_2}{p_1-m_2}} U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds \right)^{\frac{m_1(p_1-m_2)}{p_1(m_1-m_2)}} \right. \\
& \qquad \left. \times \left(\int_0^t \sigma(s)\,ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} \sigma(t)\,dt \right)^{\frac{m_1-m_2}{m_1 m_2}} \\
& + \left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty U_1(y)^{-\frac{p_1}{p_1-p_2}} U_2(y)^{\frac{p_1}{p_1-p_2}} u_1(y)\,dy \right)^{\frac{m_2(p_1-p_2)}{p_2(p_1-m_2)}} \left(\int_s^\infty w_2(y)\,dy \right)^{\frac{m_2}{p_1-m_2}} w_2(s)\,ds \right)^{\frac{m_1(p_1-m_2)}{p_1(m_1-m_2)}} \right. \\
& \qquad \left. \times \left(\int_0^t \sigma(s)\,ds \right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}} \sigma(t)\,dt \right)^{\frac{m_1-m_2}{m_1 m_2}} \\
& + \left(\int_0^\infty \left(\int_0^t \left(\int_s^t U_1(y)^{-\frac{p_1}{p_1-p_2}} U_2(y)^{\frac{p_1}{p_1-p_2}} u_1(y)\,dy \right)^{\frac{p_1-p_2}{p_1}} \sigma(s)\,ds \right)^{\frac{m_2(m_1-p_2)}{p_2(m_1-m_2)}} \right. \\
& \qquad \left. \times \left(\int_t^\infty w_2(s)\,ds \right)^{\frac{m_2}{m_1-m_2}} w_2(t)\,dt \right)^{\frac{m_1-m_2}{m_1 m_2}} .
\end{aligned}
$$

The cases when either $m_2 < p_2$ or $m_2 > p_2$, $m_1 > p_2$, $p_1 > p_2$ and $m_1 \geq p_1$ remain open. In the case when $m_2 = p_2$, the space $\mathrm{G}\Gamma^{m_2,p_2}_{u_2,w_2}$ degenerates to a classical Lorentz space of type Λ for which everything is known ([25]).

The key ingredient of the proof of Theorem 1.1 is a combination of duality techniques with embedding results for classical Lorentz spaces and estimates of optimal constants in weighted inequalities involving iterated integral and supremum operators. Detailed analysis of separate cases leads to the need of necessary and sufficient conditions for various, quite different in nature, inequalities, of which only some are known. Interestingly, some of these results have been obtained only quite recently, such as [20], for instance. Even more interestingly, some are not known at all and will appear here for the first time.

The proof can be naturally expected to be quite technical and to involve plenty of computation. There is hardly any way to avoid it. We shall therefore do our best to simplify the notation, shorten the formulas, and make the exposition as reader-friendly as possible.

The paper is organized as follows. In the next section we collect the necessary background material. We intend to save the reader plenty of tedious work since the relevant results are scattered over literature with inconsistent notation. We also characterize several inequalities involving iterated integral and supremum operators which are not available and will also be needed in the proofs. In the last section we present the proof of Theorem 1.1.

2. Background material

In this section we collect background results that will be used in the proof of the main theorem.

We begin with the well-known duality principle in weighted Lebesgue spaces. If $p \in (1,\infty)$, $f \in \mathfrak{M}_+$ and v is a weight on $(0,\infty)$, then

$$\left(\int_0^\infty f(t)^p v(t)\,dt\right)^{\frac{1}{p}} = \sup_{h\in\mathfrak{M}_+} \frac{\int_0^\infty f(t)h(t)\,dt}{\left(\int_0^\infty h(t)^{p'} v(t)^{1-p'}\,dt\right)^{\frac{1}{p'}}}. \tag{2.1}$$

Let us now recall a quantified version of classical Hardy inequalities.

Theorem 2.1 ([3, Theorem 1] and [43, Theorem 1.3.1])**.** *Let $1 < p, q < \infty$ and let u, v, w be weights on $(0,\infty)$. Let*

$$K = \sup_{f\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^t f(s)u(s)\,ds\right)^q w(t)\,dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty f(t)^p v(t)\,dt\right)^{\frac{1}{p}}}.$$

(a) *Let $1 < p \le q < \infty$. Then $K \approx A_1$, where*

$$A_1 = \sup_{t\in(0,\infty)} \left(\int_t^\infty w(s)\,ds\right)^{\frac{1}{q}} \left(\int_0^t u(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}.$$

(b) *Let $1 < q < p < \infty$. Then $K \approx A_2$, where*

$$A_2 = \left(\int_0^\infty \left(\int_t^\infty w(s)\,ds\right)^{\frac{p}{p-q}} \left(\int_0^t u(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{p(q-1)}{p-q}} u(t)^{p'} v(t)^{1-p'}\,dt\right)^{\frac{p-q}{pq}}.$$

Theorem 2.2 ([3, Theorem 2] and [43, Theorem 1.3.2])**.** *Let $1 < p, q < \infty$ and let v and w be weights on $(0,\infty)$. Let*

$$K = \sup_{f\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_t^\infty f(s)\,ds\right)^q w(t)\,dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty f(t)^p v(t)\,dt\right)^{\frac{1}{p}}}.$$

(a) *Let $1 < p \le q < \infty$. Then $K \approx A_1$, where*

$$A_1 = \sup_{t\in(0,\infty)} \left(\int_0^t w(s)\,ds\right)^{\frac{1}{q}} \left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}.$$

(b) *Let $1 < q < p < \infty$. Then $K \approx A_2$, where*

$$A_2 = \left(\int_0^\infty \left(\int_0^t w(s)\,ds\right)^{\frac{p}{p-q}} \left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{p(q-1)}{p-q}} v(t)^{1-p'}\,dt\right)^{\frac{p-q}{pq}}.$$

We now turn our attention to inequalities involving supremum operators.

Theorem 2.3 ([23, Theorem 4.1(i) and Theorem 4.4])**.** *Let $0 < p, q < \infty$. Let u be a continuous weight and let v, w and ϱ be weights such that $0 < \int_0^t v(s)\,ds < \infty$ and $0 < \int_0^t w(s)\,ds < \infty$ for every $t \in (0,\infty)$. Let*

$$K = \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \sup\limits_{s\in(t,\infty)} u(s)^q \left(\int_0^s g(y)\varrho(y)\,dy\right)^q w(t)\,dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty g(t)^p v(t)\,dt\right)^{\frac{1}{p}}}.$$

(a) *Let $1 < p \le q < \infty$. Then $K \approx A_1$, where*

$$A_1 = \sup_{t\in(0,\infty)} \left(\sup_{s\in(t,\infty)} u(s)^q \int_0^s w(y)\,dy + \int_t^\infty \sup_{y\in(s,\infty)} u(y)^q w(s)\,ds\right)^{\frac{1}{q}} \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}.$$

(b) *Let $1 \le p < \infty$ and $0 < q < p$. Then $K \approx A_2 + A_3$, where*

$$A_2 = \left(\int_0^\infty \sup_{s\in(t,\infty)} u(s)^q \left(\int_t^\infty \sup_{y\in(s,\infty)} u(y)^q w(s)\,ds\right)^{\frac{q}{p-q}} \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{q(p-1)}{p-q}} w(t)\,dt\right)^{\frac{p-q}{pq}}$$

and

$$A_3 = \left(\int_0^\infty \sup_{s\in(t,\infty)} u(s)^{\frac{pq}{p-q}} \left(\int_0^s \varrho(y)^{p'} v(y)^{1-p'}\, dy \right)^{\frac{q(p-1)}{p-q}} \left(\int_0^t w(s)\, ds \right)^{\frac{q}{p-q}} w(t)\, dt \right)^{\frac{p-q}{pq}}.$$

One of the most important ingredients of the proof of the main theorem will be the following quantified version of an embedding between classical Lorentz spaces in a certain particular case.

Theorem 2.4 ([25, Theorem 4.2])**.** *Let u, v, w be weights on $[0,\infty)$. Let $p, q \in (0,\infty)$. Assume that the following conditions are satisfied:*

- $\lim_{t\to\infty} U(t) = \infty$,
- $\int_0^\infty \frac{v(s)}{U(s)^p + U(t)^p}\, ds < \infty$ *for every* $t \in (0,\infty)$,
- $\int_0^1 \frac{v(s)}{U(s)^p}\, ds = \infty$,
- $\int_1^\infty v(s)\, ds = \infty$.

Let

$$K = \sup_{f\in\mathfrak{M}_+} \frac{\left(\int_0^\infty f^*(t)^q w(t)\, dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty f_u^{**}(t)^p v(t)\, dt\right)^{\frac{1}{p}}}.$$

(a) *If* $0 < p \le q < \infty$ *and* $1 \le q < \infty$, *then*

$$K \approx A_1,$$

where

$$A_1 = \sup_{t\in(0,\infty)} \frac{W(t)^{\frac{1}{q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\, ds\right)^{\frac{1}{p}}}.$$

(b) *If* $1 \le q < p < \infty$, *then*

$$K \approx A_2,$$

where

$$A_2 = \left(\int_0^\infty \frac{\sup_{y\in(t,\infty)} U(y)^{-\frac{pq}{p-q}} W(y)^{\frac{p}{p-q}} V(t) U(t)^{\frac{pq}{p-q}+p-1} u(t) \int_t^\infty U(s)^{-p} v(s)\, ds}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\, ds\right)^{\frac{p}{p-q}+1}}\, dt \right)^{\frac{p-q}{pq}}.$$

(c) *If* $0 < p \le q < 1$, *then*

$$K \approx A_3,$$

where

$$A_3 = \sup_{t\in(0,\infty)} \frac{W(t)^{\frac{1}{q}} + U(t) \left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\, ds\right)^{\frac{1-q}{q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\, ds\right)^{\frac{1}{p}}}.$$

(d) *If* $0 < q < 1$ *and* $0 < q < p$, *then*

$$K \approx A_4,$$

where

$$A_4 = \left(\int_0^\infty \frac{\left(W(t)^{\frac{1}{1-q}} + U(t)^{\frac{q}{1-q}} \int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\, ds\right)^{\frac{p(1-q)}{p-q}}}{\left(V(t) + U(t)^p \int_t^\infty U(s)^{-p} v(s)\, ds\right)^{\frac{p}{p-q}+1}} \times \right.$$
$$\left. \times\, V(t) U(t)^{p-1} u(t) \int_t^\infty U(s)^{-p} v(s)\, ds\, dt \right)^{\frac{p-q}{pq}}.$$

We now recall characterization of a weighted inequality involving a kernel operator.

Theorem 2.5 ([45, Theorems 1.1 and 1.2]). *Let $1 < p, q < \infty$ and let v and w be weights. Let*

$$K = \sup_{f \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^t h(s) \int_s^t u(y)\, dy\, ds\right)^q w(t)\, dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty (f(t))^p v(t)\, dt\right)^{\frac{1}{p}}}.$$

(a) *Let $1 < p \le q < \infty$. Then $K \approx A_1 + A_2$, where*

$$A_1 = \sup_{t \in (0,\infty)} \left(\int_t^\infty \left(\int_s^t u(y)\, dy\right)^q w(s)\, ds\right)^{\frac{1}{q}} \left(\int_0^t v(s)^{1-p'}\, ds\right)^{\frac{1}{p'}}$$

and

$$A_2 = \sup_{t \in (0,\infty)} \left(\int_t^\infty w(s)\, ds\right)^{\frac{1}{q}} \left(\int_0^t \left(\int_s^t u(y)\, dy\right)^{p'} v(s)^{1-p'}\, ds\right)^{\frac{1}{p'}}.$$

(b) *Let $1 < q < p < \infty$. Then $K \approx A_3 + A_4$, where*

$$A_3 = \left(\int_0^\infty \left(\left(\int_s^\infty \left(\int_s^t u(y)\, dy\right)^q w(s)\, ds\right) \left(\int_0^t v(s)^{1-p'}\, ds\right)^{q-1}\right)^{\frac{p}{p-q}} v(t)^{1-p'}\, dt\right)^{\frac{p-q}{pq}}$$

and

$$A_4 = \left(\int_0^\infty \left(\left(\int_t^\infty w(s)\, ds\right) \left(\int_0^t \left(\int_s^t u(y)\, dy\right)^{p'} v(s)^{1-p'}\, ds\right)^{p-1}\right)^{\frac{q}{p-q}} w(t)\, dt\right)^{\frac{p-q}{pq}}.$$

Now we shall present a quantified version of a weighted inequality involving a specific combination of a supremum operator and an integral operator.

Theorem 2.6 ([39, Theorem 6]). *Let v and w be weights on $(0,\infty)$ and let u be a continuous weight on $(0,\infty)$. Let*

$$K = \sup_{g \in \mathfrak{M}_+} \frac{\left(\int_0^\infty \sup\limits_{s \in (t,\infty)} u(s)^q \left(\int_s^\infty g(y)\, dy\right)^q w(t)\, dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s)\, ds\right)^{\frac{1}{p}}}.$$

(a) *Assume that $1 < p \le q < \infty$. Then*

$$K \approx A_1,$$

where

$$A_1 = \sup_{t \in (0,\infty)} \left(\int_0^t \sup_{y \in (s,t)} u(y)^q w(s)\, ds\right)^{\frac{1}{q}} \left(\int_t^\infty v(s)^{1-p'}\, ds\right)^{\frac{1}{p'}}.$$

(b) *Assume that $1 < p < \infty$ and $0 < q < p < \infty$. Then*

$$K \approx A_2 + A_3,$$

where

$$A_2 = \left(\int_0^\infty \sup_{s \in (t,\infty)} u(s)^{\frac{pq}{p-q}} W(t)^{\frac{q}{p-q}} w(t) \left(\int_t^\infty v(s)^{1-p'}\, ds\right)^{\frac{q(p-1)}{p-q}} dt\right)^{\frac{p-q}{pq}}$$

and

$$A_3 = \left(\int_0^\infty \sup_{s \in (t,\infty)} u(s)^q \left(\int_s^\infty v(y)^{1-p'}\, dy\right)^{\frac{q(p-1)}{p-q}} \left(\int_0^t \sup_{y \in (s,t)} u(y)^q w(s)\, ds\right)^{\frac{q}{p-q}} w(t)\, dt\right)^{\frac{p-q}{pq}}.$$

At one stage of the proof of the main result, a reformulation of conditions on weights will be required. This will be done through the following elementary lemma.

Lemma 2.7. *Let w, u be weights. Assume that*

$$\int_0^\infty u(t)\,dt = \infty.$$

Let $0 < q < 1$. Then, for every $t \in (0,\infty)$, one has

$$W(t)^{\frac{1}{q}} + U(t)\left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\,ds\right)^{\frac{1-q}{q}} \approx U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds\right)^{\frac{1-q}{q}},$$

in which the constants of equivalence depend only on q.

Proof. Fix $t \in (0,\infty)$. Integration by parts yields

$$\begin{aligned} &\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{1}{1-q}}\,ds \\ &= q\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds + (1-q)\left(\lim_{y\to\infty} \frac{W(y)^{\frac{1}{1-q}}}{U(y)^{\frac{q}{1-q}}} - \frac{W(t)^{\frac{1}{1-q}}}{U(t)^{\frac{q}{1-q}}}\right). \end{aligned} \tag{2.2}$$

Therefore, we immediately have

$$\begin{aligned} &\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{1}{1-q}}\,ds \\ &\le q\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds + (1-q)\lim_{y\to\infty} W(y)^{\frac{1}{1-q}} U(y)^{-\frac{q}{1-q}}. \end{aligned}$$

Next,

$$\begin{aligned} \lim_{y\to\infty} W(y)^{\frac{1}{1-q}} U(y)^{-\frac{q}{1-q}} &\le \sup_{t\le y<\infty} W(y)^{\frac{1}{1-q}} U(y)^{-\frac{q}{1-q}} \\ &= \tfrac{q}{1-q} \sup_{t\le y<\infty} W(y)^{\frac{1}{1-q}} \int_y^\infty U(s)^{-\frac{1}{1-q}} u(s)\,ds \\ &\le \tfrac{q}{1-q} \sup_{t\le y<\infty} \int_y^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds \\ &= \tfrac{q}{1-q} \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds. \end{aligned}$$

Altogether, we obtain

$$\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{1}{1-q}}\,ds \le 2q\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds. \tag{2.3}$$

We also have

$$\begin{aligned} W(t)^{\frac{1}{1-q}} &= W(t)^{\frac{1}{1-q}} U(t)^{\frac{q}{1-q}} U(t)^{-\frac{q}{1-q}} \\ &= \tfrac{1-q}{q} W(t)^{\frac{1}{1-q}} U(t)^{\frac{q}{1-q}} \int_t^\infty U(s)^{-\frac{1}{1-q}} u(s)\,ds \\ &\le \tfrac{1-q}{q} U(t)^{\frac{q}{1-q}} \int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds. \end{aligned}$$

Raising the inequality to $\frac{1-q}{q}$, we get

$$W(t)^{\frac{1}{q}} \le (\tfrac{1-q}{q})^{\frac{1-q}{q}} U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds\right)^{\frac{1-q}{q}}. \tag{2.4}$$

Altogether, (2.3) and (2.4) imply

$$W(t)^{\frac{1}{q}} + U(t)\left(\int_t^\infty W(s)^{\frac{q}{1-q}} w(s) U(s)^{-\frac{q}{1-q}}\,ds\right)^{\frac{1-q}{q}} \le C_q U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}} U(s)^{-\frac{1}{1-q}} u(s)\,ds\right)^{\frac{1-q}{q}}$$

in which

$$C_q = \left(\tfrac{1-q}{q}\right)^{\frac{1-q}{q}} + (2q)^{\frac{1-q}{q}}.$$

Conversely, by (2.2) again, we have

$$\begin{aligned}&\int_t^\infty W(s)^{\frac{1}{1-q}}U(s)^{-\frac{1}{1-q}}u(s)\,ds\\&\le \tfrac{1}{q}\int_t^\infty W(s)^{\frac{q}{1-q}}w(s)U(s)^{-\frac{q}{1-q}}\,ds+\left(\tfrac{1-q}{q}\right)^{\frac{1-q}{q}}\frac{W(t)^{\frac{1}{1-q}}}{U(t)^{\frac{q}{1-q}}}.\end{aligned}$$

Raising this estimate to $\frac{1-q}{q}$ and multiplying it by $U(t)$, we obtain

$$\begin{aligned}&U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}}U(s)^{-\frac{1}{1-q}}u(s)\,ds\right)^{\frac{1-q}{q}}\\&\le\left(\tfrac{1}{q}\right)^{\frac{1-q}{q}}U(t)\left(\int_t^\infty W(s)^{\frac{q}{1-q}}w(s)U(s)^{-\frac{q}{1-q}}\,ds\right)^{\frac{1-q}{q}}+\left(\tfrac{1-q}{q}\right)^{\frac{1-q}{q}}W(t)^{\frac{1}{q}}.\end{aligned}$$

The proof is complete. □

We finish this section with two theorems in which we characterize weighted inequalities involving iteration of two integral operators.

Theorem 2.8. *Assume that* $p,q,m\in(1,\infty)$ *and* $q<m$. *Let* u,v,w *be weights on* $(0,\infty)$. *Let*

$$K=\sup_{g\in\mathfrak{M}_+}\frac{\left(\int_0^\infty\left(\int_t^\infty\left(\int_s^\infty g(y)\,dy\right)^q u(s)\,ds\right)^{\frac{m}{q}}w(t)\,dt\right)^{\frac{1}{m}}}{\left(\int_0^\infty g(s)^p v(s)\,ds\right)^{\frac{1}{p}}}.$$

(a) *Let* $1<p\le q<\infty$. *Then*

$$K\approx A_1,$$

where

$$A_1=\sup_{t\in(0,\infty)}\left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}\left(\int_0^t\left(\int_s^t u(y)\,dy\right)^{\frac{m}{q}}w(s)\,ds\right)^{\frac{1}{m}}.$$

(b) *Let* $1<q<p<\infty$ *and* $p\le m$. *Then*

$$K\approx A_1+A_2,$$

where

$$A_2=\sup_{t\in(0,\infty)}\left(\int_t^\infty\left(\int_s^\infty u(y)\,dy\right)^{\frac{p}{p-q}}\left(\int_s^\infty v(y)^{1-p'}\,dy\right)^{\frac{p(q-1)}{p-q}}v(s)^{1-p'}\,ds\right)^{\frac{p-q}{pq}}\left(\int_0^t w(s)\,ds\right)^{\frac{1}{m}}.$$

(c) *Let* $1<q<p<\infty$ *and* $m<p$. *Then*

$$K\approx A_3+A_4,$$

where

$$A_3=\left(\int_0^\infty\left(\int_t^\infty\left(\int_s^\infty u(y)\,dy\right)^{\frac{p}{p-q}}\left(\int_s^\infty v(y)^{1-p'}\,dy\right)^{\frac{p(q-1)}{p-q}}v(s)^{1-p'}\,ds\right)^{\frac{m(p-q)}{q(p-m)}}W(s)^{\frac{m}{p-m}}w(s)\,ds\right)^{\frac{p-m}{pm}}$$

and

$$A_4=\left(\int_0^\infty\left(\int_t^\infty v(s)^{1-p'}\,ds\right)^{\frac{p(m-1)}{p-m}}\left(\int_0^t\left(\int_s^t u(y)\,dy\right)^{\frac{m}{q}}w(s)\,ds\right)^{\frac{p}{p-m}}v(t)^{1-p'}\,dt\right)^{\frac{p-m}{pm}}.$$

Proof. We first observe that, by (2.1), one has

$$K=\sup_{g\in\mathfrak{M}_+}\sup_{h\in\mathfrak{M}_+}\frac{\left(\int_0^\infty h(t)\int_t^\infty\left(\int_s^\infty g(y)\,dy\right)^q u(s)\,ds\,dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s)\,ds\right)^{\frac{1}{p}}\left(\int_0^\infty h(s)^{\frac{m}{m-q}}w(s)^{-\frac{q}{m-q}}\,ds\right)^{\frac{m-q}{mq}}}.$$

Interchanging suprema and using the Fubini theorem, we obtain

$$K = \sup_{h\in\mathfrak{M}_+} \frac{1}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\, ds\right)^{\frac{m-q}{mq}}} \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_s^\infty g(y)\, dy\right)^q \int_0^s h(t)\, dt\, u(s)\, ds\right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s)\, ds\right)^{\frac{1}{p}}}. \tag{2.5}$$

Let $1 < p \le q < \infty$. Then, by Theorem 2.2(a), we get

$$\sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_s^\infty g(y)\, dy\right)^q \int_0^s h(t)\, dt\, u(s)\, ds\right)^{\frac{1}{q}}}{\left(\int_0^\infty g(s)^p v(s)\, ds\right)^{\frac{1}{p}}} \approx \sup_{t\in(0,\infty)} \left(\int_0^t u(s) \int_0^s h(y)\, dy\, ds\right)^{\frac{1}{q}} \left(\int_t^\infty v(s)^{1-p'}\, ds\right)^{\frac{1}{p'}}.$$

Plugging this to (2.5), we get

$$K \approx \sup_{h\in\mathfrak{M}_+} \frac{\sup\limits_{t\in(0,\infty)} \left(\int_0^t u(s) \int_0^s h(y)\, dy\, ds\right)^{\frac{1}{q}} \left(\int_t^\infty v(s)^{1-p'}\, ds\right)^{\frac{1}{p'}}}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\, ds\right)^{\frac{m-q}{mq}}}.$$

Now we interchange the suprema again, apply the Fubini theorem and raise all the expressions to q. We obtain

$$K^q \approx \sup_{t\in(0,\infty)} \left(\int_t^\infty v(s)^{1-p'}\, ds\right)^{\frac{q}{p'}} \sup_{h\in\mathfrak{M}_+} \frac{\int_0^t h(s) \int_s^t u(y)\, dy\, ds}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\, ds\right)^{\frac{m-q}{m}}}.$$

By (2.1), this yields $K \approx A_1$, proving the assertion in the case (a).

Let now $1 < q < p < \infty$. Then, by Theorem 2.1(b), we have

$$K^q \approx \sup_{h\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^t u(s) \int_0^s h(y)\, dy\, ds\right)^{\frac{p}{p-q}} \left(\int_t^\infty v(s)^{1-p'}\, ds\right)^{\frac{p(q-1)}{p-q}} v(t)^{1-p'}\, dt\right)^{\frac{p-q}{p}}}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\, ds\right)^{\frac{m-q}{m}}}.$$

Consequently, by the Fubini theorem,

$$K^q \approx \sup_{h\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_0^t h(y) \int_y^t u(s)\, ds\, dy\right)^{\frac{p}{p-q}} \left(\int_t^\infty v(s)^{1-p'}\, ds\right)^{\frac{p(q-1)}{p-q}} v(t)^{1-p'}\, dt\right)^{\frac{p-q}{p}}}{\left(\int_0^\infty h(s)^{\frac{m}{m-q}} w(s)^{-\frac{q}{m-q}}\, ds\right)^{\frac{m-q}{m}}}.$$

Now, in the case (b) the assertion follows from Theorem 2.5(a) and in the case (c) from Theorem 2.5(b). □

Theorem 2.9. *Assume that $m, p, q \in (1,\infty)$ and let u, v, w and ϱ be weights on $(0,\infty)$. Assume that $q < m$. Let*

$$K = \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty \left(\int_t^\infty \left(\int_0^s g(y)\varrho(y)\, dy\right)^q u(s)\, ds\right)^{\frac{m}{q}} w(t)\, dt\right)^{\frac{1}{m}}}{\left(\int_0^\infty g(s)^p v(s)\, ds\right)^{\frac{1}{p}}}.$$

(a) *If $p \le q < m$, then*

$$\begin{aligned} K \approx{} & \sup_{t\in(0,\infty)} W(t)^{\frac{1}{m}} \left(\int_t^\infty u(s)\, ds\right)^{\frac{1}{q}} \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\, ds\right)^{\frac{1}{p'}} \\ & + \sup_{t\in(0,\infty)} \left(\int_t^\infty \left(\int_s^\infty u(y)\, dy\right)^{\frac{m}{q}} w(s)\, ds\right)^{\frac{1}{m}} \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\, ds\right)^{\frac{1}{p'}}. \end{aligned}$$

(b) *If* $q < p \le m$, *then*

$$K \approx \sup_{t\in(0,\infty)} \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy\right)^{\frac{m}{q}} w(s)\,ds\right)^{\frac{1}{m}} \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{1}{p'}}$$

$$+ \sup_{t\in(0,\infty)} W(t)^{\frac{1}{m}} \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy\right)^{\frac{p}{p-q}} \left(\int_0^s \varrho(y)^{p'} v(y)^{1-p'}\,dy\right)^{\frac{p(q-1)}{p-q}} \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{p-q}{pq}}.$$

(c) *If* $q < m < p$, *then*

$$K \approx \left(\int_0^\infty \left(\int_0^t \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{m(p-1)}{p-m}} \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy\right)^{\frac{m}{q}} w(s)\,ds\right)^{\frac{m}{p-m}} \left(\int_t^\infty u(y)\,dy\right)^{\frac{m}{q}} w(t)\,dt\right)^{\frac{p-m}{mp}}$$

$$+ \left(\int_0^\infty \left(\int_t^\infty \left(\int_s^\infty u(y)\,dy\right)^{\frac{p}{p-q}} \left(\int_0^s \varrho(y)^{p'} v(y)^{1-p'}\right)^{\frac{p(q-1)}{p-q}} \varrho(s)^{p'} v(s)^{1-p'}\,ds\right)^{\frac{m(p-q)}{q(p-m)}} W(t)^{\frac{m}{p-m}} w(t)\,dt\right)^{\frac{p-m}{mp}}.$$

Proof. The proof can be done in the same way as that of Theorem 2.8. ☐

We note that the assertion of Theorem 2.9 can be also extracted from [22], where however the characterizing conditions are formulated in modified way and where a completely different proof is presented.

3. PROOF OF THE MAIN RESULT

Proof of Theorem 1.1. As the first step of our analysis we will express the value of C in a modified way. For every fixed $g \in \mathfrak{M}_+$, set

$$A(g) = \sup_{h\in\mathfrak{M}_+} \frac{\left(\int_0^\infty h^*(t)^{\frac{p_2}{p_1}} u_2(t) \int_t^\infty g(s)\,ds\,dt\right)^{\frac{p_1}{p_2}}}{\left(\int_0^\infty h^{**}_{u_1}(t)^{\frac{m_1}{p_1}} w_1(t) U_1(t)^{\frac{m_1}{p_1}}\,dt\right)^{\frac{p_1}{m_1}}},$$

where we apply the notation introduced in (1.4). We claim that

$$C = \sup_{g\in\mathfrak{M}_+} \frac{A(g)^{\frac{1}{p_1}}}{\left(\int_0^\infty g(t)^{\frac{m_2}{m_2-p_2}} w_2(t)^{-\frac{p_2}{m_2-p_2}}\,dt\right)^{\frac{m_2-p_2}{m_2p_2}}}. \tag{3.1}$$

Indeed, fix $f \in \mathfrak{M}$. Since $\frac{m_2}{p_2} > 1$, we can apply (2.1) to $p = \frac{m_2}{p_2}$ and $v = w_2$. Then $p' = \frac{m_2}{m_2-p_2}$ and $1 - p' = -\frac{p_2}{m_2-p_2}$, and so we get

$$\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_2} u_2(s)\,ds\right)^{\frac{m_2}{p_2}} w_2(t)\,dt\right)^{\frac{1}{m_2}} = \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty g(t) \int_0^t f^*(s)^{p_2} u_2(s)\,ds\,dt\right)^{\frac{1}{p_2}}}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2-p_2}} w_2(s)^{-\frac{p_2}{m_2-p_2}}\,ds\right)^{\frac{m_2-p_2}{m_2p_2}}}.$$

By the Fubini theorem, this turns into

$$\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_2} u_2(s)\,ds\right)^{\frac{m_2}{p_2}} w_2(t)\,dt\right)^{\frac{1}{m_2}} = \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty f^*(s)^{p_2} u_2(s) \int_s^\infty g(t)\,dt\,ds\right)^{\frac{1}{p_2}}}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2-p_2}} w_2(s)^{-\frac{p_2}{m_2-p_2}}\,ds\right)^{\frac{m_2-p_2}{m_2p_2}}}.$$

Plugging this into (1.6), we get

$$C = \sup_{f\in\mathfrak{M}} \frac{1}{\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1} u_1(s)\,ds\right)^{\frac{m_1}{p_1}} w_1(t)\,dt\right)^{\frac{1}{m_1}}} \sup_{g\in\mathfrak{M}_+} \frac{\left(\int_0^\infty f^*(s)^{p_2} u_2(s) \int_s^\infty g(t)\,dt\,ds\right)^{\frac{1}{p_2}}}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2-p_2}} w_2(s)^{-\frac{p_2}{m_2-p_2}}\,ds\right)^{\frac{m_2-p_2}{m_2p_2}}}.$$

On interchanging suprema, this yields

$$C = \sup_{g\in\mathfrak{M}_+} \frac{1}{\left(\int_0^\infty g(s)^{\frac{m_2}{m_2-p_2}} w_2(s)^{-\frac{p_2}{m_2-p_2}}\,ds\right)^{\frac{m_2-p_2}{m_2p_2}}} \sup_{f\in\mathfrak{M}} \frac{\left(\int_0^\infty f^*(s)^{p_2}u_2(s)\int_s^\infty g(t)\,dt\,ds\right)^{\frac{1}{p_2}}}{\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1}u_1(s)\,ds\right)^{\frac{m_1}{p_1}} w_1(t)\,dt\right)^{\frac{1}{m_1}}}.$$

Now, for a change, fix $g \in \mathfrak{M}_+$. Given $f \in \mathfrak{M}$, set $h = |f|^{p_1}$. Then $f^* = (h^*)^{\frac{1}{p_1}}$, and we have

$$\sup_{f\in\mathfrak{M}} \frac{\left(\int_0^\infty f^*(s)^{p_2}u_2(s)\int_s^\infty g(t)\,dt\,ds\right)^{\frac{1}{p_2}}}{\left(\int_0^\infty \left(\int_0^t f^*(s)^{p_1}u_1(s)\,ds\right)^{\frac{m_1}{p_1}} w_1(t)\,dt\right)^{\frac{1}{m_1}}} = \sup_{h\in\mathfrak{M}} \frac{\left(\int_0^\infty h^*(t)^{\frac{p_2}{p_1}}u_2(t)\int_t^\infty g(s)\,ds\,dt\right)^{\frac{1}{p_2}}}{\left(\int_0^\infty h^{**}_{u_1}(t)^{\frac{m_1}{p_1}} w_1(t)U_1(t)^{\frac{m_1}{p_1}}\,dt\right)^{\frac{1}{m_1}}}.$$

The quantity on the right-hand side now equals $A(g)^{\frac{1}{p_1}}$. This establishes (3.1).

We next observe that, for every fixed $g \in \mathfrak{M}_+$, one has

$$A(g) = \sup_{h\in\mathfrak{M}} \frac{\left(\int_0^\infty h^*(t)^q w(t)\,dt\right)^{\frac{1}{q}}}{\left(\int_0^\infty h^{**}_u(t)^p v(t)\,dt\right)^{\frac{1}{p}}}$$

with

$$p = \frac{m_1}{p_1},\ q = \frac{p_2}{p_1} \tag{3.2}$$

and

$$w(t) = u_2(t)\int_t^\infty g(s)\,ds, \quad v(t) = U_1(t)^{\frac{m_1}{p_1}} w_1(t), \quad u(t) = u_1(t), \qquad t \in (0,\infty). \tag{3.3}$$

Now, the quantity $A(g)$ can be equivalently evaluated in terms of parameters p, q and weights u, v, w via Theorem 2.4 (we note that the assumptions of that theorem are fulfilled). However, the expressions in cases (c) and (d) are not in a satisfactory form and we have to modify them through Lemma 2.7. The reason will become apparent soon - roughly speaking, we need to get rid of all the expressions that involve w and have to replace them by those involving W instead. Thus, by Lemma 2.7, we get

(c) if $0 < p \le q < 1$, then

$$A(g) \approx \sup_{t\in(0,\infty)} \frac{U(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}}U(s)^{-\frac{1}{1-q}}u(s)\,ds\right)^{\frac{1-q}{q}}}{\left(V(t) + U(t)^p\int_t^\infty U(s)^{-p}v(s)\,ds\right)^{\frac{1}{p}}},$$

and

(d) if $0 < q < 1$ and $0 < q < p$, then

$$A(g) \approx \left(\int_0^\infty \frac{U(t)^{\frac{pq}{p-q}+p-1}V(t)\left(\int_t^\infty W(s)^{\frac{1}{1-q}}U(s)^{-\frac{1}{1-q}}u(s)\,ds\right)^{-\frac{p(q-1)}{p-q}}\int_t^\infty U(s)^{-p}v(s)\,ds}{\left(V(t)+U(t)^p\int_t^\infty U(s)^{-p}v(s)\,ds\right)^{\frac{p}{p-q}+1}}\,dt\right)^{\frac{p-q}{pq}}.$$

Our next step is "translation" of expressions characterizing $A(g)$ in cases (a)-(d) into the language of the parameters and weights occurring in Theorem 1.1 via (3.2) and (3.3). These expressions depend on g in a somewhat concealed way, namely through the weight w. It will be useful to note that

$$\varphi(t) = V(t) + U(t)^p\int_t^\infty U(s)^{-p}v(s)\,ds$$

and

$$W(t) = \int_0^t g(s)U_2(s)\,ds + U_2(t)\int_t^\infty g(s)\,ds.$$

We obtain the following reformulations of $A(g)$:

(a) if $p_1 \le p_2$ and $m_1 \le p_2$, then

$$A(g) \approx \sup_{t\in(0,\infty)} \frac{\left(\int_0^t g(s)U_2(s)\,ds + U_2(t)\int_t^\infty g(s)\,ds\right)^{\frac{p_1}{p_2}}}{\varphi(t)^{\frac{p_1}{m_1}}},$$

(b) if $p_1 \le p_2$ and $m_1 > p_2$, then

$$A(g) \approx \left(\int_0^\infty \sup_{s\in(t,\infty)} U_1(s)^{-\frac{m_1p_2}{p_1(m_1-p_2)}} \left(\int_0^s g(y)U_2(y)\,dy + U_2(s)\int_s^\infty g(y)\,dy\right)^{\frac{m_1}{m_1-p_2}} \sigma(t)\,dt\right)^{\frac{p_1(m_1-p_2)}{m_1p_2}},$$

(c) if $p_1 > p_2$ and $m_1 \le p_2$, then

$$A(g) \approx \sup_{t\in(0,\infty)} \frac{U_1(t)\left(\int_t^\infty \left(\int_0^s g(y)U_2(y)\,dy + U_2(s)\int_s^\infty g(y)\,dy\right)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}}u_1(s)\,ds\right)^{\frac{p_1-p_2}{p_2}}}{\varphi(t)^{\frac{p_1}{m_1}}},$$

and

(d) if $p_1 > p_2$ and $m_1 > p_2$, then

$$A(g) \approx \left(\int_0^\infty \left(\int_t^\infty \left(\int_0^s g(y)U_2(y)\,dy + U_2(s)\int_s^\infty g(y)\,dy\right)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}}u_1(s)\,ds\right)^{\frac{m_1(p_1-p_2)}{p_1(m_1-p_2)}} \sigma(t)\,dt\right)^{\frac{p_1(m_1-p_2)}{m_1p_2}}.$$

Now, let us introduce an abbreviated notation. We will write, for $g \in \mathfrak{M}$,

$$\|g\| = \left(\int_0^\infty g(t)^{\frac{m_2}{m_2-p_2}} w_2(t)^{-\frac{p_2}{m_2-p_2}}\,dt\right)^{\frac{m_2-p_2}{m_2}},$$

and set

$$D = \sup_{g\in\mathfrak{M}_+} \frac{A(g)^{\frac{p_2}{p_1}}}{\|g\|}.$$

Then, by (3.1),

$$C \approx D^{\frac{1}{p_2}}.$$

It follows from the above estimates that

(a) if $p_1 \le p_2$ and $m_1 \le p_2$, then $D \approx D_1 + D_2$, where

$$D_1 = \sup_{g\in\mathfrak{M}_+} \frac{1}{\|g\|} \sup_{t\in(0,\infty)} \frac{\int_0^t g(s)U_2(s)\,ds}{\varphi(t)^{\frac{p_2}{m_1}}}$$

and

$$D_2 = \sup_{g\in\mathfrak{M}_+} \frac{1}{\|g\|} \sup_{t\in(0,\infty)} \frac{U_2(t)\int_t^\infty g(s)\,ds}{\varphi(t)^{\frac{p_2}{m_1}}},$$

(b) if $p_1 \le p_2$ and $m_1 > p_2$, then $D \approx D_3 + D_4$, where

$$D_3 = \sup_{g\in\mathfrak{M}_+} \frac{1}{\|g\|} \left(\int_0^\infty \sup_{s\in(t,\infty)} U_1(s)^{-\frac{m_1p_2}{p_1(m_1-p_2)}} \left(\int_0^s g(y)U_2(y)\,dy\right)^{\frac{m_1}{m_1-p_2}} \sigma(t)\,dt\right)^{\frac{m_1-p_2}{m_1}},$$

and

$$D_4 = \sup_{g\in\mathfrak{M}_+} \frac{1}{\|g\|} \left(\int_0^\infty \sup_{s\in(t,\infty)} U_1(s)^{-\frac{m_1p_2}{p_1(m_1-p_2)}} U_2(s)^{\frac{m_1}{m_1-p_2}} \left(\int_s^\infty g(y)\,dy\right)^{\frac{m_1}{m_1-p_2}} \sigma(t)\,dt\right)^{\frac{m_1-p_2}{m_1}},$$

(c) if $p_1 > p_2$ and $m_1 \le p_2$, then $D \approx D_5 + D_6$, where

$$D_5 = \sup_{g\in\mathfrak{M}_+} \frac{1}{\|g\|} \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{p_2}{p_1}}\left(\int_t^\infty \left(\int_0^s g(y)U_2(y)\,dy\right)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}}u_1(s)\,ds\right)^{\frac{p_1-p_2}{p_1}}}{\varphi(t)^{\frac{p_1}{m_1}}},$$

and

$$D_6 = \sup_{g \in \mathfrak{M}_+} \frac{1}{\|g\|} \sup_{t \in (0,\infty)} \frac{U_1(t)^{\frac{p_2}{p_1}} \left(\int_t^\infty \left(U_2(s) \int_s^\infty g(y)\,dy \right)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}} u_1(s)\,ds \right)^{\frac{p_1-p_2}{p_1}}}{\varphi(t)^{\frac{p_1}{m_1}}},$$

(d) if $p_1 > p_2$ and $m_1 > p_2$, then $D \approx D_7 + D_8$, where

$$D_7 = \sup_{g \in \mathfrak{M}_+} \frac{1}{\|g\|} \left(\int_0^\infty \left(\int_t^\infty \left(\int_0^s g(y) U_2(y)\,dy \right)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}} u_1(s)\,ds \right)^{\frac{m_1(p_1-p_2)}{p_1(m_1-p_2)}} \sigma(t)\,dt \right)^{\frac{m_1-p_2}{m_1}}$$

and

$$D_8 = \sup_{g \in \mathfrak{M}_+} \frac{1}{\|g\|} \left(\int_0^\infty \left(\int_t^\infty \left(U_2(s) \int_s^\infty g(y)\,dy \right)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}} u_1(s)\,ds \right)^{\frac{m_1(p_1-p_2)}{p_1(m_1-p_2)}} \sigma(t)\,dt \right)^{\frac{m_1-p_2}{m_1}}.$$

Our final task is to establish two-sided estimates for D_1–D_8. We shall treat each case separately.

Case (a). Assume that $m_1 \le p_2$ and $p_1 \le p_2$. Interchanging the suprema, we have

$$D_1 = \sup_{t \in (0,\infty)} \frac{1}{\varphi(t)^{\frac{p_2}{m_1}}} \sup_{g \in \mathfrak{M}_+} \frac{\int_0^t g(s) U_2(s)\,ds}{\|g\|}.$$

We now fix $t \in (0,\infty)$ and apply (2.1) to

$$p = \frac{m_2}{p_2},\ f = U_2 \chi_{(0,t)} \text{ and } v = w_2.$$

We then arrive at

$$D_1 = \sup_{t \in (0,\infty)} \frac{1}{\varphi(t)^{\frac{p_2}{m_1}}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds \right)^{\frac{p_2}{m_2}}.$$

Similarly,

$$D_2 = \sup_{t \in (0,\infty)} \frac{U_2(t)}{\varphi(t)^{\frac{p_2}{m_1}}} \sup_{g \in \mathfrak{M}_+} \frac{\int_t^\infty g(s)\,ds}{\|g\|}.$$

Using (2.1) with a fixed $t \in (0,\infty)$ once again, this time to

$$p = \frac{m_2}{p_2},\ f = \chi_{(t,\infty)} \text{ and } v = w_2,$$

we get

$$D_2 = \sup_{t \in (0,\infty)} \frac{U_2(t)}{\varphi(t)^{\frac{p_2}{m_1}}} \left(\int_t^\infty w_2(s)\,ds \right)^{\frac{p_2}{m_2}}.$$

Taking the p_2-roots, we get the assertion of the theorem in case (a).

Case (b). Assume that $m_1 > p_2$ and $p_1 \le p_2$. To characterize D_3 and D_4, we have to distinguish two subcases depending on the comparison of m_1 and m_2.

Case (b-i). Assume that $m_1 \le m_2$. Then, by Theorem 2.3(a), applied to

$$p = \frac{m_2}{m_2 - p_2}, \quad q = \frac{m_1}{m_1 - p_2}, \quad u = U_1^{-\frac{p_2}{p_1}}, \quad v = U_2^{-\frac{m_2}{m_2-p_2}} w_2^{-\frac{p_2}{m_2-p_2}}, \quad \varrho = U_2 \quad \text{and } w = \sigma,$$

we arrive at

$$D_3 \approx \sup_{t \in (0,\infty)} \left(\sup_{s \in (t,\infty)} U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \int_0^s \sigma(y)\,dy + \int_t^\infty \sup_{y \in (s,\infty)} U_1(y)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \sigma(s)\,ds \right)^{\frac{m_1-p_2}{m_1}} \times$$
$$\times \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds \right)^{\frac{p_2}{m_2}}.$$

By monotonicity of U_1, we get

$$D_3 \approx \sup_{t\in(0,\infty)} \left(\sup_{s\in(t,\infty)} U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \int_0^s \sigma(y)\,dy + \int_t^\infty U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \sigma(s)\,ds \right)^{\frac{m_1-p_2}{m_1}} \times \\ \times \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds \right)^{\frac{p_2}{m_2}}.$$

By the subadditivity of the supremum, one has, for a fixed $t \in (0,\infty)$,

$$\begin{aligned} &\sup_{s\in(t,\infty)} U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \int_0^s \sigma(y)\,dy + \int_t^\infty U_1(y)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \sigma(y)\,dy \\ &\approx \sup_{s\in(t,\infty)} \left(U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \int_0^s \sigma(y)\,dy + \int_s^\infty U_1(y)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \sigma(y)\,dy \right) \\ &= \sup_{s\in(t,\infty)} \int_0^\infty \min\left\{ U_1(y)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}}, U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \right\} \sigma(y)\,dy. \end{aligned}$$

Using the monotonicity of U_1 once again, we conclude that the last expression is decreasing in $s \in (0,\infty)$. Hence,

$$\begin{aligned} &\sup_{s\in(t,\infty)} U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \int_0^s \sigma(y)\,dy + \int_t^\infty U_1(y)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \sigma(y)\,dy \\ &\approx \int_0^\infty \min\left\{ U_1(y)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}}, U_1(t)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \right\} \sigma(y)\,dy \\ &= U_1(t)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \int_0^t \sigma(y)\,dy + \int_t^\infty U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \sigma(s)\,ds. \end{aligned}$$

Altogether,

$$D_3 \approx \sup_{t\in(0,\infty)} \left(U_1(t)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \int_0^t \sigma(y)\,dy + \int_t^\infty U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}} \sigma(s)\,ds \right)^{\frac{m_1-p_2}{m_1}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds \right)^{\frac{p_2}{m_2}}.$$

Further, by Theorem 2.6(a), applied to

$$p = \frac{m_2}{m_2-p_2},\ q = \frac{m_1}{m_1-p_2},\ u = U_2 U_1^{-\frac{p_2}{p_1}},\ v = w_2^{-\frac{p_2}{m_2-p_2}} \text{ and } w = \sigma,$$

we get

$$D_4 \approx \sup_{t\in(0,\infty)} \left(\int_0^t \sup_{y\in(s,t)} U_2(y)^{\frac{m_1}{m_1-p_2}} U_1(y)^{-\frac{m_1 p_2}{p_1(m_1-p_2)}} \sigma(s)\,ds \right)^{\frac{m_1-p_2}{m_1}} \left(\int_t^\infty w_2(s)\,ds \right)^{\frac{p_2}{m_2}}.$$

Combining all the estimates obtained and taking the roots we establish the assertion of the theorem in the case (b-i).

Case (b-ii). Assume now that $m_1 > m_2$ (while still $m_1 > p_2$ and $p_1 \le p_2$). By Theorem 2.3(b), applied to

$$p = \frac{m_2}{m_2-p_2}, \quad q = \frac{m_1}{m_1-p_2}, \quad u = U_1^{-\frac{p_2}{p_1}}, \quad v = U_2^{-\frac{m_2}{m_2-p_2}} w_2^{-\frac{p_2}{m_2-p_2}}, \quad \varrho = U_2 \quad \text{and} \quad w = \sigma,$$

and observing that this time $1 < q < p < \infty$, we get

$$D_3 \approx D_{31} + D_{32},$$

where

$$D_{31}=\left(\int_0^\infty\left(\int_t^\infty U_1(s)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}}\sigma(s)\,ds\right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}}U_1(t)^{-\frac{p_2}{p_1}\frac{m_1}{m_1-p_2}}\right.$$
$$\left.\times\left(\int_0^t U_2(s)^{\frac{m_2}{p_2}}w_2(s)\,ds\right)^{\frac{m_1}{m_1-p_2}}\sigma(t)\,dt\right)^{\frac{p_2(m_1-m_2)}{m_1m_2}}$$

and

$$D_{32}=\left(\int_0^\infty\sup_{s\in(t,\infty)}U_1(s)^{-\frac{m_1m_2}{p_1(m_1-m_2)}}\left(\int_0^s U_2(y)^{\frac{m_2}{p_2}}w_2(y)\,dy\right)^{\frac{m_1}{m_1-m_2}}\left(\int_0^t\sigma(s)\,ds\right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}}\sigma(t)\,dt\right)^{\frac{p_2(m_1-m_2)}{m_1m_2}}.$$

By Theorem 2.6(b), applied to

$$p=\frac{m_2}{m_2-p_2},\ q=\frac{m_1}{m_1-p_2},\ u=U_2U_1^{-\frac{p_2}{p_1}},\ v=w_2^{-\frac{p_2}{m_2-p_2}}\text{ and }w=\sigma$$

we obtain

$$D_4\approx D_{41}+D_{42},$$

where

$$D_{41}=\left(\int_0^\infty\sup_{s\in(t,\infty)}U_2(s)^{\frac{m_1m_2}{p_2(m_1-m_2)}}U_1(s)^{-\frac{m_1m_2}{p_1(m_1-m_2)}}\left(\int_s^\infty w_2(y)\,dy\right)^{\frac{m_1}{m_1-m_2}}\right.$$
$$\left.\times\left(\int_0^t\sigma(s)\,ds\right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}}\sigma(t)\,dt\right)^{\frac{p_2(m_1-m_2)}{m_1m_2}}$$

and

$$D_{42}=\left(\int_0^\infty\sup_{s\in(t,\infty)}U_2(s)^{\frac{m_1}{m_1-p_2}}U_1(s)^{-\frac{m_1p_2}{p_1(m_1-p_2)}}\left(\int_s^\infty w_2(y)\,dy\right)^{\frac{m_1}{m_1-m_2}}\right.$$
$$\left.\times\left(\int_0^t\sup_{y\in(s,t)}U_2(y)^{\frac{m_1}{m_1-p_2}}U_1(y)^{-\frac{m_1p_2}{p_1(m_1-p_2)}}\sigma(s)\,ds\right)^{\frac{m_1(m_2-p_2)}{p_2(m_1-m_2)}}\sigma(t)\,dt\right)^{\frac{p_2(m_1-m_2)}{m_1m_2}}.$$

Combining the estimates and taking the roots, we obtain the assertion of the theorem in the case (b-ii).

Case (c). Assume that $m_1\le p_2$ and $p_1>p_2$. We start by interchanging the suprema in the definition of D_5 and D_6. We get

$$D_5=\sup_{t\in(0,\infty)}\frac{U_1(t)^{\frac{p_2}{p_1}}}{\varphi(t)^{\frac{p_1}{m_1}}}\sup_{g\in\mathfrak{M}_+}\frac{\left(\int_t^\infty\left(\int_0^s g(y)U_2(y)\,dy\right)^{\frac{p_1}{p_1-p_2}}U_1(s)^{-\frac{p_2}{p_1-p_2}}u_1(s)\,ds\right)^{\frac{p_1-p_2}{p_1}}}{\|g\|},$$

and

$$D_6=\sup_{t\in(0,\infty)}\frac{U_1(t)^{\frac{p_2}{p_1}}}{\varphi(t)^{\frac{p_1}{m_1}}}\sup_{g\in\mathfrak{M}_+}\frac{\left(\int_t^\infty\left(\int_s^\infty g(y)\,dy\right)^{\frac{p_1}{p_1-p_2}}U_2(s)^{\frac{p_1}{p_1-p_2}}U_1(s)^{-\frac{p_2}{p_1-p_2}}u_1(s)\,ds\right)^{\frac{p_1-p_2}{p_1}}}{\|g\|}.$$

We will distinguish two subcases. This time, the decisive factor is the comparison between p_1 and p_2.

Case (c-i). Assume that $p_1\le m_2$ (while still $m_1\le p_2$ and $p_1>p_2$). Fix $t\in(0,\infty)$. Applying Theorem 2.1(a) to the parameters

$$p=\frac{m_2}{m_2-p_2},\ q=\frac{p_1}{p_1-p_2}$$

and the weights

$$u = U_2,\ v = w_2^{-\frac{p_2}{m_2-p_2}} \quad \text{and} \quad w(s) = U_1(s)^{-\frac{p_1}{p_1-p_2}} u_1(s)\chi_{(t,\infty)(s)},\ s \in (0,\infty),$$

we get

$$\sup_{g\in\mathfrak{M}_+} \frac{\left(\int_t^\infty \left(\int_0^s g(y)U_2(y)\,dy\right)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}} u_1(s)\,ds\right)^{\frac{p_1-p_2}{p_1}}}{\|g\|}$$

$$\approx \sup_{s\in(0,\infty)} \left(\int_s^\infty U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y)\chi_{(t,\infty)}(y)\,dy\right)^{\frac{p_1-p_2}{p_1}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy\right)^{\frac{p_2}{m_2}}.$$

Since

$$\sup_{s\in(0,\infty)} \left(\int_s^\infty U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y)\chi_{(t,\infty)}(y)\,dy\right)^{\frac{p_1-p_2}{p_1}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy\right)^{\frac{p_2}{m_2}}$$

$$= \max\left\{ \sup_{s\in(0,t)} \left(\int_t^\infty U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y)\,dy\right)^{\frac{p_1-p_2}{p_1}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy\right)^{\frac{p_2}{m_2}};\right.$$

$$\left. \sup_{s\in(t,\infty)} \left(\int_s^\infty U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y)\,dy\right)^{\frac{p_1-p_2}{p_1}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy\right)^{\frac{p_2}{m_2}} \right\}$$

$$= \sup_{s\in(t,\infty)} \left(\int_s^\infty U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y)\,dy\right)^{\frac{p_1-p_2}{p_1}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy\right)^{\frac{p_2}{m_2}},$$

calculating the first integral we finally arrive at

$$\sup_{g\in\mathfrak{M}_+} \frac{\left(\int_t^\infty \left(\int_0^s g(y)U_2(y)\,dy\right)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}} u_1(s)\,ds\right)^{\frac{p_1-p_2}{p_1}}}{\|g\|} \approx \sup_{s\in(t,\infty)} U_1(s)^{-\frac{p_2}{p_1}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy\right)^{\frac{p_2}{m_2}}.$$

Similarly, by Theorem 2.2(a), applied to

$$p = \frac{m_2}{m_2-p_2},\ q = \frac{p_1}{p_1-p_2},\ v = w_2^{-\frac{p_2}{m_2-p_2}} \quad \text{and} \quad w(s) = U_2(s)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_1}{p_1-p_2}} u_1(s)\chi_{(t,\infty)(s)},\ s\in(0,\infty),$$

we get

$$\sup_{g\in\mathfrak{M}_+} \frac{\left(\int_t^\infty \left(\int_s^\infty g(y)\,dy\right)^{\frac{p_1}{p_1-p_2}} U_2(s)^{\frac{p_1}{p_1-p_2}} U_1(s)^{-\frac{p_2}{p_1-p_2}} u_1(s)\,ds\right)^{\frac{p_1-p_2}{p_1}}}{\|g\|} \approx$$

$$\approx \sup_{s\in(t,\infty)} \left(\int_t^s U_2(y)^{\frac{p_1}{p_1-p_2}} U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y)\,dy\right)^{\frac{p_1-p_2}{p_1}} \left(\int_s^\infty w_2(y)\,dy\right)^{\frac{p_2}{m_2}}.$$

The obtained estimates hold for every fixed $t \in (0,\infty)$. Hence, plugging them into the definitions of D_5 and D_6, we get

$$D_5 \approx \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{p_2}{p_1}}}{\varphi(t)^{\frac{p_2}{m_1}}} \sup_{s\in(t,\infty)} U_1(s)^{-\frac{p_2}{p_1}} \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy\right)^{\frac{p_2}{m_2}}$$

and

$$D_6 \approx \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{p_2}{p_1}}}{\varphi(t)^{\frac{p_2}{m_1}}} \sup_{s\in(t,\infty)} \left(\int_t^s U_2(y)^{\frac{p_1}{p_1-p_2}} U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y)\,dy\right)^{\frac{p_1-p_2}{p_1}} \left(\int_s^\infty w_2(y)\,dy\right)^{\frac{p_2}{m_2}}.$$

Combining the estimates and taking the roots, we get the assertions of the theorem in case (c-i).

Case (c-ii). Assume that $p_1 \le m_2$ (and $m_1 \le p_2$ and $p_1 > p_2$ remain in power). By Theorem 2.1(b), applied to the same set of parameters as in the case (c-i), we obtain

$$\begin{aligned} D_5 \approx & \sup_{t\in(0,\infty)} \frac{\left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds\right)^{\frac{p_2}{m_2}}}{\varphi(t)^{\frac{p_2}{m_1}}} \\ & + \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{p_2}{p_1}} \left(\int_t^\infty \left(\int_0^s U_2(y)^{\frac{m_2}{p_2}} w_2(y)\,dy\right)^{\frac{m_2}{p_1-m_2}} U_2(s)^{\frac{m_2}{p_2}} w_2(s) U_1(s)^{-\frac{m_2}{p_1-m_2}}\,ds\right)^{\frac{p_2(p_1-m_2)}{p_1 m_2}}}{\varphi(t)^{\frac{p_2}{m_1}}}. \end{aligned}$$

By Theorem 2.2(b), again applied to the same array of parameters as in the case (c-i), we get

$$D_6 \approx \sup_{t\in(0,\infty)} \frac{U_1(t)^{\frac{p_2}{p_1}} \left(\int_t^\infty \left(\int_t^s U_2(y)^{\frac{p_1}{p_1-p_2}} U_1(y)^{-\frac{p_1}{p_1-p_2}} u_1(y)\,dy\right)^{\frac{m_2(p_1-p_2)}{p_2(p_1-m_2)}} \left(\int_s^\infty w_2(y)\,dy\right)^{\frac{m_2}{p_1-m_2}} w_2(t)\,dt\right)^{\frac{p_2(p_1-m_2)}{p_1 m_2}}}{\varphi(t)^{\frac{p_2}{m_1}}}.$$

Case (d). Assume that $m_1 > p_2$ and $p_1 > p_2$. Here we shall distinguish three subcases.

Case (d-i). Assume that $p_2 < m_1 < p_1 \le m_2$.

By Theorem 2.9(a), applied to

$$p = \frac{m_2}{m_2 - p_2},\ q = \frac{p_1}{p_1 - p_2},\ m = \frac{m_1}{m_1 - p_2},\ \varrho = U_2,\ w = \sigma,\ u = U_1^{-\frac{p_1}{p_1-p_2}} u_1 \text{ and } v = w_2^{-\frac{p_2}{m_2-p_2}},$$

we get

$$\begin{aligned} D_7 \approx & \sup_{t\in(0,\infty)} \left(\int_0^t \sigma(s)\,ds\right)^{\frac{m_1-p_2}{m_1}} U_1(t)^{-\frac{p_2}{p_1}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds\right)^{\frac{p_2}{m_2}} \\ & + \sup_{t\in(0,\infty)} \left(\int_t^\infty U_1(s)^{-\frac{m_1 p_2}{p_1(m_1-p_2)}} \sigma(s)\,ds\right)^{\frac{m_1-p_2}{m_1}} \left(\int_0^t U_2(s)^{\frac{m_2}{p_2}} w_2(s)\,ds\right)^{\frac{p_2}{m_2}} \end{aligned}$$

By Theorem 2.8(a), applied to

$$p = \frac{m_2}{m_2 - p_2},\ q = \frac{p_1}{p_1 - p_2},\ m = \frac{m_1}{m_1 - p_2},\ w = \sigma,\ u = U_2^{\frac{p_1}{p_1-p_2}} U_1^{-\frac{p_1}{p_1-p_2}} u_1,\ v = w_2^{-\frac{p_2}{m_2-p_2}},$$

we get

$$D_8 \approx \sup_{t\in(0,\infty)} \left(\int_t^\infty w_2(s)\,ds\right)^{\frac{p_2}{m_2}} \left(\int_0^t \left(\int_s^t U_1(y)^{-\frac{p_1}{p_1-p_2}} U_2(y)^{\frac{p_1}{p_1-p_2}} u_1(y)\,dy\right)^{\frac{m_1(p_1-p_2)}{p_1(m_1-p_2)}} \sigma(s)\,ds\right)^{\frac{m_1-p_2}{m_1}}.$$

The assertion of the theorem in the case (d-i) now follows by the usual combination of estimates and taking the roots.

Case (d-ii). Assume that $p_2 < m_1 \le m_2 < p_1$.

We follow the same line of argument as in case (d-i), applying this time Theorem 2.9(b) to evaluate D_7 and Theorem 2.8(b) to evaluate D_8.

Case (d-iii). Assume that $p_2 < m_2 < m_1 < p_1$.

Again, the assertion can be proved as in the case (d-i). This time we use Theorem 2.9(c) for D_7 and Theorem 2.8(c) for D_8.

The proof of the theorem is complete. □

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