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FRACTIONAL POWERS OF THE STOKES OPERATOR WITH BOUNDARY CONDITIONS INVOLVING THE PRESSURE

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ABSTRACT. Since the pioneer work of Leray [21] and Hopf [15], Stokes and Navier-Stokes problems have been often studied with Dirichlet boundary condition. Nevertheless, in the opinion of engineers and physicists such a condition is not always realistic in industrial and applied problems of origin. Thus arises naturally the need to carry out a mathematical analysis of these systems with different boundary conditions, which best represent the underlying fluid dynamic phenomenology. Based on the study of the complex and fractional powers of the Stokes operator with pressure boundary condition, we carry out a systematic treatment of the Stokes problem with the corresponding boundary conditions in L^p-spaces.

Keyword. Stokes Problem, pressure boundary conditions, Complex and fractional powers of operators.
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CONTENTS

1. INTRODUCTION AND MOTIVATION

Stokes and Navier Stokes problems have often played a fundamental role in engineering, fluid mechanics and mathematical physics. Original contributions trace back to H. Navier [22] and G. Stokes [27] in the mid-nineteenth century. With three seminal articles of J. Leray published in 1930's (cf. [21]), this issue received proper attention from mathematicians, who began a systematic investigation of these systems of partial differential equations.

The cases studied by Leray, and archetype for the further development of mathematical analysis of the equations of fluid dynamics in the twentieth century, are the Stokes and Navier-Stokes Problem with the Dirichlet boundary condition also called non-slip boundary condition on the boundary. However, as stated by J. Serrin [26] this condition is not always realistic since in some cases it doesn't reflect the behaviour of the fluid on or near the boundary. It is then important, in many real life situation, to study these problems with other boundary conditions that describe better the behaviour of the fluid on the wall. An overview of the state of the art in the nineties concerning the Stokes and Navier-Stokes problem subject

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to several nonstandard boundary conditions can be found in [12] where the author also discussed time-dependent flows and gave methods for starting these flows in the context of numerical simulations.

Based on the study of the complex and the fractional powers of the Stokes operator we carry out a systematic treatment of the Stokes problem in a bounded domain Ω of $\mathbb{R}^3$ with boundary condition involving the pressure. It should be mentioned that there is no physical justification for prescribing only a pressure boundary condition on the boundary and it must be completed by adding some boundary condition involving the velocity. In this paper we will consider the tangential part of the velocity on the boundary together with the pressure :

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, \qquad \pi = 0 \qquad \text{on} \quad \Gamma \times (0, T). \tag{1.1}$$

In [7] the incompressible Stokes and Navier-Stokes problems with a large class of non-standard boundary conditions were studied, in particular the boundary condition (1.1) and inhomogeneous pressure boundary conditions. The pressure Poisson equation equipped with a Neumann pressure boundary condition is studied in [13] where the authors formulated a hypothesis that the Navier-Stokes problem and the pressure Poisson equation would be equivalent. Conca et al. studied in [9, 10] the stationary Stokes and Navier-Stokes systems, with boundary conditions involving the pressure in a bounded three-dimensional domain and got existence and uniqueness results. In [14] the authors reviewed the mathematical formulations for unbounded domains for a class of problems that involve the prescription of pressure drops or net flux conditions. They also emphasis the advantage of pressure boundary conditions for modeling and numerical simulations. Pressure boundary conditions were also discussed in [18] where the author considered the initial Navier-Stokes problem in a class of space-time domains with a dynamic pressure on a part of the boundary. A study of incompressible Newtonian fluids in a finite number of pipes with impermeable walls can be found in [19]. The non-stationary Navier-Stokes problem subject to the boundary condition (1.1) in a part of the boundary and in a bounded domain of $\mathbb{R}^2$ was studied in [20] and a local existence and uniqueness results were proved. The Stokes operator subject to the pressure boundary condition (1.1) in a bounded domain Ω with a boundary Γ of class C^3 was studied in [17, Chapter 3]. The author proved using abstract L^p theory of parabolic evolution equation that for all $1 < p < \infty$ with $p \neq \frac{3}{2}$ and $p \neq 3$ and under some compatibility conditions the Stokes operator with pressure boundary conditions has the property of maximal L^p regularity on finite time interval. He deduced from this maximal L^p regularity that the Stokes operator is sectorial and generates a bounded analytic semigroup.

The authors [5, 6] proved that, when the boundary Γ is not connected, the Stokes operator with the above mentioned boundary condition has a non trivial kernel included in all the $\boldsymbol{L}^p$ spaces for $p \in (1, \infty)$. They also showed that this kernel may be characterized as follows

$$\boldsymbol{K}_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} = 0,\ \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}. \tag{1.2}$$

Thanks to [5, Proposition 3.18] and to [6, Corollary 4.2] we know that the space $\boldsymbol{K}_N(\Omega)$ is independent of p, is of finite dimension and is spanned by the functions ∇q_i^N, $i = 1 \ldots, I$, where q_i^N is the unique solution in $W^{2,p}(\Omega)$ of the problem

$$\begin{cases} -\Delta q_i^N = 0 \qquad \text{in } \Omega, \\ q_i^N|_{\Gamma_0} = 0 \quad \text{and} \quad q_i^N|_{\Gamma_k} = \text{constant}, \ \ 1 \leq k \leq I, \\ \big\langle \partial_n q_i^N,\, 1 \big\rangle_{\Gamma_k} = \delta_{i\,k}, \ \ 1 \leq k \leq I, \quad \text{and} \quad \big\langle \partial_n q_i^N,\, 1 \big\rangle_{\Gamma_0} = -1, \end{cases} \tag{1.3}$$

where Γ_i, $0 \leq i \leq I$, are the connected component of Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \backslash \overline{\Omega}$.

More recently, the author together with C. Amrouche, N. Seloula [4] proved the analyticity of the Stokes semigroup with the pressure boundary conditions (1.1) in L^p-spaces for all $1 < p < \infty$. The proof is based on a deep study of the following complex resolvent of the Stokes operator

$$\lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \qquad \mathrm{div} \boldsymbol{u} = 0 \quad \text{in } \Omega$$

with the boundary condition (1.1). This results was exploited later in [4] to treat the time dependent problem using a semigroup approach.

The outline of the paper will be as follows. In Section 2 we give the functional framework and some preliminary results at the basis of our proofs. We also define the Stokes operators with pressure boundary conditions (respectively with flux boundary conditions) and recall some of their properties. Section 3 is devoted to the study of the pure imaginary and fractional powers of the Stokes operator with the above mentioned boundary conditions. Then, in Section 4, we solve the Stokes problem and the Stokes problem with flux under different assumptions on the initial data $\boldsymbol{u}_0$ and the function $\boldsymbol{f}$ and we deduce a maximal regularity result to these problems.

2. Preliminary results

In this section we review some of the basic notations, definitions and functional framework which are essential in our work.

2.1. **Sectorial operator and maximal regularity.**

2.1.1. *Some properties of sectorial and non-negative operators.* Let X denotes a Banach space and $\mathcal{A} : D(\mathcal{A}) \subset X \mapsto X$ is a closed linear densely defined operator with domain $D(\mathcal{A})$ and range $R(\mathcal{A})$.

Definition 2.1 (Non-negative operator)**.** *An operator $\mathcal{A}$ is said to be a non-negative operator if its resolvent set contains all negative real numbers and*

$$\sup_{t>0} t \, \|(t\, I + \mathcal{A})^{-1}\|_{\mathcal{L}(X)} < \infty.$$

For a non-negative operator $\mathcal{A}$ it is possible to define its complex power $\mathcal{A}^z$ for every $z \in \mathbb{C}$ as a densely defined closed linear operator in the closed subspace $X_{\mathcal{A}} = \overline{D(\mathcal{A})} \cap \overline{R(\mathcal{A})}$ in X. Observe that, if both $D(\mathcal{A})$ and $R(\mathcal{A})$ are dense in X, then $X_{\mathcal{A}} = X$. We refer to [16, 28] for the definition and some relevant properties of the complex power of a non-negative operator.

For a non-negative bounded operator whose inverse $\mathcal{A}^{-1}$ exists and is bounded (*i.e.* $0 \in \rho(\mathcal{A})$), the complex power $\mathcal{A}^z$ can be defined for all $z \in \mathbb{C}$ by the means of the Dunford integral ([29]):

$$\mathcal{A}^z f = \frac{1}{2\,\pi\, i} \int_{\Gamma_\theta} (-\lambda)^z \, (\lambda\, I + \mathcal{A})^{-1} f \mathrm{d}\, \lambda,$$

where Γ_θ runs in the resolvent set of $-\mathcal{A}$ from $\infty\, e^{i(\theta-\pi)}$ to zero and from zero to $\infty\, e^{i(\pi-\theta)}$, $0 < \theta < \pi/2$ in $\mathbb{C}$ avoiding the non negative real axis. The branch of $(-\lambda)^z$ is taken so that $\mathrm{Re}((-\lambda)^z) > 0$ for $\lambda < 0$. It is proved by Triebel [28, Theorem 1.15.2, part (e)] that when the operator $\mathcal{A}$ is of bounded inverse, the complex powers $\mathcal{A}^z$ for $\mathrm{Re}\, z > 0$ are isomorphisms from $D(\mathcal{A}^z)$ to $X_{\mathcal{A}}$. Furthermore the boundedness of $\mathcal{A}^{is}$, $s \in \mathbb{R}$ allows us to determine the domain of definition of $D(\mathcal{A}^\alpha)$, for complex number α satisfying $\mathrm{Re}\, \alpha > 0$ using complex interpolation (see [28, Theorem 1.15.3]).

Theorem 2.2. *Let $\mathcal{A}$ be a non-negative operator with bounded inverse. We suppose that there exist two positive numbers ε and C such that $\mathcal{A}^{is}$ is bounded for $-\varepsilon \leq s \leq \varepsilon$ and $\|\mathcal{A}^{is}\|_{\mathcal{L}(X_A)} \leq C$. If α is a complex number such that $0 < \operatorname{Re}\alpha < \infty$ and $0 < \theta < 1$ then*

$$[X\,,\, D(\mathcal{A}^{\alpha})]_{\theta} \;=\; D(\mathcal{A}^{\alpha\theta}).$$

Let $0 \leq \theta < \pi/2$ and let Σ_θ be the sector

$$\Sigma_\theta \;=\; \Big\{\lambda \in \mathbb{C}^*; \;\; |\arg\lambda| < \pi - \theta\Big\}.$$

Definition 2.3 (Sectorial operator)**.** *A linear densely defined operator $\mathcal{A}$ is sectorial if there exits constant $M > 0$ and $0 \leq \theta < \pi/2$ such that*

$$\forall\,\lambda \in \Sigma_\theta, \qquad \|R(\lambda,\,\mathcal{A})\|_{\mathcal{L}(X)} \;\leq\; \frac{M}{|\lambda|},$$

where $R(\lambda,\,\mathcal{A}) \,=\, (\lambda\,I \,-\, \mathcal{A})^{-1}$.

Theorem 2.4. *An operator $\mathcal{A}$ generates a bounded analytic semigroup if and only if it is sectorial in the sense of Definition 2.3.*

When $-\mathcal{A}$ is the infinitesimal generator of a bounded analytic semigroup $(e^{-t\mathcal{A}})_{t\geq 0}$, the following proposition holds (see [16, Theorem 12.1] for instance)

Proposition 2.5. *Let $-\mathcal{A}$ be the infinitesimal generator of a bounded analytic semigroup $(e^{-t\mathcal{A}})_{t\geq 0}$. For any complex number α such that $\operatorname{Re}\alpha > 0$ one has*

$$\forall t > 0, \qquad \|\mathcal{A}^{\alpha} e^{-t\mathcal{A}}\|_{\mathcal{L}(X)} \leq C\,t^{-\operatorname{Re}\alpha}.$$

Remark 2.6 (interpolation extrapolation scale)**.** Let $-\mathcal{A}$ be the infinitesimal generator of a bounded analytic semigroup on a Banach space $X_0 := (X_0, \|\cdot\|_0)$. Assume for simplicity that $\mathcal{A}$ has a bounded inverse on X_0. Set $X_1 := D(\mathcal{A})$, then $u \mapsto \|u\|_1 := \|\mathcal{A}u\|_0$ defines a norm on X_1 which is equivalent to the graph norm. Furthermore $X_1 := (D(\mathcal{A}), \|\cdot\|_1)$ is a Banach space such that

$$X_1 \hookrightarrow^d X_0,$$

where $\hookrightarrow^d$ denotes continuous and dense embedding.

This allows us to introduce a superspace $X_{-1} := (X_{-1}, \|\cdot\|_{-1})$ of X_0 by choosing for X_{-1} a completion of X_0 in the norm $u \mapsto \|u\|_{-1} := \|\mathcal{A}^{-1}u\|_0$. Then $\mathcal{A}_0 := \mathcal{A}$ extends continuously to an operator $\mathcal{A}_{-1}$ that generates a bounded analytic semigroup on X_{-1}. If $\mathcal{A}$ is not invertible, it suffices to replace $\mathcal{A}$ by $\lambda I + \mathcal{A}$ with $\lambda > 0$.

Let $\theta \in (0, 1)$ and $(\cdot,\cdot)_\theta$ the interpolation functor of exponent θ. Set $X_\theta := (X_0, X_1)_\theta$ and $X_{\theta-1} := (X_{-1}, X_0)_\theta$. This defines a scale of Banach spaces

$$X_1 \hookrightarrow^d X_\alpha \hookrightarrow^d X_\beta \hookrightarrow^d X_{-1}, \qquad -1 < \beta < \alpha < 1.$$

Denote by $\mathcal{A}_{\alpha-1}$ with $\alpha \in [0, 1]$ the $X_{\alpha-1}-$realization of the $\mathcal{A}_{-1}$. Thus $\mathcal{A}_{\alpha-1}$ generates a bounded analytic semigroup on $X_{\alpha-1}$. These extensions are natural in the sense that $e^{-t\mathcal{A}_{\alpha-1}}$ is the restriction to $X_{\alpha-1}$ of $e^{-t\mathcal{A}_{\beta-1}}$ for $0 \leq \beta \leq \alpha \leq 1$.

If $\mathcal{A}$ belongs to $\mathcal{BIP}(X_0)$ the class of operator with bounded imaginary powers on X_0, then the interpolation-extrapolation scale $(X_0, \mathcal{A})_{\alpha\in\mathbb{R}}$ is equivalent to the fraction power scale generated by $(X_0, \mathcal{A})$ (see [2, Chapter V, Theorem 1.5.4]). If in addition $\mathcal{A} \in \mathcal{BIP}(X_0, M, \vartheta)$ the class of operator with uniformly bounded imaginary powers on X_0 with constant M and angle ϑ, then $\mathcal{A}_\alpha \in \mathcal{BIP}(X_\alpha, M, \vartheta)$ (see [2, Chapter V, Theorem 1.5.5]).

2.1.2. *Maximal regularity.* In this subsection we recall some basic definitions on maximal regularity. This will be used in the sequel in order to obtain strong solution to the time dependent Stokes problem.

Consider the following abstract Cauchy problem

$$u_t = \mathcal{A}u + f(t), \ \ t > 0, \qquad u(0) = 0, \tag{2.1}$$

where $\mathcal{A}$ is a closed linear densely defined unbounded operator in a Banach space X with domain $D(\mathcal{A})$ and $f : \mathbb{R}^+ \to X$ is a given function.

Definition 2.7. *Let $1 < p < \infty$. The abstract Cauchy problem (2.1) is said to enjoy the maximal L^p-regularity property on $[0, T)$, $0 < T \le \infty$, if for every $f \in L^p([0, T); X)$, there exists a unique u satisfying (2.1) almost everywhere and such that $u \in W^{1,p}([0, T); X) \cap L^p([0, T); D(\mathcal{A}))$.*

It is well known that that if (2.1) has a maximal L^p-regularity then $\mathcal{A}$ generates an analytic semigroup in X. Unfortunately the converse is not true unless X is a Hilbert space. Weis obtained in [30] a necessary and sufficient condition for maximal L^p-regularity when X is a UMD Banach space in terms of $\mathcal{R}$-sectoriality of the operator $\mathcal{A}$ (cf. [23] for the definition and further properties of $\mathcal{R}$-sectorial operator). Using a perturbation argument Giga and Sohr [11] proved the existence of a unique solution to (2.1) satisfying the maximal regularity property, when the pure imaginary powers of $\mathcal{A}$ are uniformly bounded in X and satisfy

$$\forall s \in \mathbb{R}, \quad \|\mathcal{A}^{is}\|_{\mathcal{L}(X)} \le M\, e^{\theta\,|s|}, \tag{2.2}$$

for some constant $M > 0$ and $0 < \theta < \pi/2$.

Let us recall the definition of a UMD Banach space (or ζ-convex Banach space).

Definition 2.8 (UMD Banach space)**.** *By a UMD Banach space (or ζ-convex Banach space) we mean a Banach space X where the truncated Hilbert transform*

$$(H_\varepsilon f)(t) = \frac{1}{\pi}\int_{|\tau|>\varepsilon} \frac{f(t-\tau)}{\tau}\,\mathrm{d}\tau$$

converges as $\varepsilon \to 0$, for almost all $t \in \mathbb{R}$, for all $f \in L^s(\mathbb{R}; X)$, and there is a constant $C = C(s, X)$ independent of f such that

$$\|Hf\|_{L^s(\mathbb{R},\, X)} \le C\, \|f\|_{L^s(\mathbb{R};\, X)},$$

where $(Hf)(t) = \lim_{\varepsilon\to 0}(H_\varepsilon f)(t)$.

For further readings on UMD Banach spaces we refer to [8]. We recall the following proposition.

Proposition 2.9. *Every closed subspace of a ζ-convex space is also ζ-convex. In addition for any open domain Ω of $\mathbb{R}^3$ the space $L^p(\Omega)$ is ζ-convex if and only if $1 < p < \infty$.*

The theorem of Giga and Sohr is the following (see [11, Theorem 2.1])

Theorem 2.10. *Let X be a UMD Banach space. Assume that $0 < T \le \infty$, $1 < p < \infty$ and that $-\mathcal{A}$ satisfy (2.2) for some $M \ge 0$, $0 \le \theta < \pi/2$. Then for every $f \in L^p(0, T; X)$ there exists a unique solution $\boldsymbol{u}$ of the Cauchy-Problem (2.1) satisfying the properties:*

$$u \in L^p(0, T_0;\, D(\mathcal{A})), \quad T_0 \le T \ \textit{ if } T < \infty \ \text{ and } \ T_0 < T \ \textit{ if } T = \infty,$$

$$\frac{\partial u}{\partial t} \in L^p(0, T;\, X)$$

and

$$\int_0^T \Big\|\frac{\partial u}{\partial t}\Big\|_X^p \,\mathrm{d}\,t + \int_0^T \|\mathcal{A}u(t)\|_X^p \,\mathrm{d}\,t \le C \int_0^T \|f(t)\|_X^p \,\mathrm{d}\,t$$

with $C = C(p, \theta, K, X)$ independent of f and T.

2.2. **Functional framework and Stokes operator.**

2.2.1. *Functional framework.* In what follows, if we do not state otherwise, Ω will be considered as an open bounded domain of $\mathbb{R}^3$ with a boundary Γ of class $C^{2,1}$. We do not assume that Ω is simply-connected neither that its boundary Γ is connected but we suppose that there exist J connected open surfaces Σ_j, $1 \leq j \leq J$, called "cuts", contained in Ω, such that each surface Σ_j is an open subset of a smooth manifold, the boundary of Σ_j is contained in Γ. The intersection $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ is empty for $i \neq j$ and finally the open set $\Omega^\circ = \Omega \backslash \cup_{j=1}^J \Sigma_j$ is simply connected and pseudo-$C^{1,1}$. We denote by Γ_i, $0 \leq i \leq I$, the connected component of Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \backslash \overline{\Omega}$. We also fix a smooth open set ϑ with a connected boundary (a ball, for instance), such that $\overline{\Omega}$ is contained in ϑ, and we denote by Ω_i, $0 \leq i \leq I$, the connected component of $\vartheta \backslash \overline{\Omega}$ with boundary Γ_i ($\Gamma_0 \cup \partial\vartheta$ for $i = 0$). A unit normal vector to the boundary can be defined almost everywhere it will be denoted by $\boldsymbol{n}$. The generic point in Ω is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$.

Vector fields, matrix fields and their corresponding spaces defined on Ω will be denoted by bold character. In all this paper, if we do not state otherwise C denotes a constant that may differ from one inequality to another. Let us introduce some functional spaces. Let $\boldsymbol{\mathcal{D}}(\Omega)$ be the set of infinitely differentiable functions with compact support in Ω and let $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ the restriction to Ω of infinitely differentiable functions with compact support in $\mathbb{R}^3$. Let $\boldsymbol{\mathcal{D}}_\sigma(\Omega) := \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega);\ \mathrm{div}\,\boldsymbol{\varphi} = 0 \text{ in } \Omega\}$ and $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) := \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\overline{\Omega});\ \mathrm{div}\,\boldsymbol{\varphi} = 0 \text{ in } \Omega\}$. The space $\boldsymbol{L}^p(\Omega)$ denote the usual vector valued $\boldsymbol{L}^p$-space over Ω. We define also the following spaces equipped with the graph norm:

$\boldsymbol{L}^p_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega\}$,
$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}$,
$\boldsymbol{H}^p(\mathbf{curl},\Omega) = \big\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\big\}$,
$\boldsymbol{H}^p(\mathrm{div},\Omega) = \big\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} \in L^p(\Omega)\big\}$,
$\boldsymbol{X}^p(\Omega) = \boldsymbol{H}^p(\mathbf{curl},\Omega) \cap \boldsymbol{H}^p(\mathrm{div},\Omega)$,
$\boldsymbol{H}^p_0(\mathbf{curl},\Omega) = \big\{\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\big\}$,
$\boldsymbol{H}^p_0(\mathrm{div},\Omega) = \big\{\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\big\}$,
$\boldsymbol{X}^p_N(\Omega) = \big\{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\big\}$,
$\boldsymbol{X}^p_{N,\sigma}(\Omega) = \Big\{\boldsymbol{v} \in \boldsymbol{X}^p_N(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega\Big\}$,

It is known that $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\boldsymbol{L}^p_\sigma(\Omega)$ and that $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. It is also known that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\mathbf{curl},\Omega)$, $\boldsymbol{H}^p(\mathrm{div},\Omega)$ and $\boldsymbol{X}^p(\Omega)$ and that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}^p_0(\mathbf{curl},\Omega)$ and in $\boldsymbol{H}^p_0(\mathrm{div},\Omega)$. We recall that for all function $\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl},\Omega)$ (respectively $\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div},\Omega)$), the tangential trace $\boldsymbol{v}\times\boldsymbol{n}_{|\Gamma}$ (respectively the normal trace $\boldsymbol{v}\cdot\boldsymbol{n}_{|\Gamma}$) exists and belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)$ (respectively to $W^{-1/p,p}(\Gamma)$), (see [6, Section 2] and [5, Proposition 2.3]). The spaces $[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'$ and $[\boldsymbol{H}^p_0(\mathrm{div},\Omega)]'$ represent the dual spaces of $\boldsymbol{H}^p_0(\mathbf{curl},\Omega)$ and $\boldsymbol{H}^p_0(\mathrm{div},\Omega)$ respectively.

Thanks to [6, Theorem 3.2], we know that the space $\boldsymbol{X}^p_N(\Omega)$ is continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$ and

$$\forall\, \boldsymbol{v} \in \boldsymbol{X}^p_N(\Omega), \quad \|\boldsymbol{v}\|_{\boldsymbol{X}^p_N(\Omega)} \simeq \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

Notice that the space $\boldsymbol{X}^p_{N,\sigma}(\Omega)$ is equal to the following space

$$\boldsymbol{W}^{1,p}_{\sigma,N}(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\ \mathrm{div}\,\boldsymbol{u} = 0 \text{ in } \Omega \text{ and } \boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}. \tag{2.3}$$

Finally consider the space

$$\boldsymbol{V}^p_{\sigma,N}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^p_{\sigma,N}(\Omega);\ \langle \boldsymbol{v}\cdot\boldsymbol{n}\,,\,1\rangle_{\Gamma_i} = 0,\ 1 \leq i \leq I\}. \tag{2.4}$$

Thanks to [6, Corollary 3.2] we know that, for all $\boldsymbol{v} \in \boldsymbol{V}^p_{\sigma,N}(\Omega)$ the norm of $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ is equivalent to the norm $\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$.

We recall that the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\,\boldsymbol{v} = \mathbf{grad}\,(\operatorname{div}\boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}.$$

The dual space $[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'$ of $\boldsymbol{H}^p_0(\mathbf{curl},\Omega)$ can be characterized as follows (cf. [25, Proposition 1.0.5]): A distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'$ if and only if there exist functions functions $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{\xi} \in \boldsymbol{L}^{p'}(\Omega)$, $(\frac{1}{p} + \frac{1}{p'} = 1)$, such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\,\boldsymbol{\xi}$. Moreover one has

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'} = \inf_{\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{curl}\,\boldsymbol{\xi}} \max\,(\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)}, \|\boldsymbol{\xi}\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

We end this section with the following remark (see [6])

Remark 2.11 (Vector potential). **(i)** A vector field $\boldsymbol{u}$ in $\boldsymbol{H}^p(\operatorname{div},\Omega)$ satisfies

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in}\,\Omega \quad \text{and} \qquad \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_{\mathrm{i}}} = 0, \quad 0 \le \mathrm{i} \le \mathrm{I}, \tag{2.5}$$

if and only if there exists a vector potential $\boldsymbol{\psi}_0$ *in* $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}_0.$$

Moreover, we can choose $\boldsymbol{\psi_0}$ such that $\operatorname{div}\boldsymbol{\psi}_0 = 0$ and we have the estimate

$$\|\boldsymbol{\psi}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\,\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)},$$

where $C > 0$ depends only on p and Ω.

(ii) A function $\boldsymbol{u}$ in $\boldsymbol{H}^p(\operatorname{div},\Omega)$ satisfies (2.5) if and only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \operatorname{div}\boldsymbol{\psi} = 0 \text{ in } \Omega,\\ &\boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma,\\ &\langle \boldsymbol{\psi}\cdot\boldsymbol{n},\, \mathit{1}\rangle_{\Sigma_j} = 0,\ 1 \le j \le J. \end{aligned}$$

This function $\boldsymbol{\psi}$ is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

(iii) A function $\boldsymbol{u}$ in $\boldsymbol{H}^p(\operatorname{div},\Omega)$ satisfies:

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in}\,\Omega,\ \ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on}\ \Gamma \quad \text{and} \quad \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_{\mathrm{j}}} = 0, \quad 1 \le \mathrm{j} \le \mathrm{J}, \tag{2.6}$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \operatorname{div}\boldsymbol{\psi} = 0 \text{ in } \Omega,\\ &\boldsymbol{\psi}\times\boldsymbol{n} = \boldsymbol{0} \qquad \text{on } \Gamma,\\ &\langle \boldsymbol{\psi}\cdot\boldsymbol{n},\, \mathit{1}\rangle_{\Gamma_i} = 0,\ \text{for any}\ 1 \le i \le I. \end{aligned}$$

This function $\boldsymbol{\psi}$ is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

2.2.2. *Stokes problem with pressure boundary condition.* Throughout this work we will denote by A_p the Stokes operator with pressure boundary conditions (1.1) on $\boldsymbol{L}^p_\sigma(\Omega)$, $1 < p < \infty$. As described in [4, section 2], due to boundary conditions (1.1) the pressure can be decoupled from the Stokes problem using the following Dirichlet problem

$$\Delta\pi = \operatorname{div}\boldsymbol{f} \quad \text{in } \Omega, \qquad \pi = 0 \quad \text{on } \Gamma. \tag{2.7}$$

Thus the Stokes operator A_p is a linear closed densely defined operator A_p : $\mathbf{D}(A_p) \subset \boldsymbol{L}^p_\sigma(\Omega) \longmapsto \boldsymbol{L}^p_\sigma(\Omega)$, where

$$\mathbf{D}(A_p) = \left\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\times\boldsymbol{n} \ = \ \boldsymbol{0} \text{ on } \Gamma\right\}, \tag{2.8}$$

$$\forall\, \boldsymbol{u} \in \mathbf{D}(A_p), \quad A_p\,\boldsymbol{u} = -\Delta\boldsymbol{u} \quad \text{in } \Omega.$$

Furthermore the Stokes operator A_p satisfy the following natural weak formulation: for all $\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\sigma,N}(\Omega)$ and for all $\boldsymbol{v} \in \boldsymbol{W}^{1,p'}_{\sigma,N}(\Omega)$

$$\langle A_p\,\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{(\boldsymbol{W}^{1,p'}_{\sigma,N}(\Omega))'\times\boldsymbol{W}^{1,p'}_{\sigma,N}(\Omega)} \ = \ \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x.$$

Finally we note that when the boundary Γ is not connected, the range $R(A_p)$ of the Stokes operator is not dense in $\boldsymbol{L}^p_\sigma(\Omega)$, since in this case the dimension of the kernel $\boldsymbol{K}_N(\Omega)$ of the Stokes operator A_p on $\boldsymbol{L}^p_\sigma(\Omega)$ is finite and greater than or equal to 1. This can be seen using that

$$\overline{R(A_p)} = [Ker(A_p)^*]^\perp = [Ker(A_{p'})]^\perp.$$

Consider the complex resolvent of the Stokes operator with the pressure boundary condition

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} \,+\, \nabla\pi = \boldsymbol{f}, & \operatorname{div}\boldsymbol{u} = 0 \qquad \text{in} \quad \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}, & \pi = 0 \qquad \text{on} \quad \Gamma. \end{cases} \tag{2.9}$$

We can show that the operator $-A_p$ generates a bounded analytic semigroup on $\boldsymbol{W}^{1,p}_{\sigma,N}(\Omega)$ (given by (2.3)).

Proposition 2.12. *Let* $\lambda \in \mathbb{C}^*$ *such that* $\mathrm{Re}\lambda \geq 0$ *and* $\boldsymbol{f} \in \boldsymbol{W}^{1,p}_{\sigma,N}(\Omega)$*. The unique solution* $\boldsymbol{u}$ *to Problem* (2.9) *satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \ \leq \ \frac{C(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{W}^{1,p}(\Omega)}, \tag{2.10}$$

with a constant C *independent of* λ *and* $\boldsymbol{f}$*.*

Proof. Thanks to [4, Theorems 3.5-3.7] we know that Problem (2.9) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ satisfying estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \ \leq \ \frac{C(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.11}$$

Set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$ in Ω and observe that $\boldsymbol{z}$ satisfies

$$\begin{cases} \lambda\boldsymbol{z} - \Delta\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{f}, & \operatorname{div}\boldsymbol{z} = 0 \qquad \text{in} \quad \Omega, \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{0} \qquad \text{on} \quad \Gamma. \end{cases}$$

Using the result of [3, Theorems 4.10-4.11], we deduce that $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} \ \leq \ \frac{C(\Omega,p)}{|\lambda|}\,\|\mathbf{curl}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.12}$$

Thus Putting together (2.11) and (2.12) we obtain (2.10). □

Consider the space:

$$[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma := \{\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]';\ \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega\}.$$

We define the operator B_p as the extension of the Stokes operator A_p to the space $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$ by:

$$B_p\ :\ \mathbf{D}(B_p) \subset [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma \longmapsto [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$$

and

$$\forall\, \boldsymbol{u} \in \mathbf{D}(B_p), \qquad B_p\, \boldsymbol{u} = -\Delta \boldsymbol{u} \qquad \text{in } \Omega.$$

The domain of B_p is given by

$$\mathbf{D}(B_p) = \big\{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta \boldsymbol{u} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]',\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\big\}. \quad (2.13)$$

The operator $-B_p$ generates a bounded analytic semigroup on $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$ for all $1 < p < \infty$ (see [4, Theorem 3.13]).

2.2.3. *Stokes operator with flux boundary conditions.* This section is devoted to the study of the Stokes operator A_p in the the orthogonal of its kernel. This is quite natural since as we have already mentioned, when the boundary Γ is not connected, the operator A_p has a non trivial finite dimensional kernel $\boldsymbol{K}_N(\Omega)$.

First notice that for any function $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$, the condition

$$\forall\ \boldsymbol{v} \in \boldsymbol{K}_N(\Omega), \qquad \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\, x = 0, \quad (2.14)$$

is equivalent to the condition:

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Gamma_i} = 0, \quad 0 \leq i \leq I. \quad (2.15)$$

Indeed, observe that a function $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$ satisfying the condition (2.15), satifies the compatibility condition (2.14). Conversly, let $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$ satisfying the compatibility condition (2.14) then $\boldsymbol{u}$ can be written as

$$\boldsymbol{u} = \boldsymbol{\psi} - \sum_{i=1}^{I} \langle \boldsymbol{\psi} \cdot \boldsymbol{n}\,;\, 1\rangle_{\Gamma_i} \nabla q_i^N,$$

for some function $\boldsymbol{\psi} \in \boldsymbol{L}^p_\sigma(\Omega)$. Since for all $0 \leq i \leq I$, ∇q_N^i satisfies (1.3) we can check that forl all $1 \leq k \leq I$

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}\,;\, 1\rangle_{\Gamma_k} = 0.$$

It turns out that, for a function $\boldsymbol{u} \in L^p_\sigma(\Omega)$, to be in the orthogonal of $\boldsymbol{K}_N(\Omega)$ is equivalent to the condition (2.15). It is then equivalent for our purpose to consider the operator A_p with the supplementary flux condition (2.15).

Consider the space

$$\boldsymbol{\mathcal{X}}_p = \{\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega);\ \forall\, \boldsymbol{v} \in \boldsymbol{K}_N(\Omega),\ \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\, x = 0\}. \quad (2.16)$$

Notice that

$$\forall\, 1 < p < \infty, \quad \boldsymbol{L}^p_\sigma(\Omega) = \boldsymbol{K}_N(\Omega) \oplus \boldsymbol{\mathcal{X}}_p \quad \text{and} \quad (\boldsymbol{X}_p)' = \boldsymbol{X}_{p'}. \quad (2.17)$$

The operator $A_{|\boldsymbol{\mathcal{X}}_p}$ which is the restriction of A_p to $\boldsymbol{\mathcal{X}}_p$ is a well defined operator with domain

$$\mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p}) = \big\{\boldsymbol{u} \in \mathbf{D}(A_p);\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I\big\}.$$

Proposition 2.13. *The operator $A_{|\boldsymbol{\mathcal{X}}_p}$ is a densely defined operator.*

Proof. Let $\boldsymbol{f} \in \boldsymbol{\mathcal{X}}_p$, thanks to Remark 2.11, (see also [6, Theorem 4.7]), we know that there exists $\boldsymbol{\xi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\xi}$ in Ω, $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω and $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ. Since $\boldsymbol{\mathcal{D}_\sigma}(\Omega)$ is dense in the space

$$\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega) = \big\{\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega);\ \operatorname{div}\boldsymbol{\xi} = 0 \,\text{in}\,\Omega,\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0 \ \text{on}\ \Gamma\big\},$$

there exists a sequence $(\boldsymbol{\xi}_k)_k$ in $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ such that $\boldsymbol{\xi}_k \longrightarrow \boldsymbol{\xi}$ in $\boldsymbol{W}^{1,p}(\Omega)$. Thus $\mathbf{curl}\,\boldsymbol{\xi}_k \longrightarrow \mathbf{curl}\,\boldsymbol{\xi}$ in $\boldsymbol{L}^p(\Omega)$. As a result, $\mathbf{curl}\,\boldsymbol{\xi}_k \longrightarrow \boldsymbol{f}$ in $\boldsymbol{\mathcal{X}}_p$ and $(\mathbf{curl}\,\boldsymbol{\xi}_k)_k \subset \mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})$, this ends the proof. □

The operator $-A_{|\boldsymbol{\mathcal{X}}_p}$ generates a bounded analytic semigroup on $\boldsymbol{\mathcal{X}}_p$ for $1 < p < \infty$. In addition, it is invertible with bounded inverse (*i.e.* $0 \in \rho(A_{|\boldsymbol{\mathcal{X}}_p})$). Notice that, obviously, A_p coincides with $A_{|\boldsymbol{\mathcal{X}}_p}$ in the case where Γ is connected. We denote $e^{-tA_{|\boldsymbol{\mathcal{X}}_p}}$ this semigroup. Obviously $e^{-tA_{|\boldsymbol{\mathcal{X}}_p}} = e^{-tA_p}_{|\boldsymbol{\mathcal{X}}_p}$, where e^{-tA_p} is the semigroup generated by $-A_p$ on $\boldsymbol{L}^p_\sigma(\Omega)$.

Remark 2.14. Let $\boldsymbol{f} \in \boldsymbol{V}^p_{\sigma,N}(\Omega)$ and let $\boldsymbol{u} \in \mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})$ such that $(\lambda\, I + A_{|\boldsymbol{\mathcal{X}}_p})^{-1}\boldsymbol{f} = \boldsymbol{u}$ in Ω. This means that $\boldsymbol{u}$ is a solution of the problem

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, & \operatorname{div}\boldsymbol{u} = 0 & \text{in} & \Omega,\\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}, & \pi = 0 & \text{on} & \Gamma,\\ & \langle\boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, & 1 \le i \le I. & \end{cases}$$

Set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$ in Ω. Then $\boldsymbol{z}$ satisfies

$$\begin{cases} \lambda\boldsymbol{z} - \Delta\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{f}, & \operatorname{div}\boldsymbol{z} = 0 & \text{in} & \Omega,\\ \boldsymbol{z}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{0} & \text{on} & \Gamma,\\ & \langle\boldsymbol{z}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0, & 1 \le j \le J. & \end{cases}$$

Consider the space

$$\boldsymbol{\mathcal{Y}}_p = \Big\{\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'_\sigma;\ \forall\,\boldsymbol{v} \in \boldsymbol{K}^{p'}_N(\Omega),\ \ \langle\boldsymbol{f},\,\boldsymbol{v}\rangle_{[\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'\times[\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'} = 0\Big\}.$$

The operator $B_{|\boldsymbol{\mathcal{Y}}_p}$ which is the restriction of B_p to the space $\boldsymbol{\mathcal{Y}}_p$ is a well defined operator with domain

$$\mathbf{D}(B_{|\boldsymbol{\mathcal{Y}}_p}) = \big\{\boldsymbol{u} \in \mathbf{D}(B_p);\ \langle\boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, \quad 1 \le i \le I\big\}.$$

Proposition 2.15. *The operator $B_{|\boldsymbol{\mathcal{Y}}_p}$ is a densely defined operator in Y_p.*

Proof. Let $\boldsymbol{f} \in \boldsymbol{\mathcal{Y}}_p$, thanks to [6, Theorem 4.7] we know that there exists $\boldsymbol{\xi}$ in $\boldsymbol{L}^p(\Omega)$ such that $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\xi}$ in Ω, $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω and $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ. Since $\boldsymbol{\mathcal{D}_\sigma}(\Omega)$ is dense in the space $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ there exists a sequence $(\boldsymbol{\xi}_k)_k$ in $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ such that $\boldsymbol{\xi}_k \longrightarrow \boldsymbol{\xi}$ in $\boldsymbol{L}^p(\Omega)$. Thus $\mathbf{curl}\,\boldsymbol{\xi}_k \longrightarrow \mathbf{curl}\,\boldsymbol{\xi}$ in $\boldsymbol{L}^p(\Omega)$. As a result, $\mathbf{curl}\,\boldsymbol{\xi}_k \longrightarrow \boldsymbol{f}$ in $\boldsymbol{\mathcal{Y}}_p$ and $(\mathbf{curl}\,\boldsymbol{\xi}_k)_k \subset \mathbf{D}(\mathcal{B}_{p|\boldsymbol{\mathcal{Y}}_p})$, this ends the proof. □

Remark 2.16. Putting together Propositions 2.13 and 2.15 we deduce the density of $\boldsymbol{\mathcal{X}}_p$ in $\boldsymbol{\mathcal{Y}}_p$.

Thanks to [4] the operator $-B_{|\boldsymbol{\mathcal{Y}}_p}$ generates a bounded analytic semigroup on $\boldsymbol{\mathcal{Y}}_p$ for $1 < p < \infty$.

3. Fractional powers of the Stokes operator

In this section we give some properties and key estimates on the complex and fractional powers of the Stokes operator. The existence of these powers is guaranteed since the Stokes operator with the boundary conditions (1.1) generates a bounded analytic semigroup on $\boldsymbol{L}^p_\sigma(\Omega)$, $1 < p < \infty$. Nevertheless since the Stokes operator A_p is not invertible with bounded inverse, its complex powers can not be expressed through an integral formula and it is not easy in general to compute calculus inequality involving these powers. To avoid this difficulty we prove the desired results for the operator $A_{|\boldsymbol{\mathcal{X}}_p}$. The knowledge of the complex powers plays an important role in the associated parabolic problem and will be used later to prove sharp regularity results for the non-homogeneous Stokes problem.

Theorem 3.1. *For all $s \in \mathbb{R}$, the operators $(A_{|\boldsymbol{\mathcal{X}}_p})^{i\,s}$ are uniformly bounded in $\boldsymbol{\mathcal{X}}_p$ and there exists an angle $0 < \theta_0 < \pi/2$ and a constant $M > 0$ such that for all $s \in \mathbb{R}$ we have:*

$$\|(A_{|\boldsymbol{\mathcal{X}}_p})^{i\,s}\|_{\mathcal{L}(\boldsymbol{\mathcal{X}}_p)} \,\leq\, M\, e^{|s|\,\theta_0}. \tag{3.1}$$

Proof. Since the operator $A_{|\boldsymbol{\mathcal{X}}_p}$ is a non-negative bounded operator with bounded inverse in $\boldsymbol{\mathcal{X}}_p$ its complex powers can be expressed through the following Dunford integral formula (cf. [16]):

$$(A_{|\boldsymbol{\mathcal{X}}_p})^{z} \,=\, \frac{1}{2\,\pi\,i}\int_{\Gamma_{\theta_0}} (-\lambda)^z\,(\lambda\, I + A_{|\boldsymbol{\mathcal{X}}_p})^{-1}\,\mathrm{d}\,\lambda, \tag{3.2}$$

where

$$\Gamma_{\theta_0} \,=\, \{\rho\, e^{i\,(\pi-\theta_0)};\ 0\leq\rho\leq\infty\} \,\cup\, \{-\,\rho\, e^{i\,(\theta_0-\pi)};\ -\infty\leq\rho\leq 0\}.$$

First let $\boldsymbol{f} \in \boldsymbol{V}^p_{\sigma,N}(\Omega)$ and let $\boldsymbol{u} \in \mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})$ such that $(\lambda\, I + A_{|\boldsymbol{\mathcal{X}}_p})^{-1}\boldsymbol{f} = \boldsymbol{u}$ in Ω. Applying the **curl** operator to (3.2) and using Remark 2.14 yield:

$$\begin{aligned}
\mathbf{curl}[(A_{|\boldsymbol{\mathcal{X}}_p})^{z}\boldsymbol{f}] \;&=\; \frac{1}{2\,\pi\,i}\int_{\Gamma_{\theta_0}} (-\lambda)^z\,\mathbf{curl}[(\lambda\, I + A_{|\boldsymbol{\mathcal{X}}_p})^{-1}\boldsymbol{f}]\,\mathrm{d}\,\lambda\\
&=\; \frac{1}{2\,\pi\,i}\int_{\Gamma_{\theta_0}} (-\lambda)^z\,\mathbf{curl}\,\boldsymbol{u}\,\mathrm{d}\,\lambda\\
&=\; \frac{1}{2\,\pi\,i}\int_{\Gamma_{\theta_0}} (-\lambda)^z\,(\lambda\, I + A_{NT})^{-1}\mathbf{curl}\,\boldsymbol{f}\,\mathrm{d}\,\lambda\\
&=\; (A_{NT})^z\mathbf{curl}\,\boldsymbol{f}.
\end{aligned}$$

We have denote by A_{NT} the Stokes operator with Navier-type boundary conditions:

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma.$$

In particular for all $s \in \mathbb{R}$

$$\mathbf{curl}[(A_{|\boldsymbol{\mathcal{X}}_p})^{i\,s}\boldsymbol{f}] = (A_{NT})^{i\,s}\mathbf{curl}\boldsymbol{f}.$$

Using [3, Theorem 6.6], we deduce that there exist constants $M > 0$ and $0 < \theta_0 < \pi/2$ such that

$$\begin{aligned}
\|\mathbf{curl}[(A_{|\boldsymbol{\mathcal{X}}_p})^{i\,s}\boldsymbol{f}]\|_{\boldsymbol{L}^p(\Omega)} \;&=\; \|(A_{NT})^{i\,s}\mathbf{curl}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\\
&\leq\; M\, e^{|s|\theta_0}\|\mathbf{curl}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}$$

Using [6, Corollary 3.2], we get

$$\|(A_{|\boldsymbol{\mathcal{X}}_p})^{i\,s}\boldsymbol{f}\|_{\boldsymbol{W}^{1,p}(\Omega)} \,\simeq\, \|\mathbf{curl}[(A_{|\boldsymbol{\mathcal{X}}_p})^{i\,s}\boldsymbol{f}]\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, M\, e^{|s|\theta_0}\|\boldsymbol{f}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

Thus $A_{|\boldsymbol{\mathcal{X}}_p} \in \mathcal{BIP}(\boldsymbol{V}^p_{\sigma,N}(\Omega), M, \theta_0)$ and satisfies

$$\|(A_{|\boldsymbol{\mathcal{X}}_p})^{i\,s}\|_{\mathcal{L}(\boldsymbol{V}^p_{\sigma,N}(\Omega))} \leq M\, e^{|s|\theta_0},$$

with $M > 0$ and $0 < \theta_0 < \pi/2$.

Taking the adjoint we obtain using [2, Lemma 1.4.11] that

$$[A_{|\boldsymbol{\mathcal{X}}_p}]^* \in \mathcal{BIP}([\boldsymbol{V}^p_{\sigma,N}(\Omega)]^*, M, \theta_0)$$

and satisfies

$$\|([A_{|\boldsymbol{\mathcal{X}}_p}]^*)^{i\,s}\|_{\mathcal{L}([\boldsymbol{V}^p_{\sigma,N}(\Omega)]^*)} \leq M\, e^{|s|\theta_0},$$

with $M > 0$ and $0 < \theta_0 < \pi/2$.

Using that $[A_{|\boldsymbol{\mathcal{X}}_p}]^* = A_{|\boldsymbol{\mathcal{X}}_{p'}}$ we deduce that $A_{|\boldsymbol{\mathcal{X}}_p} \in \mathcal{BIP}([\boldsymbol{V}^{p'}_{\sigma,N}(\Omega)]^*, M, \theta_0)$ for all $1 < p < \infty$.

Interpolating we obtain that $\Big[[\boldsymbol{V}^{p'}_{\sigma,N}(\Omega)]^*\,,\, \boldsymbol{V}^{p}_{\sigma,N}(\Omega)\Big]_{1/2} = \boldsymbol{\mathcal{X}}_p$. Thus $A_{|\boldsymbol{\mathcal{X}}_p} \in \mathcal{BIP}(\boldsymbol{\mathcal{X}}_p, M, \theta_0)$ and estimate (3.1) follows directly. □

In the following proposition we prove that the pure imaginary powers of the operators $B_{|\boldsymbol{\mathcal{Y}}_p}$ are uniformly bounded on $\boldsymbol{\mathcal{Y}}_p$. We recall that the operators B_p given by (2.2.2) is the extensions of the Stokes operator to the space $[\boldsymbol{H}^{p'}_0(\mathbf{curl}, \Omega)]'_\sigma$

Proposition 3.2. *There exists $0 < \theta_0 < \pi/2$ and a constant $C > 0$ such that for all $s \in \mathbb{R}$*

$$\|(B_{|\boldsymbol{\mathcal{Y}}_p})^{i\,s}\|_{\mathcal{L}(\boldsymbol{\mathcal{Y}}_p)} \,\leq\, C\, e^{|s|\,\theta_0}. \tag{3.3}$$

Proof. Let $\boldsymbol{f} \in \boldsymbol{\mathcal{X}}_p$. Notice that

$$\begin{aligned} \|(B_{|\boldsymbol{\mathcal{Y}}_p})^{i\,s}\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'} = \|(A_{|\boldsymbol{\mathcal{X}}_p})^{i\,s}\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'} &\leq \|(A_{|\boldsymbol{\mathcal{X}}_p})^{i\,s}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \\ &\leq C\, e^{|s|\,\theta_0}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \\ &= C\, e^{|s|\,\theta_0}\|\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'}. \end{aligned}$$

Next, using the density of $\boldsymbol{\mathcal{X}}_p$ in $\boldsymbol{\mathcal{Y}}_p$ (see Remark 2.16) estimate (3.3) follows directly. □

The boundedness of the pure imaginary powers of the Stokes operator $A_{|\boldsymbol{\mathcal{X}}_p}$ allows us to characterise the domains of its fractional powers through an interpolation argument and also to obtain an embedding of Sobolev type for these domains. Since the operator $A_{|\boldsymbol{\mathcal{X}}_p}$ has a bounded inverse, for all $\alpha \in \mathbb{C}^*$, the operator $(A_{|\boldsymbol{\mathcal{X}}_p})^\alpha$ is an isomorphism from $\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^\alpha)$ to $\boldsymbol{\mathcal{X}}_p$ and for all $\alpha \in \mathbb{R}$, the map $\boldsymbol{v} \longmapsto \|(A_{|\boldsymbol{\mathcal{X}}_p})^\alpha\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$ is a norm on $\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^\alpha)$ (cf. [28, Theorem 1.15.2, part (e)]).

Theorem 3.3. *For all $1 < p < \infty$, $\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}}) = \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$ with equivalent norms. Furthermore, for every $\boldsymbol{v} \in \mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}})$, the norm $\|(A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$ is a norm on $\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}})$ which is equivalent to the norm $\|\mathbf{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$.*

Proof. First observe that

$$\begin{aligned} \mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}}) &= \big[\boldsymbol{\mathcal{X}}_p\,;\,\mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})\big]_{1/2} \\ &\subset \big[\boldsymbol{L}^p(\Omega)\,;\,\boldsymbol{W}^{2,p}(\Omega)\big]_{1/2} \,=\, \boldsymbol{W}^{1,p}(\Omega), \end{aligned}$$

with continuous embedding.

Next let $\boldsymbol{u} \in \mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}})$ and let $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ a sequence in $\mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})$ that converges to $\boldsymbol{u}$ in $\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}})$. This is true, since $\mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})$ is dense in $\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}})$. As a result, $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges to $\boldsymbol{u}$ in $\boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{u}_k \times \boldsymbol{n} \longrightarrow \boldsymbol{u} \times \boldsymbol{n}$ in $\boldsymbol{W}^{1-1/p,p}(\Gamma)$ as $k \to \infty$. Since $(\boldsymbol{u}_k)_{k\in\mathbb{N}} \subset \mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})$, then for all $k \in \mathbb{N}$, $\boldsymbol{u}_k \times \boldsymbol{n} = \mathbf{0}$ on Γ and $\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ. In addition for all $1 \leq i \leq I$, $\langle \boldsymbol{u}_k \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}$ converges to $\langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}$ as $k \to \infty$. Thus for all $1 \leq i \leq I$, $\langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0$ and

$$\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}}) \hookrightarrow \boldsymbol{V}^p_{\sigma,N}(\Omega).$$

In order to prove the second inclusion we prove that for all $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,\tau}(\Omega)$

$$\|(A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

Indeed let $\boldsymbol{F} \in \boldsymbol{\mathcal{X}}_{p'}$ and let $\boldsymbol{v} \in \mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_{p'}})^{\frac{1}{2}})$ solution of

$$(A_{|\boldsymbol{\mathcal{X}}_{p'}})^{1/2}\boldsymbol{v} \,=\, \boldsymbol{F}. \tag{3.4}$$

As a result for all $\boldsymbol{u} \in \mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})$ we have

$$\begin{aligned} \|(A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &= \sup_{\boldsymbol{F}\in\boldsymbol{\mathcal{X}}_{p'},\, \boldsymbol{F}\neq\boldsymbol{0}} \frac{\left|\langle (A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}}\boldsymbol{u}\,,\, \boldsymbol{F}\rangle_{\boldsymbol{\mathcal{X}}_p\times\boldsymbol{\mathcal{X}}_{p'}}\right|}{\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}} \\ &= \sup_{\boldsymbol{F}\in\boldsymbol{\mathcal{X}}_{p'},\, \boldsymbol{F}\neq\boldsymbol{0}} \frac{\left|\langle (A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}}\boldsymbol{u}\,,\, (A_{|\boldsymbol{\mathcal{X}}_{p'}})^{\frac{1}{2}}\boldsymbol{v}\rangle_{\boldsymbol{\mathcal{X}}_p\times\boldsymbol{\mathcal{X}}_{p'}}\right|}{\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}}, \end{aligned}$$

where $\boldsymbol{v}$ is the unique solution of (3.4).

As a result,

$$\begin{aligned} \|(A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &= \sup_{\boldsymbol{v}\in\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_{p'}})^{1/2}),\, \boldsymbol{v}\neq\boldsymbol{0}} \frac{\left|\langle (A_{|\boldsymbol{\mathcal{X}}_p})\boldsymbol{u}\,,\, \boldsymbol{v}\rangle_{\boldsymbol{\mathcal{X}}_p\times\boldsymbol{\mathcal{X}}_{p'}}\right|}{\|(A_{|\boldsymbol{\mathcal{X}}_{p'}})^{\frac{1}{2}}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}} \\ &= \sup_{\boldsymbol{v}\in\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_{p'}})^{1/2}),\, \boldsymbol{v}\neq\boldsymbol{0}} \frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x\right|}{\|(A_{|\boldsymbol{\mathcal{X}}_{p'}})^{\frac{1}{2}}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}} \\ &\leq C(\Omega,p)\,\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \end{aligned} \tag{3.5}$$

Since $\mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})$ is dense in $\boldsymbol{V}^p_{\sigma,N}(\Omega)$ one gets inequality (3.5) for all $\boldsymbol{u} \in \boldsymbol{V}^p_{\sigma,N}(\Omega)$. Thus

$$\boldsymbol{V}^p_{\sigma,N}(\Omega) \hookrightarrow \mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^{\frac{1}{2}})$$

and the result is prove. □

The following Theorem shows an embedding of Sobolev type for the domains of fractional powers of the Stokes operator with flux boundary conditions. This embedding give us the $\boldsymbol{L}^p - \boldsymbol{L}^q$ estimates for the corresponding homogeneous problem.

Theorem 3.4. *for all $1 < p < \infty$ and for all $\alpha \in \mathbb{R}$ such that $0 < \alpha < 3/2p$ the following Sobolev embedding holds*

$$\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^\alpha) \hookrightarrow \boldsymbol{L}^q(\Omega), \qquad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{3}, \tag{3.6}$$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega,p)\,\|(A_{|\boldsymbol{\mathcal{X}}_p})^\alpha\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.7}$$

Proof. For $0 < \alpha < \min(1, 3/2p)$ the Sobolev embedding (3.6) is a consequence of [1, Theorem 7.57] since

$$\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^\alpha) = \left[\boldsymbol{\mathcal{X}}_p\,;\,\mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})\right]_\alpha \hookrightarrow \left[\boldsymbol{L}^p(\Omega)\,;\,\boldsymbol{W}^{2,p}(\Omega)\right]_\alpha = \boldsymbol{W}^{2(1-\alpha),p}(\Omega).$$

Next, to extend (3.6) to any real α such that $0 < \alpha < 3/2p$ we write $\alpha = k + \theta$, where k is a non negative integer and $0 < \theta < 1$. We set

$$\frac{1}{q_0} = \frac{1}{p} - \frac{2\theta}{3} \qquad \text{and} \qquad \frac{1}{q_j} = \frac{1}{q_0} - \frac{2j}{3}, \quad j = 0,1,....,k.$$

It is clear that $\frac{1}{q_j} = \frac{1}{q_{j-1}} - \frac{2}{3}$ and that $q_k = q$. Moreover, by assumptions on p and α we have for $j = 0,1,....,k$, $\theta + j < 3/2p$. It follows that

$$\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^\theta) \hookrightarrow \boldsymbol{L}^{q_0}(\Omega)$$

and for all $1 \leq j \leq k$

$$\mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_{q_{j-1}}}) \hookrightarrow \boldsymbol{L}^{q_j}(\Omega).$$

For all $\boldsymbol{u} \in \mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^\infty) = \cap_{m\in\mathbb{N}}\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^m)$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq \|A_{|\boldsymbol{\mathcal{X}}_{q_{k-1}}}\boldsymbol{u}\|_{\boldsymbol{L}^{q_{k-1}}(\Omega)} \leq ... \leq \|(A_{|\boldsymbol{\mathcal{X}}_{q_0}})^k\boldsymbol{u}\|_{\boldsymbol{L}^{q_0}(\Omega)} \leq \|(A_{|\boldsymbol{\mathcal{X}}_p})^\alpha\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.8}$$

By density of $\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^\infty)$ in $\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^\alpha)$ on gets the Sobolev embeddings (3.6) and the estimate (3.7) follows directly. □

4. Application to the Stokes problem

This section is devoted to time dependent Stokes Problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0,T), \\ \qquad\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \quad \Omega. \end{cases} \tag{4.1}$$

Of course the main goal is to analyze Problem (4.1) with the pressure boundary condition (1.1). The starting point is an existence and uniqueness result using semigroup theory. As described above due the boundary condition (1.1), the pressure can be decoupled from the Stokes Problem (4.1) using the Dirichlet Problem (2.7). As a result our work is reduced to study the following vectorial Laplace like problem under a free-divergence condition and the boundary conditions $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ. Indeed taking the divergence of the first equation in (4.1), the pressure is a solution of the following Dirichlet problem

$$\Delta \pi = \operatorname{div} \boldsymbol{f} \quad \text{in } \Omega \times (0,T), \qquad \pi = 0 \quad \text{on } \Gamma \times (0,T).$$

Assuming that $\operatorname{div} \boldsymbol{f} = 0$, π is equal to zero and our work is reduced to study

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \end{cases} \tag{4.2}$$

4.1. **Homogeneous Problem.** Suppose that $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$ and $\boldsymbol{f} = \boldsymbol{0}$. The analyticity of the semigroup generated by $-A_p$ gives a unique solution u to (4.2) satisfying

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{L}^p_\sigma(\Omega)) \cap C(]0, +\infty[, \mathbf{D}(A_p)) \cap C^1(]0, +\infty[, \boldsymbol{L}^p_\sigma(\Omega)),$$
$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(A_p^\ell)), \qquad \forall\, k,\, \ell \in \mathbb{N}.$$

Furthermore the initial data $\boldsymbol{u}_0$ can be written uniquely in the form:

$$\boldsymbol{u}_0 = \boldsymbol{w}_0 + \widetilde{\boldsymbol{u}}_0,$$

where $\boldsymbol{w}_0 \in \boldsymbol{K}_N(\Omega)$, $\widetilde{\boldsymbol{u}}_0 \in \boldsymbol{\mathcal{X}}_p$ and

$$\boldsymbol{w}_0 = \sum_{i=1}^{I} \langle \boldsymbol{u}_0 \cdot \boldsymbol{n}\,,\, 1 \rangle_{\Gamma_i} \nabla q_i^N,$$

$$\widetilde{\boldsymbol{u}}_0 = \boldsymbol{u}_0 - \sum_{i=1}^{I} \langle \boldsymbol{u}_0 \cdot \boldsymbol{n}\,,\, 1 \rangle_{\Gamma_i} \nabla q_i^N.$$

It follows that the unique solution to Problem (4.2) with $\boldsymbol{f} = \boldsymbol{0}$ can be written in the form

$$\boldsymbol{u}(t) = \boldsymbol{w}_0 + e^{-tA_p} \widetilde{\boldsymbol{u}}_0, \tag{4.3}$$

where e^{-tA_p} is the Stokes semigroup on $\boldsymbol{L}^p_\sigma(\Omega)$. The unique solution $\boldsymbol{u}$ of Problem satisfies also

$$\forall\, t > 0, \quad \forall\, 0 < \alpha < 1, \quad \boldsymbol{u}(t) \in \mathbf{D}(A_p) \hookrightarrow \mathbf{D}(A_p^\alpha).$$

Consider first the case where the initial data $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{X}}_p$. We have the following theorem

Theorem 4.1. *Suppose that $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{X}}_p$ and $\boldsymbol{f} = \boldsymbol{0}$. Let $\boldsymbol{u}$ be the unique solution to Problem (4.1)-(1.1). Then for all $q \in [p, \infty)$, and for all integers $m, n \in \mathbb{N}$, such that $m + n > 0$, there exists constants $M > 0$ and $\mu > 0$, such that the following estimates holds:*

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq M\, e^{-\mu\, t}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.4}$$

$$\|\mathbf{curl}\, \boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq M\, e^{-\mu\, t}\, t^{-3/2(1/p-1/q)-1/2} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{4.5}$$

and

$$\Big\|\frac{\partial^m}{\partial t^m}\Delta^n \boldsymbol{u}(t)\Big\|_{\boldsymbol{L}^q(\Omega)} \leq M\, e^{-\mu\, t}\, t^{-(m+n)-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.6}$$

Proof. The estimates (4.4)–(4.6) follow for the cases where $q = p$ and $m = 1, n = 0$ or $m = 0, n = 0, 1, 2$ using [4, Theorem 4.2]. Suppose that $p \neq q$ and let $s \in \mathbb{R}$ such that $\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) < s < \frac{3}{2p}$. Set $\frac{1}{p_0} = \frac{1}{p} - \frac{2s}{3}$, it is clear that $p < q < p_0$.

Since for all $t > 0$, $\boldsymbol{u}(t) \in \mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^\infty)$, then using Corollary 3.4 we have $\boldsymbol{u}(t) \in \mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^s) \hookrightarrow \boldsymbol{L}^{p_0}(\Omega)$. Now set $\alpha = \frac{1/p-1/q}{1/p-1/p_0} \in]0,1[$, we can easily verify that $\frac{1}{q} = \frac{\alpha}{p_0} + \frac{1-\alpha}{p}$. Thus $\boldsymbol{u}(t) \in \boldsymbol{L}^q(\Omega)$ and there exists constants $M > 0$ and $\mu > 0$, such that

$$\begin{aligned}
\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} &\leq C\|\boldsymbol{u}(t)\|^{\alpha}_{\boldsymbol{L}^{p_0}(\Omega)}\|\boldsymbol{u}(t)\|^{1-\alpha}_{\boldsymbol{L}^p(\Omega)}\\
&\leq C\|(A_{|\boldsymbol{\mathcal{X}}_p})^s e^{-tA_p}\boldsymbol{u}_0\|^{\alpha}_{\boldsymbol{L}^p(\Omega)}\|e^{-tA_p}\boldsymbol{u}_0\|^{1-\alpha}_{\boldsymbol{L}^p(\Omega)}\\
&\leq C\, e^{-\mu t}t^{-\alpha s}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \qquad (4.7)\\
&= C\, e^{-\mu t}t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \qquad (4.8)
\end{aligned}$$

Estimate (4.7) follows from the fact that, (cf. [24, Chapter 2, Theorem 6.13, page 76]),

$$\|(A_{|\boldsymbol{\mathcal{X}}_p})^\alpha\, e^{-tA_p}\|_{\mathcal{L}(\boldsymbol{\mathcal{X}}_p)} \leq M\,\kappa_1(\Omega, p)\,\frac{e^{-\mu\, t}}{t^\alpha}.$$

Next, let $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{X}}_p \cap \boldsymbol{\mathcal{X}}_q$ then $\mathbf{curl}\,\boldsymbol{u}(t) \in \boldsymbol{L}^q(\Omega)$ and

$$\begin{aligned}
\|\mathbf{curl}\,\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C\,\|(A_{|\boldsymbol{\mathcal{X}}_q})^{\frac{1}{2}}\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} &= \|(A_{|\boldsymbol{\mathcal{X}}_q})^{\frac{1}{2}}\, e^{-\frac{t}{2}A_p}e^{-\frac{t}{2}A_p}\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\Omega)}\\
&\leq C\, e^{-\mu t}t^{-1/2}\,\|e^{-\frac{t}{2}A_p}\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\Omega)}\\
&\leq C\, e^{-\mu t}t^{-1/2}\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}$$

Now let $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{X}}_p$, using the density of $\boldsymbol{\mathcal{X}}_p \cap \boldsymbol{\mathcal{X}}_q$ in $\boldsymbol{\mathcal{X}}_p$ we know that there exists a sequence $(\boldsymbol{u}_{0_m})_{m\geq 0}$ in $\boldsymbol{\mathcal{X}}_p \cap \boldsymbol{\mathcal{X}}_q$ that converges to $\boldsymbol{u}_0$ in $\boldsymbol{\mathcal{X}}_p$. For all $m \in \mathbb{N}$ we set $\boldsymbol{u}_m(t) = e^{-tA_p}\boldsymbol{u}_{0_m}$, as a result the sequences $(\boldsymbol{u}_m(t))_{m\geq 0}$ and $(\mathbf{curl}\,\boldsymbol{u}_m(t))_{m\geq 0}$ converges to $\boldsymbol{u}(t)$ and $\mathbf{curl}\,\boldsymbol{u}(t)$ respectively in $\boldsymbol{L}^p(\Omega)$, where $u(t) = e^{-tA_p}\boldsymbol{u}_0$. On the other hand, for all $m, n \in \mathbb{N}$ one has

$$\|\mathbf{curl}(\boldsymbol{u}_n(t) - \boldsymbol{u}_m(t))\|_{\boldsymbol{L}^q(\Omega)} \leq C\, e^{-\mu t}t^{-1/2}\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_{0_n} - \boldsymbol{u}_{0_m}\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus $(\mathbf{curl}\,\boldsymbol{u}_m(t))_{m\geq 0}$ is a Cauchy sequence in $\boldsymbol{L}^q(\Omega)$ and converges to $\mathbf{curl}\,\boldsymbol{u}(t)$ in $\boldsymbol{L}^q(\Omega)$. This means that $\mathbf{curl}\,\boldsymbol{u}(t) \in \boldsymbol{L}^q(\Omega)$ and by passing to the limit as $m \to \infty$ one gets estimate (4.5).

Finally, using [4, Theorem 4.2] we have

$$\forall\, m, n \in \mathbb{N}, \quad \frac{\partial^m}{\partial t^m}\Delta^n \boldsymbol{u} \in C^\infty((0,\infty),\, \mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})).$$

Thus $\frac{\partial^m}{\partial t^m}\Delta^n\boldsymbol{u}(t)$ belongs to $\boldsymbol{L}^q(\Omega)$ and

$$\begin{aligned}
\Big\|\frac{\partial^m}{\partial t^m}\Delta^n \boldsymbol{u}(t)\Big\|_{\boldsymbol{L}^q(\Omega)} &= \|(A_{|\boldsymbol{\mathcal{X}}_p})^{(m+n)}\, e^{-tA_p}\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\Omega)}\\
&\leq C\, e^{-\mu t}t^{-(m+n)-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}$$

□

Next we consider the general case where $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$.

Theorem 4.2. *Let* $1 < p \leq q < \infty$, $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$ *and* $\boldsymbol{f} = \mathbf{0}$. *Let* $\boldsymbol{u}(t)$ *be the unique solution of Problem* (4.1) *with* (1.1). *For all integers* $m, n \in \mathbb{N}$, *such that* $m+n > 0$, *there exists constants* $M > 0$ *and* $\mu > 0$, *such that the following estimates holds:*

$$\|\boldsymbol{u}(t) - \boldsymbol{w}_0\|_{\boldsymbol{L}^q(\Omega)} \leq C\, e^{-\mu\, t}\, t^{-3/2(1/p-1/q)}\|\widetilde{\boldsymbol{u}}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.9}$$

$$\|\mathbf{curl}\,\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \,\leq\, M\,e^{-\mu t}\,t^{-3/2(1/p-1/q)-1/2}\|\widetilde{\boldsymbol{u}}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{4.10}$$

and

$$\Big\|\frac{\partial^m}{\partial t^m}\Delta^n\boldsymbol{u}(t)\Big\|_{\boldsymbol{L}^q(\Omega)} \,\leq\, M\,e^{-\mu t}\,t^{-(m+n)-3/2(1/p-1/q)}\|\widetilde{\boldsymbol{u}}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.11}$$

Proof. The case $p=q$ follows from [4, Theorem 4.2], so let us suppose that $p\neq q$. As described above we write $\boldsymbol{u}_0 = \boldsymbol{w}_0 + \widetilde{\boldsymbol{u}}_0$, with $\boldsymbol{w}_0\in\boldsymbol{K}_N(\Omega)$ and $\widetilde{\boldsymbol{u}}_0\in\boldsymbol{\mathcal{X}}_p$. It follows that the unique solution to Problem (4.2) can be written in the form (4.3) where $e^{-tA_p}\widetilde{\boldsymbol{u}}_0 \in C^k(]0,+\infty[,\,\mathbf{D}((A_{|\boldsymbol{\mathcal{X}}_p})^\ell))$, for all $k,\,\ell\in\mathbb{N}$ and satisfies (4.4)-(4.6).

The estimate (4.9) follows from (4.3) and (4.4).

Estimate (4.10) follows from (4.5) using that

$$\mathbf{curl}\,\boldsymbol{u}(t) = \mathbf{curl}\,\boldsymbol{w}_0 + \mathbf{curl}\,(e^{-tA_p}\widetilde{\boldsymbol{u}}_0) = \mathbf{curl}\,(e^{-tA_p}\widetilde{\boldsymbol{u}}_0).$$

Finally, for all $m,n\in\mathbb{N}$, such that $m+n>0$ we have

$$\frac{\partial^m}{\partial t^m}\Delta^n\boldsymbol{u}(t) \,=\, A_p^{m+n}\boldsymbol{w}_0 \,+\, A_p^{m+n}(e^{-tA_p}\widetilde{\boldsymbol{u}}_0) \,=\, (A_{|\boldsymbol{\mathcal{X}}_p})^{m+n}\,(e^{-tA_p}\widetilde{\boldsymbol{u}}_0).$$

As a result, using Theorem 4.1 one has estimate (4.11). □

4.2. **The non homogeneous problem.** Consider now the non-homogeneous case where $\boldsymbol{f}$ belongs to $L^q(0,T;\,\boldsymbol{L}^p_\sigma(\Omega))$ and $\boldsymbol{u}_0=\mathbf{0}$. It is known that the unique solution $\boldsymbol{u}$ of Problem (4.1) with the boundary conditions (1.1) exists and belongs to $C([0,T]\,;\,\boldsymbol{L}^p_\sigma(\Omega))$ for $T<\infty$. For such $\boldsymbol{f}$ the analyticity of the semigroup is not sufficient to obtain a solution $\boldsymbol{u}$ satisfying what is called the maximal L^p-L^q regularity property :

$$\boldsymbol{u}\in L^q(0,T;\,\boldsymbol{W}^{2,p}(\Omega)),\qquad \frac{\partial\boldsymbol{u}}{\partial t}\in L^q(0,T;\,\boldsymbol{L}^p_\sigma(\Omega)).$$

The following theorem gives the maximal L^p-L^q regularity property for the inhomogeneous Stokes problem (4.1) with pressure boundary conditions.

Theorem 4.3. *Let $T\in(0,\infty]$, $1<p,q<\infty$, $\boldsymbol{f}\in\boldsymbol{L}^q(0,T;\boldsymbol{L}^p(\Omega))$ and $\boldsymbol{u}_0=\mathbf{0}$. The Problem (4.1) with (1.1) has a unique solution $(\boldsymbol{u},\pi)$ such that*

$$\boldsymbol{u}\in L^q(0,T_0;\,\boldsymbol{W}^{2,p}(\Omega)),\;\; T_0\leq T \;\textit{ if } T<\infty \;\text{ and }\; T_0<T\;\textit{ if } T=\infty, \tag{4.12}$$

$$\pi\in L^q(0,T;\,W^{1,p}_0(\Omega)),\qquad \frac{\partial\boldsymbol{u}}{\partial t}\in L^q(0,T;\,\boldsymbol{L}^p(\Omega)) \tag{4.13}$$

and

$$\int_0^T\Big\|\frac{\partial\boldsymbol{u}}{\partial t}\Big\|^q_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,t + \int_0^T\|\Delta\boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,t + \int_0^T\|\pi(t)\|^q_{W^{1,p}(\Omega)/\mathbb{R}}\,\mathrm{d}\,t \\ \leq C(p,q,\Omega)\int_0^T\|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,t. \tag{4.14}$$

Proof. The proof is done in three steps.

(i) Let $\boldsymbol{f}\in\boldsymbol{L}^q(0,T;\boldsymbol{\mathcal{X}}_p)$. Since the pure imaginary powers of the operator are uniformly bounded on $\boldsymbol{\mathcal{X}}_p$, the Stokes problem (4.1) with (1.1) has a unique solution $(\boldsymbol{u},\pi)$ that satisfies the maximal regularity (4.12)-(4.13) (see [11, Theorem 2.1]).

(ii) For $\boldsymbol{f}\in\boldsymbol{L}^q(0,T;\boldsymbol{L}^p_\sigma(\Omega))$. We write $\boldsymbol{f}$ in the form, $\boldsymbol{f}=\boldsymbol{f}_1+\boldsymbol{f}_2$ where $\boldsymbol{f}_1\in L^q(0,T;\,\boldsymbol{\mathcal{X}}_p)$ and $\boldsymbol{f}_2\in L^q(0,T;\,\boldsymbol{K}_N(\Omega))$. Thus the solution $\boldsymbol{u}$ to problem (4.2) is equal to $\boldsymbol{u}=\boldsymbol{u}_1+\boldsymbol{u}_2$, where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ satisfy

$$\begin{cases} \frac{\partial\boldsymbol{u}_1}{\partial t}-\Delta\boldsymbol{u}_1=\boldsymbol{f}_1, & \operatorname{div}\boldsymbol{u}_1=0 \quad \text{in} \quad \Omega\times(0,T),\\ & \boldsymbol{u}_1\times\boldsymbol{n}=\mathbf{0} \quad \text{on} \quad \Gamma\times(0,T),\\ & \boldsymbol{u}_1(0)=\mathbf{0} \quad \text{in} \quad \Omega \end{cases} \tag{4.15}$$

and

$$\begin{cases} \frac{\partial \boldsymbol{u}_2}{\partial t} - \Delta \boldsymbol{u}_2 = \boldsymbol{f}_2, & \operatorname{div} \boldsymbol{u}_2 = 0 \quad \text{in} \quad \Omega \times (0,T), \\ & \boldsymbol{u}_2 \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}_2(0) = \boldsymbol{0} \quad \text{in} \quad \Omega \end{cases} \tag{4.16}$$

respectively.

From the previous step we know that $\boldsymbol{u}_1$ satisfies

$$\boldsymbol{u}_1 \in L^q(0,T_0;\, \mathbf{D}(A_{|\boldsymbol{\mathcal{X}}_p})) \cap W^{1,q}(0,T;\, \boldsymbol{\mathcal{X}}_p), \tag{4.17}$$

$$\int_0^T \left\| \frac{\partial \boldsymbol{u}_1}{\partial t} \right\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t + \int_0^T \|\Delta \boldsymbol{u}_1(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t \le C(p,q,\Omega) \int_0^T \|\boldsymbol{f}_1(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t. \tag{4.18}$$

Set $\boldsymbol{z}_2 = \mathbf{curl}\boldsymbol{u}_2$. Then $\boldsymbol{z}_2$ is a solution of the problem

$$\begin{cases} \frac{\partial \boldsymbol{z}_2}{\partial t} - \Delta \boldsymbol{z}_2 = \boldsymbol{0}, & \operatorname{div} \boldsymbol{z}_2 = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{z}_2 \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{z}_2 \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma \times (0,T), \\ & \boldsymbol{z}_2(0) = \boldsymbol{0} \quad \text{in} \quad \Omega. \end{cases}$$

Using [3, Theorem 7.1] we know that $\mathbf{curl}\boldsymbol{u}_2 = \boldsymbol{z}_2 = \boldsymbol{0}$ in Ω. This means that $\boldsymbol{u}_2 \in \boldsymbol{K}_\tau(\Omega)$ and then

$$\forall t \ge 0, \quad \frac{\partial \boldsymbol{u}_2(t)}{\partial t} = \boldsymbol{f}_2(t) \quad \text{in } \Omega. \tag{4.19}$$

As a result $\boldsymbol{u}_2$ satisfies

$$\boldsymbol{u}_2 \in L^q(0,T_0;\, \mathbf{D}(A_p)) \cap W^{1,q}(0,T;\, \boldsymbol{L}^p_\sigma(\Omega)) \tag{4.20}$$

and

$$\int_0^T \left\| \frac{\partial \boldsymbol{u}_2}{\partial t} \right\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t = \int_0^T \|\boldsymbol{f}_2(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t \le C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)}. \tag{4.21}$$

Thus putting together (4.17)-(4.18) and (4.20)-(4.21) we deduce our result.

(iii) Let $\boldsymbol{f} \in \boldsymbol{L}^q(0,T;\boldsymbol{L}^p(\Omega))$. For almost all $t > 0$ the problem

$$\Delta \pi = \operatorname{div} \boldsymbol{f} \quad \text{in } \Omega \times (0,T), \qquad \pi = 0 \quad \text{on } \Gamma \times (0,T)$$

has a unique solution $\pi \in L^q(0,T;W^{1,p}_0(\Omega))$ satisfying

$$\|\pi\|_{W^{1,p}(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Notice that $\boldsymbol{f} - \nabla\pi \in L^q(0,T;\, \boldsymbol{L}^p_\sigma(\Omega))$. Thus from the previous step we know that the problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{f} - \nabla\pi, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, & \pi = 0 \quad \text{on} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \end{cases}$$

has a unique solution $(\boldsymbol{u}, \pi)$ satisfying (4.12)-(4.14). □

In order to get weak solution to the inhomogeneous Stokes problem with the boundary conditions (1.1) satisfying the maximal $L^p - L^q$ regularity we prove the ζ-convexity of $[\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'$.

Proposition 4.4. *Let $1 < p < \infty$, the dual space $[\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'$ is a ζ-convex Banach space.*

Proof. Let $\boldsymbol{f} \in L^s(\mathbb{R};\, [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]')$, then for almost all $t \in \mathbb{R}$, there exists $\boldsymbol{\psi}(t) \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{\xi}(t) \in \boldsymbol{L}^p(\Omega)$ such that $\boldsymbol{f}(t) = \boldsymbol{\psi}(t) + \mathbf{curl}\boldsymbol{\xi}(t)$. It is clear that $\boldsymbol{\psi} \in L^s(\mathbb{R};\, \boldsymbol{L}^p(\Omega))$ and $\boldsymbol{\xi} \in L^s(\mathbb{R};\boldsymbol{L}^p(\Omega))$. On the other hand we can easily verify that

$$(H_\varepsilon \boldsymbol{f})(t) = (H_\varepsilon \boldsymbol{\psi})(t) + \mathbf{curl}(H_\varepsilon \boldsymbol{\xi})(t).$$

Next since $\boldsymbol{L}^p(\Omega)$ is ζ-convex, then $(H_\varepsilon \boldsymbol{\psi})(t)$ (respectively $(H_\varepsilon \boldsymbol{\xi})(t)$) converges as $\varepsilon \to 0$ to $H\boldsymbol{\psi}(t)$ (respectively to $H\boldsymbol{\xi}(t)$). Moreover we have the estimates

$$\|H\boldsymbol{\psi}(t)\|_{L^s(\mathbb{R};\,\boldsymbol{L}^p(\Omega))} \leq C(s,\Omega,p)\,\|\boldsymbol{\psi}\|_{L^s(\mathbb{R};\,\boldsymbol{L}^p(\Omega))},$$

$$\|H\boldsymbol{\xi}(t)\|_{L^s(\mathbb{R};\,\boldsymbol{L}^p(\Omega))} \leq C(s,\Omega,p)\,\|\boldsymbol{\xi}\|_{L^s(\mathbb{R};\,\boldsymbol{L}^p(\Omega))}.$$

Thus $(H_\varepsilon \boldsymbol{f})(t)$ converges as $\varepsilon \to 0$ to $H\boldsymbol{f}(t) = H\boldsymbol{\psi}(t) + \mathbf{curl}\, H\boldsymbol{\xi}(t)$. Moreover we have the estimate

$$\|H\boldsymbol{f}(t)\|_{L^s(\mathbb{R};\,[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]')} \leq C(s,\Omega,p)\,\|\boldsymbol{f}\|_{L^s(\mathbb{R};\,[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]')},$$

which ends the proof. □

The following theorem gives weak solutions for problem (4.2) satisfying the maximal regularity.

Theorem 4.5. *Let* $1 < p, q < \infty$, $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]')$ *with* $0 < T \leq \infty$ *and* $\boldsymbol{u}_0 = \boldsymbol{0}$. *The Problem* (4.1) *with* (1.1) *has a unique solution* $(\boldsymbol{u}, \pi)$ *satisfying*

$$\boldsymbol{u} \in L^q(0,T_0;\; \boldsymbol{W}^{1,p}(\Omega)),\;\; T_0 \leq T \;\textit{ if }\; T < \infty \;\text{ and }\; T_0 < T \;\textit{ if }\; T = \infty, \tag{4.22}$$

$$\pi \in L^q(0,T;\; W_0^{1,p}(\Omega)), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]') \tag{4.23}$$

and

$$\int_0^T \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|^q_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]'}\,\mathrm{d}\,t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]'}\,\mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{W_0^{1,p}(\Omega)}\,\mathrm{d}\,t \leq C(p,q,\Omega)\Big(\int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'}\,\mathrm{d}\,t\Big). \tag{4.24}$$

Proof. Suppose first $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma)$. As in the proof of Theorem 4.3 we may write $\boldsymbol{f}$ as, $\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2$ where $\boldsymbol{f}_1 \in L^q(0,T;\boldsymbol{\mathcal{Y}}_p)$ and $\boldsymbol{f}_2 \in L^q(0,T;\,\boldsymbol{K}_N(\Omega))$. Proceeding as in the proof of Theorem 4.3 we deduce that the solution $\boldsymbol{u}$ to problem (4.1)–(1.1) is such that $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$, where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are weak solutions of (4.15) and (4.16) respectively and that $\boldsymbol{u}_2 \in \boldsymbol{K}_N(\Omega)$ for almost all $0 < t \leq T$. Using Proposition 3.2 we deduce that the solution $\boldsymbol{u}$ satisfies the maximal regularity (4.22)-(4.24).

Next, when the external force $\boldsymbol{f}$ is not a divergence free function, the pressure doesn't vanish in Ω and can be decoupled from the problem using a Dirichlet problem. Indeed π is solution of the following Dirichlet problem:

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} \quad \text{in}\,\Omega \qquad \text{and} \qquad \pi = 0 \quad \text{on}\,\Gamma. \tag{4.25}$$

Since $\boldsymbol{f}(t) \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'$ for all $0 < t < T$, then $\operatorname{div}\boldsymbol{f} \in W^{-1,p}(\Omega)$ and by standard L^p regularity, there exists a unique $\pi \in W_0^{1,p}(\Omega)$ solution of the Dirichlet Problem (4.25) satisfying for a.e. $t \in (0,T)$:

$$\|\pi(t)\|_{W_0^{1,p}(\Omega)} \leq C(\Omega)\|\boldsymbol{f}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'}.$$

Notice that $(\boldsymbol{f} - \nabla\pi) \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'_\sigma)$ and the Stokes problem (4.1) with the boundary condition (1.1) has a unique solution $(\boldsymbol{u},\pi)$ satisfying (4.22)-(4.24). □

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