WHY Y-C.C.

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Abstract. We outline a portfolio of novel iterable properties of c.c.c. and proper forcing notions and study its most important instantiations, Y-c.c. and Y-properness. These properties have interesting consequences for partition-type forcings and anticliques in open graphs. Using Neeman's side condition method it is possible to obtain PFA variations and prove consistency results for them.

1. Introduction

A recent work of Yorioka [15] implicitly contains the following definition.

Definition 1.1. A poset P satisfies *Y-c.c.* if for every countable elementary submodel $M \prec H_\theta$ containing P and every condition $q \in P$ there is a filter $F \in M$ on the completion $RO(P)$ such that $\{p \in RO(P) \cap M : p \geq q\} \subset F$.

We will show that this is a property intermediate between σ-centered and c.c.c. whose verification follows typical Δ-system arguments. Y-c.c. holds for many natural examples, such as Aronszajn tree specialization forcing (Corollary 3.3), gap specialization forcing (Corollary 3.4), Todorcevic's posets used for the resolution of Horn–Tarski problem (Corollary 3.6) and other partition style c.c.c. posets. Y-c.c. has a number of pleasing consequences, such as not adding random reals (Corollary 2.6) and branches into ω_1-trees (Corollary 2.9), preserving uncountable chromatic number of open graphs and not adding uncountable anticliques for them (Corollary 2.5). It is preserved under the finite support iteration (Theorem 6.2), which means that the forcing axiom for Y-c.c. posets can be forced via a Y-c.c. poset (Corollary 6.4). There is also a non-c.c.c. variant:

Definition 1.2. A poset P is *Y-proper* if for every countable elementary submodel $M \prec H_\theta$ containing P and every condition $p \in P \cap M$ there is $q \leq p$ (a *Y-master condition*) which is master for M and such that for every $r \leq q$ there is a filter $F \in M$ on the completion $RO(P)$ such that $\{s \in RO(P) \cap M : s \geq r\} \subset F$.

Again, a number of natural posets are Y-proper, such as the Laver forcing (Theorem 4.8), the ideal based forcings (Theorem 4.4), or the PID forcings (Theorem 4.6). Implications of Y-properness are similar to Y-c.c. Preservation of Y-properness under the countable support iteration is unclear, but still the Neeman method allows one to produce a Y-proper forcing which forces the forcing axiom YPFA for Y-proper posets–Theorem 6.6. YPFA does not imply OCA.

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Both Y-c.c. and Y-properness are instances of a wide-ranging portfolio of iterable forcing properties quite distinct from those considered so far in the literature. The general scheme for these properties results from replacing the requirement that the sets $F \subset RO(P)$ be filters with some other regularity demand on F. Section 5 provides the fairly involved general iteration theorems for the resulting concepts.

The present paper owes a great deal to previous work of Yorioka. In particular, most of the examples were known to Yorioka, with proofs that inspired our proofs. Our contribution consists of isolating abstract, axiomatically useful classes of partial orders, and proving general preservation and forcing axiom theorems about them.

We use set theoretic notational standard of [3]. The forcing notation follows the western convention: $p \geq q$ means that q is stronger or more informative than p. If P is a (separative) partial order then $RO(P)$ denotes the completion of P, the unique complete Boolean algebra in which P is dense. If B is a complete Boolean algebra and ϕ is a statement of its forcing language, $\|\phi\|$ denotes the Boolean value of ϕ in B; that is, $\|\phi\|$ is the supremum of all $b \in B$ such that $b \Vdash \phi$. In all arguments, θ denotes a large enough regular cardinal, and H_θ the collection of all sets whose transitive closure has size less than θ. OCA denotes the Open Coloring Axiom [10, Section 8], the statement that every open graph on a second countable space is either countably chromatic or else contains an uncountable clique.

2. Y-c.c.: consequences

With a novel property such as Y-c.c., it appears necessary to explore its basic consequences.

Theorem 2.1. *σ-centeredness implies Y-c.c. implies c.c.c.*

Proof. Suppose first that a poset P is σ-centered, and fix a covering $\{F_n : n \in \omega\}$ of $RO(P)$ by countably many filters. Let $M \prec H_\theta$ be a countable elementary submodel containing P, F_n for every $n \in \omega$. and let $q \in P$ be an arbitrary condition. There must be $n \in \omega$ such that $q \in F_n$; clearly, the filter $F_n \in M$ has the properties required in Y-c.c. The Y-c.c. of P has been verified.

The other implication is more involved. Suppose for contradiction that P is a Y-c.c. poset and $A \subset P$ is an antichain of size $\aleph_1$. Let $M \prec H_\theta$ be a countable elementary submodel containing P, A. Let $q \in A \setminus M$ be any element and let $F \in M$ be the filter guaranteed by Y-c.c. Let $G = \{B \subset A\colon \sum B \in F\}$; so $G \in M$ is a collection of subsets of A.

Claim 2.2. *For every set $B \subset A$ in M, $B \in G \leftrightarrow q \in B$.*

Proof. Suppose that $q \in B$. Then $\sum B \geq q$ is an element of M which must belong to F by the choice of F, and so $B \in G$.

Suppose that $q \notin B$ and for contradiction assume that $B \in G$. By the previous paragraph, $A \setminus B \in G$, and so both $\sum B$ and $\sum(A \setminus B)$ belong to the filter F. This is a contradiction–the conjunction of the two is zero since A is an antichain. □

We can now argue that G is a nonprincipal σ-complete ultrafilter on A. By the elementarity of M, it is enough to show that G is closed under countable intersections in M, and that for every set $B \subset A$ in M, exactly one of $B \in G$, $A \setminus B \in G$. Both of these statements follow immediately from the claim.

However, in ZFC there are no nonprincipal countably complete ultrafilters on sets of size $\aleph_1$, the final contradiction. □

An important feature of Y-c.c. posets is their interaction with open graphs. This is encapsulated in the following theorem.

Theorem 2.3. *Suppose that X is a second countable topological space and $H \subset X^\omega$ is an open set. If P is Y-c.c. then every H-anticlique in the extension is covered by a ground model countable set of H-anticliques.*

Here, an H-anticlique is just a set $A \subset X$ such that $A^\omega \cap H = 0$.

Proof. Let X be the space and $H \subset X^\omega$ be an open set. Suppose that $P \Vdash \dot{A} \subset X$ is an anticlique. For every filter $F \subset RO(P)$, let $B(\dot{A}, F) = \{x \in X :$ for every open neighborhood $O \subset X$ of x, the Boolean value $\|\check{O} \cap \dot{A} \neq 0\|$ is in the filter $F\}$.

Claim 2.4. *The set $B(\dot{A}, F)$ is an H-anticlique.*

Proof. For contradiction assume that this fails and let $\langle x_n : n \in \omega\rangle \in H$ be a sequence of points in $B(\dot{A}, F)$. Use the fact that H is open to find a number $l \in \omega$ and open sets $O_k \subset X$ for $k \in l$ such that $x_k \in O_k$ and $\prod_{k\in l} O_k \times X^\omega \subset H$. By the definition of the set $B(\dot{A}, F)$, the Boolean values $\|\check{O}_k \cap \dot{A} \neq 0\|$ for $k \in l$ are all in the filter F and have a lower bound $p \in P$. But then, $p \Vdash (\prod_{k\in l} O_k \times X^\omega) \cap \dot{A}^\omega \neq 0$ so $\dot{A}$ is not an H-anticlique. This is a contradiction. □

Now, let $M \prec H_\theta$ be a countable elementary submodel containing $P, X, H, \dot{A}$; we claim that $\dot{A}$ is forced to be covered by the anticliques in the model M. Suppose that this fails, and let $q \in P$ be a condition and $x \in X$ a point such that x belongs to no anticliques in the model M, and $q \Vdash \check{x} \in \dot{A}$. Let $F \subset RO(P)$ be a filter in the model M containing all elements of $RO(P) \cap M$ weaker than q. Since the space X is second countable, it has a basis all of whose elements belong to the model M. For every such basic open set $O \subset X$ containing the point x, the Boolean value $\|\check{O} \cap \dot{A} \neq 0\|$ is weaker than q, and it belongs to the model M. Therefore, $x \in B(\dot{A}, F)$ which an anticlique in the model M by the claim, a contradiction to the choice of the point x. □

Theorem 2.3 has a number of prominent corollaries. The following is immediate:

Corollary 2.5. *Let P be a Y-c.c. poset. Let X be a second countable space and let $H \subset X^\omega$ be an open set.*

1. *If in the extension X is covered by countably many H-anticliques, then already in the ground model it is covered by countably many anticliques;*
2. *if in the extension H has an uncountable anticlique, then H has an uncountable anticlique in the ground model.*

Thus, Y-c.c. posets cannot be used to force an instance of OCA in clopen graphs.

Corollary 2.6. *Let P be a Y-c.c. poset. If X is a compact Polish space and C is an ω_1-cover consisting of G_δ-sets, then in the extension C remains an ω_1-cover.*

Here, a set $C \subset \mathcal{P}(X)$ is an ω_1-cover if every countable subset of X is a subset of one element of C. Corollary 2.6 needs to be understood in the context of interpretations of descriptive set theoretic notions in generic extensions: the space X as well as the G_δ elements of the cover C are naturally interpreted in the P-extension as a compact Polish space and its G_δ-subsets again.

Proof. Let $p \in P$ be a condition and $\dot{x}_n$ for $n \in \omega$ be names for elements of X; we must find a set $B \in C$ and a condition $q \leq p$ such that $q \Vdash \{\dot{x}_n : n \in \omega\} \subset \dot{B}$.

Claim 2.7. *There is a condition $q \leq p$ and a countable set $\{y_n : n \in \omega\}$ such that for every compact set $K \subset X$,*

$$q \Vdash \dot{K} \cap \{\dot{x}_n : n \in \omega\} \neq 0 \textit{ implies } K \cap \{y_n : n \in \omega\} \neq 0.$$

Proof. Consider the set $H \subset (K(X))^\omega$ consisting of all sequences $\langle K_n : n \in \omega \rangle$ such that $\bigcap_n K_n = 0$. A compactness argument shows that if the hyperspace $K(X)$ of compact subsets of X is equipped with the Polish Vietoris topology, the set H is open. For each $n \in \omega$ let $\dot{A}_n$ be the P-name for the collection of compact ground model subsets of X which (whose canonical interpretations) contain the point $\dot{x}_n$. Clearly, $\dot{A}_n \subset K(X)^V$ is forced to be an H-anticlique. By Theorem 2.3, there is a condition $q \leq p$ and a countable set $\{D_n : n \in \omega\}$ of H-anticliques such that $q \Vdash \bigcup_n \dot{A}_n \subset \bigcup_n D_n$. A compactness argument shows that the intersection of each H-anticlique is nonempty, and for each $n \in \omega$ there is a point $y_n \in \bigcap D_n$. It is immediate that the set $\{y_n : n \in \omega\}$ works. □

Pick a condition $q \leq p$ and a set $\{y_n : n \in \omega\}$ as in the claim. Let $B \in C$ be a G_δ-set such that $\{y_n : n \in \omega\} \subset B$. It will be enough to prove that $q \Vdash \{\dot{x}_n : n \in \omega\} \subset \dot{B}$. Suppose that this fails. As B is G_δ, there must be a ground model open superset of B not containing the set $\{\dot{x}_n : n \in \omega\}$. Let $K \subset X$ be the compact complement of this open set. Then $K \cap \{\dot{x}_n : n \in \omega\} \neq 0$ while $K \cap \{y_n : n \in \omega\} = 0$, a contradiction. □

The corollary has numerous consequences: Y-c.c. posets do not add random reals since the G_δ Lebesgue null sets form an ω_1-cover. Y-c.c. posets do not separate gaps of uncountable cofinality, since each such gap induces a natural ω_1-cover of G_δ-sets.

Theorem 2.3 did not use the fact that the filters F of Definition 1.1 come from the model M; it was enough to assume that they come from some fixed countable set of filters on $RO(P) \cap M$. The assumption that the filters come from the model M is used in the preservation of Y-c.c. under the finite support iteration, as well as in the following two features.

Theorem 2.8. *Suppose that P has Y-c.c. and κ is a cardinal. For every function $f \in \kappa^\kappa$ in the P-extension, if $f \restriction a$ is in the ground model for every ground model countable set a, then f is in the ground model.*

Proof. Let $p \in P$ be a condition and $\dot{f}$ a name such that $p \Vdash \dot{f} \in \kappa^\kappa$ is a function such that $\dot{f} \restriction \check{a} \in V$ for every countable set $a \in V$. We must find a condition $q \leq p$ and a function $g \in \kappa^\kappa$ such that $q \Vdash \check{g} = \dot{f}$.

Let M be a countable elementary submodel of H_θ containing $P, p, \dot{f}, \kappa$. Let $q \leq p$ be a condition deciding all values of $\dot{f}(\alpha)$ for $\alpha \in \kappa \cap M$. We will show that there is a function g in the model M such that $q \Vdash \dot{f} = \check{g}$.

Let $F \subset RO(P)$ be a filter in the model M obtained by an application of Y-c.c. to M, q. By the c.c.c. of P, for every ordinal $\alpha \in \kappa \cap M$, the ordinal β such that $q \Vdash \dot{f}(\check{\alpha}) = \check{\beta}$ must be in the model M. Therefore, the Boolean value $\|\dot{f}(\check{\alpha}) = \check{\beta}\|$ is in the model M, it is weaker than q, and therefore belongs to the filter F. Let $g = \{\langle \alpha, \beta \rangle \in \kappa \times \kappa : \|\dot{f}(\check{\alpha}) = \check{\beta}\| \in F\}$. Since F is a filter, this is a partial function from κ to κ. By the elementarity of the model M, $g \in M$. We just argued that g

is defined for every ordinal $\alpha \in M$, and so by the elementarity of the model M, g is a total function from κ to κ. We have also argued that $q \Vdash \dot{f} \restriction M = \check{g} \restriction M$, and by the elementarity of M and the c.c.c. of P, $q \Vdash \dot{f} = \check{g}$ as desired. □

Corollary 2.9. *If P has Y-c.c., then P does not add any new cofinal branches into ω_1-trees.*

It is well known that an atomless σ-centered poset adds an unbounded real, and the proof translates with the obvious changes to Y-c.c. posets.

Theorem 2.10. *If P is an atomless poset satisfying Y-c.c., then P adds an unbounded real.*

Together with the preservation of Y-c.c. under complete subalgebras, this reproves the fact that Y-c.c. posets add no random reals. If an Y-c.c. poset did add a random real, then the measure algebra would be Y-c.c. which contradicts Theorem 2.10.

Proof. Let $M \prec H_\theta$ be a countable elementary submodel. Let $\langle F_i : i \in \omega \rangle$ be an enumeration of all ultrafilters on $RO(P)$ that belong to the model M. Each of them is nowhere dense in $RO(P)$ and so one can find a maximal antichain $A_i \subset RO(P) \setminus F_i$ in the model M for every $i \in \omega$. The antichain is infinite and countable by the c.c.c. of P. Let $\{a_i^j : j \in \omega\}$ be an enumeration of A_i for each $i \in \omega$ and define the name τ for an element of ω^ω by $\tau(i) = j$ if a_i^j belongs to the generic filter. We claim that this is a name for an unbounded real.

Suppose not, and find a condition $q \leq p$ such that for every $i \in \omega$, q is compatible with only finitely many elements of the antichain A_i. Let $F \subset RO(P)$ be a filter in M granted by the application of Y-c.c. to M, q. Use the axiom of choice in M to find $i \in \omega$ such that $F \subset F_i$. Let $B \subset A_i$ be the finite set of all elements of A_i compatible with q. Thus, $B \in M$, $\sum B \in M$, and necessarily $\sum B \geq q$ since no elements of $A_i \setminus B$ are compatible with q. Now, $B \cap F_i = 0$ and so $\sum B \notin F_i$ as F_i is an ultrafilter. On the other hand, $\sum B \geq q$ and so $\sum B \in F \subset F_i$ by the choice of i. This is a contradiction. □

The existence of unbounded reals in Y-c.c. extensions can be derived also abstractly from Theorem 2.3 and the following argument.

Theorem 2.11. *Let P be a bounding poset adding a new point $\dot{x} \in 2^\omega$. Then*

1. *either some condition forces $\dot{x}$ to be c.c.c. over the ground model and then some ω_1-cover of G_δ sets on a compact Polish space is not preserved;*
2. *or $\dot{x}$ is forced not to be c.c.c. over the ground model, and then there is a compact Polish space X, an open graph $H \subset [X]^2$ and an H-anticlique in the extension which is not covered by countably many ground model H-anticliques.*

Here, a point $x \in X$ is c.c.c. over the ground model if there is a σ-ideal I on X in the ground model which is c.c.c. (i.e. there is no uncountable collection of Borel pairwise disjoint I-positive sets) and x belongs to no Borel set in I coded in the ground model.

Proof. Suppose that $p \in P$ is a condition. Let I_p be the σ-ideal on 2^ω consisting of all analytic sets $A \subset 2^\omega$ such that $p \Vdash \dot{x} \notin \dot{A}$.

Claim 2.12. *Every I_p-positive analytic set has an I_p-positive compact subset.*

Proof. Let $A \notin I_p$ be an analytic set, and let $T \subset (2 \times \omega)^{<\omega}$ be a tree such that $A = \text{proj}([T])$. Let $q \leq p$ be a condition forcing $\dot{x} \in \dot{A}$, and let $\dot{y}$ be a name for a function in ω^ω such that $q \Vdash \langle \dot{x}, \dot{y} \rangle \in [\check{T}]$. Use the bounding assumption to find a condition $r \leq q$ and a function $z \in \omega^\omega$ such that $r \Vdash \dot{y}$ is dominated by $\check{z}$. Let S be the tree obtained from T by erasing all nodes which exceed the function z at some point in their domain. Then S is a finitely branching tree, $\text{proj}([S]) \subset A$ is compact, $r \Vdash \dot{x} \in \text{proj}([S])$ and so $\text{proj}([S])$ is I_p-positive. The claim follows. □

Now, suppose that there is a condition $p \in P$ such that the σ-ideal I_p is c.c.c. Then, every Borel set $B \in I_p$ is covered by a G_δ set $C \in I_p$, namely $C = 2^\omega \setminus \bigcup A$ where A is some (countable) maximal antichain of compact I_p-positive subsets of 2^ω disjoint from B. It follows that the collection of all G_δ sets in the ideal I_p is a ω_1-cover, and the condition p forces it not to be an ω_1-cover in the extension–none of its elements contain the point $\dot{x}$.

Suppose on the other hand that the ideal I_p is not c.c.c. for any condition $p \in P$. Let $X = K(2^\omega)$ and consider the open graph $H \subset X^2$ consisting of all pairs $\langle K, L \rangle$ such that $K \cap L = 0$. Let $\dot{A}$ be a name for the H-anticlique consisting of all compact sets in the ground model containing the point $\dot{x}$. We claim that it is forced not to be covered by countably many anticliques in the ground model. Suppose that $p \in P$ is a condition and B_n for $n \in \omega$ are H-anticliques; we will find a condition $q \leq p$ and a compact set $K \subset 2^\omega$ such that $K \notin \bigcup_n B_n$ and $q \Vdash \dot{x} \in \dot{K}$. Just observe that the σ-ideal I_p is not c.c.c. and use the claim to produce an uncountable collection C of pairwise disjoint I_p-positive compact sets. Note that this collection is an H-clique, and therefore for each $n \in \omega$ the intersection $B_n \cap C$ can contain at most one set. As C is uncountable, there must be $K \in C \setminus \bigcup_n B_n$ and then any condition $q \leq p$ forcing $\dot{x} \in \dot{K}$ is as required. □

3. Y-c.c.: examples

Y-c.c. in all of our examples is verified using the same general theorem.

Theorem 3.1. *Let P be a poset. Suppose that there is a function a defined on P such that*

1. *for every $p \in P$, $a(p)$ is a finite set;*
2. *if $p, q \in P$ are compatible, then they have a lower bound $r \leq p, q$ such that $a(r) = a(p) \cup a(q)$;*
3. *whenever $\{p_\alpha : \alpha \in \omega_1\}$ and $\{q_\alpha : \alpha \in \omega_1\}$ are subsets of P such that $\{a(p_\alpha) : \alpha \in \omega_1\}$ and $\{a(q_\alpha) : \alpha \in \omega_1\}$ are Δ-systems with the same root, then there are ordinals $\alpha, \beta \in \omega_1$ such that p_α and q_β are compatible.*

Then P is Y-c.c.

Proof. Let a be a finite set. Say that a set $A \subset P$ is a-large if for every countable set $b \supset a$ there is a condition $p \in A$ such that $a(p) \cap b = a$.

Claim 3.2. *The set $\{\sum A : A$ is a-large$\} \subset RO(P_\pi)$ is centered.*

The trivial case where there are no a-large sets at all is included in the statement of the claim.

Proof. Let $\{A_i : i \in n\}$ be finitely many a-large sets; we must produce a condition $q \in P$ which for each i has an element of A_i above it. To this end, use transfinite induction and the largeness assumption to find conditions $p^i_\alpha \in A_i$ for each $\alpha \in \omega_1$ and $i \in n$ so that $\{a(p^i_\alpha) : \alpha \in \omega_1\}$ is a Δ-system with root a for each $i \in n$.

By induction on $i \in n$ find sets $\{q^i_\alpha : \alpha \in \omega_1\}$ so that $\{a(q^i_\alpha) : \alpha \in \omega_1\}$ forms a Δ-system with root a and for every $\alpha \in \omega_1$ and every $j \in i+1$ there is $\beta \in \omega_1$ such that $p^j_\beta \geq q^i_\alpha$. The step $i = 0$ is trivially satisfied with $p^i_\alpha = q^i_\alpha$. To perform the induction step, by transfinite recursion on $\gamma \in \omega_1$ use item (3) of the assumptions repeatedly to find countable ordinals α_γ and β_γ such that $q^i_{\alpha_\gamma}$ and $p^i_{\beta_\gamma}$ are compatible and whenever $\delta \neq \gamma$ then $(a(p^i_{\beta_\gamma}) \cup a(q^i_{\alpha_\gamma})) \cap (a(p^i_{\beta_\delta}) \cup a(q^i_{\alpha_\delta})) = a$. Then, use (2) to find conditions $q^{i+1}_\gamma \leq p^i_{\beta_\gamma}, q^i_{\alpha_\gamma}$ such that $a(q^{i+1}_\gamma) = a(p^i_{\beta_\gamma}) \cup a(q^i_{\alpha_\gamma})$; this concludes the induction step.

In the end the condition $q = q^{n-1}_0$ works as required. □

Now suppose that $M \prec H_\theta$ is a countable elementary submodel containing X, P, and suppose that $q \in P$ is any condition. Let $a = M \cap a(p) \in M$, and find a filter $F \in M$ on $RO(P_\pi)$ extending the centered system $\{\sum A : A \text{ is } a\text{-large}\}$. We will show that for every $p \in M \cap RO(P)$, if $p \geq q$ then $p \in F$. This will complete the proof of Y-c.c. for P.

Let $A = \{r \in P : r \leq p\} \in M$. It will be enough to show that A is a-large, since $P \subset RO(P_\pi)$ is dense and so $\sum A = p$ and $q \in F$. Suppose for contradiction that A is not a-large. A counterexample, a countable set $b \subset X$, can be found in the model M by elementarity. But then, the condition $q \in A$ satisfies $a(q) \cap b = a$, contradicting the assumption that b is a counterexample. Thus, the poset P is Y-c.c. □

The first specific example of a Y-c.c. forcing is the specialization forcing for a tree without branches of length ω_1. Let T be a such a tree and consider the specialization poset $P(T)$ consisting of all finite functions $p : T \to \omega$ such that for all $s < t$ in $\mathrm{dom}(p)$ the values $p(s), p(t)$ are distinct. The ordering is that of reverse inclusion.

Corollary 3.3. *If T is a tree without branches of length ω_1, then the specialization forcing $P(T)$ satisfies Y-c.c.*

Proof. For every condition $p \in P(T)$ let $a(p) = p$. It will be enough to show that item (3) of Theorem 3.1 is satisfied. Suppose that $\{p_\alpha : \alpha \in \omega_1\}$ and $\{q_\alpha : \alpha \in \omega_1\} \subset P(T)$ are sets such that $\{a(p_\alpha) : \alpha \in \omega_1\}$ and $\{a(q_\alpha) : \alpha \in \omega_1\}$ are Δ-systems with the same root a. By transfinite recursion on $\gamma \in \omega_1$ find ordinals α_γ and β_γ such that the sets $b_\gamma = \mathrm{dom}(p_{\alpha_\gamma}) \cup \mathrm{dom}(q_{\beta_\gamma}) \setminus a$ for $\gamma \in \omega_1$ are pairwise disjoint and contain no element of T which is below some element of a.

For each $\gamma \in \omega_1$ consider a condition $r_\gamma \in P(T)$ whose domain is the set of minimal elements of b_γ and which assigns to every element of its domain value 0. Since the poset $P(T)$ is c.c.c. there must be countable ordinals $\delta \neq \gamma$ for which r_γ and r_δ are compatible, i.e. no elements of b_γ is compatible with any element of b_δ. It is immediate that the conditions p_{α_γ} and $q_{\beta_\delta} \in P$ are compatible as well. □

The usual gap specialization forcing satisfies Y-c.c. as well. To this end, recall basic definitions. An (ω_1, ω_1)-pregap is a sequence $\langle a_\alpha, b_\alpha : \alpha \in \omega_1\rangle$ of subsets of ω such that $a_\alpha \cap b_\alpha = 0$ and $\beta \in \alpha$ implies that $a_\beta \subset^* a_\alpha$ and $b_\beta \subset^* b_\alpha$, each time up to finitely many exceptional natural numbers. A set $c \subset \omega$ separates the pregap if for every ordinal $\alpha \in \omega_1$, $a_\alpha \subset^* c$ and $b_\alpha \cap c =^* 0$ holds. A gap is a pregap that cannot be separated.

A pregap is a gap if and only if for every uncountable set $D \subset \omega_1$ there are distinct ordinals $\alpha, \beta \in D$ such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq 0$. A gap is special if there is an uncountable set $D \subset \omega_1$ such that for all distinct ordinals $\alpha, \beta \in D$ $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq 0$ holds. For a special gap is it impossible to introduce a separating set without collapsing ω_1.

There is a natural specializing forcing for gaps [2, Lemma3.6]. Suppose that $H = \langle a_\alpha, b_\alpha : \alpha \in \omega_1 \rangle$ is a gap. Let $P(H)$ be the poset of all finite sets $p \subset \omega_1$ such that for distinct ordinals $\alpha, \beta \in p$ $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq 0$ holds. It turns out that $P(H)$ is c.c.c. It follows that there is a condition $p \in P(H)$ which forces that the union of the generic filter is uncountable; this is the specializing set.

Corollary 3.4. *If H is a (ω_1, ω_1)-gap then the specialization forcing $P(H)$ satisfies Y-c.c.*

Proof. For every condition $p \in P(H)$ let $a(p) = p$. It will be enough to show that item (3) of Theorem 3.1 is satisfied. Suppose that $\{p_\alpha : \alpha \in \omega_1\}$ and $\{q_\alpha : \alpha \in \omega_1\} \subset P(H)$ are sets such that $\{a(p_\alpha) : \alpha \in \omega_1\}$ and $\{a(q_\alpha) : \alpha \in \omega_1\}$ are Δ-systems with the same root a. By transfinite recursion on $\gamma \in \omega_1$ find ordinals α_γ and β_γ such that the sets $b_\gamma = p_{\alpha_\gamma} \cup q_{\beta_\gamma} \setminus a$ for $\gamma \in \omega_1$ are pairwise disjoint and contain no ordinals less or equal to $\max(a)$.

Use a counting argument to find an uncountable set $D \subset \omega_1$ and a number $k \in \omega$ such that for every ordinal $\gamma \in D$, the sets $\{a_\delta \setminus k : \delta \in b_\gamma\}$ and the sets $\{b_\delta \setminus k : \delta \in b_\gamma\}$ are linearly ordered by inclusion. For each $\gamma \in D$, write $\delta_\gamma = \min(\beta_\gamma)$ and let $c_\gamma = a_{\delta_\gamma} \setminus k$ and $d_\gamma = b_{\delta_\gamma} \setminus k$. The object $\langle c_\gamma, d_\gamma : \gamma \in D \rangle$ is a gap since any set separating this gap would also separate the original gap. Therefore, there must be ordinals $\gamma \neq \gamma' \in D$ such that $(c_\gamma \cap d_{\gamma'}) \cup (c_{\gamma'} \cap d_\gamma) \neq 0$. It is easy to verify that the conditions $p_{\alpha_\gamma}, p_{\beta_\gamma} \in P(H)$ are compatible as required. □

Todorcevic [10, Theorem 7.8] introduced a partition-type forcing associated with unbounded sequences of functions of length ω_1; this poset satisfies Y-c.c. as well. Let $F = \langle f_\alpha : \alpha \in \omega_1 \rangle$ be a modulo finite increasing, unbounded sequence of increasing functions in ω^ω. Let $P(\vec{f})$ be the poset of all finite sets $p \subset \omega_1$ such that for all ordinals $\alpha \in \beta$ in the set p, there is n such that $f_\alpha(n) > f_\beta(n)$. The ordering of $P(\vec{f})$ is that of inclusion.

Corollary 3.5. *If the sequence F is unbounded then the poset $P(F)$ satisfies Y-c.c.*

Proof. For every condition $p \in P(F)$ let $a(p) = p$. It will be enough to show that item (3) of Theorem 3.1 is satisfied. Suppose that $\{p_\alpha : \alpha \in \omega_1\}$ and $\{q_\alpha : \alpha \in \omega_1\} \subset P(F)$ are sets such that $\{a(p_\alpha) : \alpha \in \omega_1\}$ and $\{a(q_\alpha) : \alpha \in \omega_1\}$ are Δ-systems with the same root a. By transfinite recursion on $\gamma \in \omega_1$ find ordinals α_γ and β_γ such that the sets $b_\gamma = p_{\alpha_\gamma} \cup q_{\beta_\gamma} \setminus a$ for $\gamma \in \omega_1$ are pairwise disjoint and contain no ordinals less or equal to $\max(a)$.

Use a counting argument to find an uncountable set $D \subset \omega_1$ and a number $k \in \omega$ such that for every ordinal $\gamma \in D$ the functions $\{f_\delta : \delta \in b_\gamma\}$ are linearly ordered by domination everywhere above k. Let $\delta_\gamma = \min(b_\gamma)$. The collection $\langle f_{\delta_\gamma} : \gamma \in \omega_1 \rangle$ is unbounded, and therefore there is a number $n > k$ such that for every m there is an ordinal $\gamma(m) \in D$ such that $f_{\delta_{\gamma(m)}}(n) > m$. Let $\gamma' \in D$ be an ordinal larger than all $\gamma(m)$ for $m \in \omega$, let $m = \max\{f_\delta(n) \colon \delta \in b_{\gamma'}\}$ and observe that the conditions $p_{\alpha_{\gamma(m)}}, g_{\gamma'}$ are compatible as desired. □

Balcar, Pazák, and Thümmel [1], following Todorcevic [11], defined a natural ordering $T(Y)$ for every topological space Y. Thümmel used these orderings to settle an old problem of Horn and Tarski [7]. There are several closely related definitions of $T(Y)$; we will use the following. $T(Y)$ consists of all sets $p \subset Y$ such that p is a union of finitely many converging sequences together with their limits. For $p \in T(Y)$, write $a(p)$ for the set of its accumulation points. The ordering is defined by $q \leq p$ if $p \subset q$ and $a(q) \cap p = a(p)$.

The poset $T(Y)$ may or may not be c.c.c., σ-centered etc, depending on the topological space Y. However, if it is c.c.c. then it automatically assumes Y-c.c. This improves a result of Yorioka [15].

Corollary 3.6. *Let Y be a topological space. If $T(Y)$ is c.c.c. then $T(Y)$ has Y-c.c.*

Proof. We will show that the function $p \mapsto a(p)$ has the required properties. It will be enough to show that item (3) of Theorem 3.1 is satisfied. Suppose that $\{p_\alpha \colon \alpha \in \omega_1\}$ and $\{q_\alpha \colon \alpha \in \omega_1\} \subset T(Y)$ are sets such that $\{a(p_\alpha) \colon \alpha \in \omega_1\}$ and $\{a(q_\alpha) \colon \alpha \in \omega_1\}$ are Δ-systems with the same root a. By transfinite recursion on $\gamma \in \omega_1$ find ordinals α_γ and β_γ such that the sets $b_\gamma = a(p_{\alpha_\gamma}) \cup a(q_{\beta_\gamma}) \setminus a$ for $\gamma \in \omega_1$ are pairwise disjoint.

For each $\gamma \in \omega_1$ consider a condition $r_\gamma = p_{\alpha_\gamma} \cup q_{\beta_\gamma} \in T(Y)$. Since the poset P is c.c.c. there must be countable ordinals $\delta \neq \gamma$ for which r_γ and r_δ are compatible, i.e. no elements of b_γ is compatible with any element of b_δ. It is immediate that the conditions p_{α_γ} and $q_{\beta_\delta} \in P$ are compatible as well. □

As a final remark in this section, note that the OCA partition posets in general do not have Y-c.c. by Theorem 2.3.

Question 3.7. Suppose that I is a suitably definable ideal on a Polish space X and let P_I be the quotient Boolean algebra of Borel subsets of X modulo I. Are the following equivalent?

(1) P_I is Y-c.c.;
(2) P_I is σ-centered.

Question 3.8. Suppose that P, Q are Y-c.c. posets such that $P \times Q$ is c.c.c. Must $P \times Q$ be Y-c.c.?

4. Y-properness

The proper variation of Y-c.c. yields much greater variety of posets. The basic consequences of Y-properness remain mostly the same as for Y-c.c. and also the proofs of Section 2 immediately adapt to give the following:

Theorem 4.1. *Suppose that P is a Y-proper poset.*

(1) *Whenever κ is a cardinal and $f \in \kappa^\kappa$ is a function in the P-extension which is not in the ground model, then there is a ground model countable set $a \subset \kappa$ such that $f \restriction a$ is not in the ground model;*
(2) *P preserves ω_1-covers consisting of G_δ sets on compact Polish spaces;*
(3) *if X is a second countable topological space and $H \subset X^\omega$ is an open set, then every H-anticlique in the extension is covered by countably many ground model anticliques;*

(4) *if P is atomless, then it adds an unbounded real.*

The notion of Y-properness is particularly suitable for side condition type proper forcings. We discuss two classes of examples.

[16] introduced the notion of ideal based forcings. Yorioka showed that ideal based forcings do not add random reals. We will now show that ideal based forcings are Y-proper. This class of forcings includes posets used for destroying S-spaces, forcing a five-element classification of directed partial orders of size $\aleph_1$, and others.

First, the rather involved definitions must be carefully stated.

Definition 4.2. An *ideal based triple* is a triple $\langle U, \sqsubseteq, I\rangle$ such that the following are satisfied for $\sqsubseteq$:

(1) U is a collection of finite subsets of ω_1 and $\sqsubseteq$ is an ordering on it refining inclusion;
(2) whenever $a \in U$ and $\beta \in \omega_1$ then $a \cap \beta \in U$ and $a \cap \beta \sqsubseteq a$;

and the following are satisfied about I:

(3) I is an ideal on ω_1 including all singletons;
(4) every I-positive set has a countable I-positive subset;
(5) for every $a \in U$ the set $\{\beta \in \omega_1 \colon a \sqsubseteq a \cup \{\beta\}\}$ is not covered by countably many elements of I;
(6) for every $a \in U$ the set $\{\beta \in \omega_1 \colon a \cap \beta \sqsubseteq (a \cap \beta) \cup \{\beta\}$ and $a \not\sqsubseteq a \cup \{\beta\}\}$ is in I.

Definition 4.3. Given an ideal based triple $\langle U, \sqsubseteq, I\rangle$, the associated ideal based forcing P is defined as follows. A condition $p \in P$ is a finite set of ordered pairs $\langle M, \alpha\rangle$ such that $M \prec H_\lambda$ is a countable elementary submodel for some fixed λ such that $I \in H_\lambda$, α is a countable ordinal which does not belong to $\bigcup(I \cap M)$, and

- $w(p) = \{\alpha \colon \exists M \ \langle M, \alpha\rangle \in p\} \in U$;
- whenever $\langle M, \alpha\rangle$ and $\langle N, \beta\rangle$ are distinct elements of p then either $M \in N$ and $\alpha \in N$, or $N \in M$ and $\beta \in M$.

The ordering on the poset P is defined by $p \geq q$ if $p \subset q$ and $w(p) \sqsubseteq w(q)$.

Theorem 4.4. *If P is an ideal based forcing then P is Y-proper.*

Proof. Fix the ideal based triple $\langle U, \sqsubseteq, I\rangle$ generating the poset P. Let $p \in P$ be a condition. Say that a set $A \subset P$ is p-large if Player II has a winning strategy in the following game $G(A, p)$. Player I starts out with a countable set $z \in H_{\aleph_2}$. Then, Players I and II alternate for ω many rounds, Player I starts round k with a set $b_k \in I$ and Player II answers with a countable ordinal $\alpha_k \notin b_k$ such that $\alpha_0 \in \alpha_1 \in \dots$ Player II wins if there is a number l and a condition $q \leq p$ such that $w(q) = w(p) \cup \{\alpha_k : k \in l\}$, the $\in$-least model M on $q \setminus p$ contains z as an element and $w(q) \cap M = w(p)$, and there is $r \in A$ such that $r \geq q$. Note that the game is open for Player II and therefore determined.

Claim 4.5. *The collection $\{\sum A : A$ is p-large$\} \subset RO(P)$ is centered.*

Proof. Let $\{A_i \colon i \in n\}$ be a collection of p-large sets; we must find conditions $p_i \in A_i$ for each $i \in n$ with a common lower bound.

Let $\langle M_i \colon i \in n+1\rangle$ be an $\in$-chain of countable elementary submodels of some large H_θ with P, p, I, A_i for $i \in n$ all elements of M_0. By induction on $i \in n$, we will construct conditions $p_i \in A_i, q_i$ such that

- p, p_i are both weaker than q_i, $q_i \in M_{n-i}$, and the $\in$-least model on $q_i \setminus p$ contains $M_{n-i-1} \cap H_\lambda$ as an element;
- for each i, $r_i = \bigcup_{j \in i} q_j$ is a condition in P which is a lower bound of all conditions q_j for $j \in i$. Moreover, $w(r_i) \cap M_{n-i} = w(p)$.

Suppose that conditions p_j, q_j have been constructed for $j \in i$. Write $a_i = w(r_i)$. Work in the model M_{n-i}. Let σ_i be a winning strategy for Player II in the game $G(A_i, p)$. We will produce an infinite play of the game such that

- the initial move of Player I is $M_{n-i-1} \cap H_\lambda$;
- all moves are in the model M_{n-i};
- writing $e_l = \{\alpha_k : k \in l\}$, for every l we have $a_i \sqsubseteq a_i \cup e_l$.

The play is easy to construct by induction on $l \in \omega$. Suppose the first l moves have been constructed, producing a play t_l. Write $c = \{\alpha \in \omega_1 :$ for some set $b \in I$ the strategy σ_i answers the play $t_l^\frown b$ with $\alpha\}$. Thus, $c \in M_{n-i}$. Note that for every ordinal $\alpha \in c$, $w(p) \cup e_l \sqsubseteq w(p) \cup e_l \cup \{\alpha\}$: since there is a play in which Player II wins, producing a condition r such that $w(r)$ contains $w(p) \cup e_l \cup \{\alpha\}$ as an initial segment, this follows from (2) of Definition 4.2. Also, c is an I-positive set: if it were an element of I, then Player I could play a set containing c, forcing the strategy σ_i to answer with an ordinal out of c, contradicting the definition of c. By (4) of Definition 4.2, the set c contains an I-positive countable subset $d \subset c$, and this set d can be found in the model M_{n-i}. By (6), there is an ordinal $\alpha_l \in d$ such that $a_i \cup \{\alpha_k : k \in l\} \sqsubseteq a_i \cup \{\alpha_k : k \in l+1\}$. Find a move $b_l \in M_{n-i}$ provoking the strategy σ_i to answer α_l and let $t_{l+1} = t_l^\frown b_l^\frown \alpha_l$. This concludes the induction step and the construction of the play.

Now, since σ_i is a winning strategy for Player II, there is a natural number l and conditions $p_i \in A_i$ and q_i such that $q_i \leq p_i, p$, $q_i \in M_{n-i}$, and $w(q_i) = w(p) \cup e_l$. Consider the set $r_{i+1} = \bigcup_{j \leq i} q_j$. The set r_{i+1} is a condition in the poset P smaller than r_i by the third item of the construction of the infinite play. Also, $r_{i+1} \leq q_i$ by (2) of Definition 4.2. This concludes the induction step of the induction on i and the proof of the claim. □

Now we are ready to verify Y-properness for the poset P. Let $M \prec H_\theta$ be a countable elementary submodel containing $U, \sqsubseteq, I$, let $p \in P \cap M$. Find an ordinal $\alpha \in \omega_1 \setminus \bigcup(I \cap M)$ such that $w(p) \sqsubseteq w(p) \cup \{\alpha\}$; such ordinal exists by Definition 4.2(5). Let $q = p \cup \{\langle M \cap H_{\aleph_2}, \alpha\rangle\}$. It is not difficult to see that $q \leq p$. [16] shows that q is a master condition for M. We shall show that q is a Y-master condition for the model M.

Suppose that $r \leq q$ is an arbitrary condition. Observe that $r \cap M \in P$ is a condition weaker than r. Let $F \in M$ be a filter on $RO(P)$ extending the centered system $\{\sum A : A$ is $r \cap M$-large$\}$. We claim that for every condition $s \in RO(P) \cap M$ such that $s \geq r$, $s \in F$ holds. This will conclude the proof.

Indeed, let $A = \{t \in P : t \leq s\}$. Since P is dense in $RO(P)$, it is clear that $\sum A = s$. To conclude the proof, it will be enough to show that A is $r \cap M$-large. Suppose that it is not. The game $G(A, r \cap M)$ is determined, Player II has no winning strategy, therefore Player I has a winning strategy, and such a strategy σ has to exist in the model M as $r \cap M, s \in M$. Note that the strategy σ is in $H_{\aleph_2}$, and so it belongs to all models on $r \setminus M$. The definition of the poset P shows that Player II can defeat the strategy by playing the ordinals in $w(r) \setminus M$ in increasing order, since then the condition $r \leq s$ will witness the defeat of Player I at the appropriate finite stage. This is the final contradiction. □

Another class of Y-proper posets comes from the usual way of forcing the P-ideal dichotomy, PID [12, 13]. Let X be a set and $I \subset [X]^{\leq\aleph_0}$ be a P-ideal containing all singletons. This means that for every countable set $J \subset I$ there is a set $a \in I$ such that for every $b \in J$, $b \subset^* a$. Suppose that X is not a countable union of sets Y_n for $n \in \omega$ such that $\mathcal{P}(Y_n) \cap I = [Y_n]^{<\aleph_0}$. Then there is a proper poset P adding an uncountable set $Z \subset X$ such that $[Z]^{\aleph_0} \subset I$, which we now proceed to define.

For simplicity assume that the underlying set X is a cardinal κ. Let K be the σ-ideal on X generated by those sets $Y \subset X$ such that $I \cap \mathcal{P}(Y) = [Y]^{<\aleph_0}$. Thus, the assumptions imply that $X \notin K$. The poset P consists of conditions p, which are finite sets of triples $\langle M, x, a\rangle$ such that $M \prec H_{\kappa^+}$ is a countable elementary submodel, $x \in X$ is a point which does not belong to $\bigcup(K \cap M)$, and $a \in I$ is a set which modulo finite contains all sets in $I \cap M$. Moreover, if $\langle M, x, a\rangle$ and $\langle N, y, b\rangle$ are distinct elements of p, then either $M, x, a \in N$ or $N, y, b \in M$. The ordering is defined by $q \leq p$ if $p \subseteq q$ and whenever $\langle M, x, a\rangle \in q \setminus p$ and $\langle N, y, b\rangle \in p$ are such that $M \in N$ then $x \in b$. As in the ideal-based case, for a condition $p \in P$ we write $w(p) = \{x \in X : \exists M, a\ \langle M, x, a\rangle \in p\}$.

Theorem 4.6. *The PID poset P is Y-proper.*

Proof. Suppose that $p \in P$ is a condition and $A \subset P$ is a set. Say that A is p-large if Player II has a winning strategy in the following game $G(A, p)$. In the game, Player I starts with a set $z \in H_{\kappa^+}$, and then Player I and II alternate for ω many rounds. At round k, Player I plays a set $Y_k \in K$ and Player II answers with a point $x_k \in X \setminus Y_k$. Player II wins if at some round $l \in \omega$ there are conditions $q \in P$ and $r \in A$ such that q is a lower bound of p, r, $w(q) = w(p) \cup \{x_k : k \in l\}$, and the $\in$-first model M on $q \setminus p$ contains the set z, and $w(q) \cap M = w(p)$. Note that the game is open for Player II and therefore determined.

Claim 4.7. *The set $\{\sum A : A$ is p-large$\} \subset RO(P)$ is centered.*

Proof. Let $\{A_i : i \in n\}$ be a collection of p-large sets; we must find conditions $p_i \in A_i$ for each $i \in n$ with a common lower bound.

Let $\langle M_i : i \in n+1\rangle$ be an $\in$-chain of countable elementary submodels of some large H_θ with M_0 containing X, I, P, p, A_i for $i \in n$ as elements. By induction on $i \in n$, we will construct conditions $p_i \in A_i, q_i$ such that

- p, p_i are both weaker than q_i, $q_i \in M_{n-i}$, and the $\in$-least model on $q_i \setminus p$ contains $M_{n-i-1} \cap H_{\kappa^+}$ as an element;
- for each i, $r_i = \bigcup_{j \in i} q_j$ is a condition in P which is a lower bound of all conditions q_j for $j \in i$, and $w(r_i) \cap M_{n-i} = w(p)$.

Suppose that conditions p_j, q_j have been constructed for $j \in i$. Write $a_i \subset X$ for the intersection of all sets in the P-ideal I which occur on $r_i \setminus p$. Observe that every set in $I \cap M_{n-i}$ is contained in a_i up to finitely many exceptions. Work in the model M_{n-i}. Let σ_i be a winning strategy for Player II in the game $G(A_i, p)$. We will produce an infinite play of the game such that

- the initial move of Player I is $M_{n-i-1} \cap H_{\kappa^+}$;
- all moves are in the model M_{n-i};
- all moves of Player II belong to the set a_i.

The play is easy to construct by induction on $l \in \omega$. Suppose the first l moves have been constructed, producing a play t_l. Write $c = \{x \in X :$ for some set $b \in K$ the strategy σ_i answers the play $t_l^\frown b$ with $x\}$. Thus, $c \in M_{n-i}$. Observe that $c \notin K$: if $c \in K$, then Player I could play the set c, forcing the strategy σ_i to answer with a point out of c, contradicting the definition of c. By the definition of the ideal K, the set c contains an infinite countable set $d \subset c$ in the P-ideal I, and this set d can be found in the model M_{n-i}. Thus, the intersection $a_i \cap d$ is nonempty, containing some element $x \in M_{n-i}$. Find a move $b_l \in M_{n-i}$ provoking the strategy σ_i to answer with x and let $t_{l+1} = t_l^\frown b_l^\frown x$. This concludes the induction step and the construction of the play.

Now, since σ_i is a winning strategy for Player II, there is a natural number l and conditions $p_i \in A_i$ and q_i such that $q_i \leq p_i, p$, $q_i \in M_{n-i}$, and all points in X appearing on $q_i \setminus p$ belong to the set a_i. It is immediate to verify that $r_{i+1} = \bigcup_{j \leq i} q_i$ is a lower bound of r_i and q_i. This concludes the induction step of the induction on i and the proof of the claim. □

Now suppose that $M \prec H_\theta$ is a countable elementary submodel containing X, I, and let $p \in P \cap M$ be any condition. We must produce a Y-master condition $q \leq p$ for the model M. Let $x \in X$ be some point not in $\bigcup(K \cap M)$, and let $a \in I$ be some set which modulo finite contains all sets in $I \cap M$; these objects exist by initial assumptions on the ideal I. Let $q = p \cup \{\langle M \cap H_{\kappa^+}, x, a\rangle\}$. [12] shows that q is a master condition for M. We will show that q is a Y-master condition for the model M.

Let $r \leq q$ be a condition. Note that $r \cap M \in P$ is a condition weaker than r. Let $F \in M$ be any filter on $RO(P)$ extending the centered system $\{\sum A : A \subset P$ is $r \cap M$-large$\}$. We will show that for every condition $s \in RO(P) \cap M$, if $s \geq r$ then $s \in F$. This will conclude the proof.

Indeed, suppose that $s \in RO(P) \cap M$ is a condition weaker than r. Let $A = \{t \in P : t \leq s\} \in M$ and argue that A is $r \cap M$-large. This will conclude the proof since P is dense in $RO(P)$ and so $\sum A = s$ and $s \in F$. Suppose for contradiction that A is not $r \cap M$-large. Since the game $G(A, r \cap M)$ is determined, there must be a winning strategy $\sigma \in M$ for Player I in it. Now, let $\langle M_k, x_k, a_k\rangle : k \in l$ enumerate $r \setminus M$ in $\in$-increasing order and consider the counterplay of Player II against the strategy σ in which Player II's moves are x_k for $k \in l$ in this order. Note that the strategy σ belongs to all models M_k for $k \in l$ and so this is a legal counterplay. At the end of it, Player II is in a winning position, as witnessed by the condition $r \in A$. This contradicts the assumption that σ was a winning strategy for Player I. □

It is natural to ask which traditional fusion-type forcings are Y-proper. We do not have a comprehensive answer to this question. Instead, we prove a rather limited characterization theorem which nevertheless illustrates the complexity of the question well. For an ideal I on ω let $P(I)$ be the poset of all trees $T \subset \omega^{<\omega}$ which have a trunk t and for every node $s \in T$ extending the trunk, the set $\{n \in \omega : s^\frown n \in T\}$ does not belong to I. The ordering is that of inclusion. Rather standard fusion arguments show that posets of this form are all proper, and they preserve ω_1 covers on compact Polish spaces consisting of G_δ sets.

Theorem 4.8. *Let I be an ideal on ω. If I is the intersection of F_σ-ideals, then $P(I)$ is Y-proper.*

In particular, Laver forcing is Y-proper. The implication in Theorem 4.8 cannot be reversed already for $\mathbf{\Sigma}^0_4$ ideals. The ideal I on $\omega \times \omega$ generated by vertical sections and sets with all vertical sections finite is not the intersection of F_σ-ideals, but the poset $P(I)$ is Y-proper. The simplest example in which we do not know how to check the status of Y-properness is I =the ideal of nowhere dense subsets on $2^{<\omega}$.

Theorem 4.9. *Let I be an analytic P-ideal on ω. The following are equivalent:*

1. *I is the intersection of F_σ-ideals;*
2. *$P(I)$ is Y-proper;*
3. *for every compact Polish space X and every open set $H \subset X^\omega$, every H-anticlique in the $P(I)$-extension is covered by countably many H-anticliques in the ground model.*

An analytic P-ideal is an intersection of F_σ-ideals if and only if it is the intersection of countably many F_σ-ideals. A good example of an analytic P-ideal which can be written as such an intersection and yet is not F_σ is $I = \{a \subset \omega \colon \forall \varepsilon > 0 \sum_{n \in a} n^{-\varepsilon} < \infty\}$. An example of an analytic P-ideal which is not an intersection of F_σ-ideals is the ideal of sets of asymptotic density zero; in fact, the only F_σ ideal containing the density ideal is trivial, containing ω as an element.

Proof of Theorem 4.8. We will need a bit of notation. Write $P = P(I)$. Let $T \in P$ be a tree with trunk t. For a function $f \colon \omega^{<\omega} \to I$ write $T_f = \{s \in T \colon \forall i \in \mathrm{dom}(s \setminus t)\ s(i) \notin f(s \restriction i)\}$, observe that the trees $\{T_f \colon f \in \omega^{(\omega^{<\omega})}\}$ form a centered system in P, and pick an ultrafilter $F(T) \subset RO(P)$ extending this centered system.

Claim 4.10. *For every element $p \in F(T)$ there is a tree $S \subset T$ with trunk t such that $S \leq p$.*

Proof. Suppose this fails for some p. Let U be the set of all nodes $s \in T$ such that $t \subseteq s$ and there is no tree $S \subset T$ with trunk s such that $S \leq p$. Observe that $t \in U$, and if $s \in U$ then the set $\{i \in \omega : s^\frown i \in T \text{ and } s^\frown i \notin U\}$ is in I. If this failed, then there would be a I-positive set $a \subset \omega$ such that for each $i \in a$, $s^\frown i \in T$ and there is a tree $S_i \subset T \restriction s^\frown i$ with trunk $s^\frown i$ which is below p. Now, $S = \bigcup_{i \in a} S_i = \sum_{i \in a} S_i \leq p$, contradicting the assumption that $s \in U$.

Now, let $V = \{s \in T : \text{if } \mathrm{dom}(t) \leq i \leq \mathrm{dom}(s) \text{ then } s \restriction i \in U\}$ and use the previous paragraph to see that $V = T_f$ for some $f : \omega^{<\omega} \to I$. Since $p \in F(T)$, p is compatible with V and there is some tree $W \subset V$ such that $W \leq p$. Let s be the trunk of W and obtain a contradiction with the fact that $s \in U$. □

The (ultra)filters on $RO(P)$ critical for Y-properness of P will be obtained in the following way. If $T \in P$ is a tree with trunk t, write $a = \{i \in \omega \colon t^\frown i \in T\} \notin I$, use the assumption on the ideal I to find an F_σ-ideal $I(T)$ such that $I \subset I(T)$ and $a \notin I(T)$, and an ultrafilter $U(T)$ on ω such that $a \in U(T)$ and $I(T) \cap U(T) = 0$. Finally, let $G(T) = \{p \in RO(P) \colon \{i \in a : p \in F(T \restriction t^\frown i)\} \in U(T)\}$. It is not difficult to see that $G(T)$ is an ultrafilter.

Now we are ready for the fusion argument. Let $M \prec H_\theta$ be a countable elementary submodel containing U, G, F.

Claim 4.11. *If $T \in M \cap P$ is a tree with trunk t, then there is a tree $S \subset T$ with the same trunk such that*

1. *for all $i \in \omega$, if $t^\frown i \in S$ then $S \restriction t^\frown i \in M$;*

(2) *for every* $p \in G(T) \cap M$, *for all but finitely many* i, *if* $t^\frown i \in S$ *then* $S \restriction t^\frown i \leq p$.

Proof. Let $\{p_i : i \in \omega\}$ be an enumeration of $G(T) \cap M$. Let μ be a lower semicontinuous submeasure on ω such that $I(T) = \{a : \mu(a) < \infty\}$. By induction on $j \in \omega$ find finite pairwise disjoint sets $a_j \subset \omega$ and trees $S_k \subset T$ for each $k \in a_j$ so that

- $\mu(a_j) \geq j$;
- $S_k \in M$ is a tree with trunk $t^\frown k$, below $\bigwedge_{i<j} p_i$.

This is easy to do using Claim 4.10 and elementarity of the model M repeatedly. In the end, let $S = \bigcup_j S_j$. □

Now, an obvious fusion argument using Claim 4.11 repeatedly gives the following. For every tree $T \in M \cap P$ there is a tree $S \subset T$ in P with the same trunk such that for every node $s \in S$ there is an ultrafilter $F(s) \in M$ on $RO(P)$ such that for every $p \in F(s) \cap M$ for all but finitely many $i \in \omega$, either $s^\frown i \notin S$ or $S \restriction s^\frown i \leq p$. We will verify that the condition $S \leq T$ is Y-master for the model M.

Indeed, let $U \leq S$ be any tree, and let s be its trunk. We claim that for every $p \in RO(P) \cap M$, if $p \geq U$ then $p \in F(s)$. Indeed, if this failed then $1 - p \in F(s)$, by the properties of the tree S one can erase finitely many immediate successors of s in the tree U to get some $V \subset U$ such that $V \leq 1 - p$, and then V would be a common lower bound of p and $1 - p$, a contradiction. □

Proof of Theorem 4.9. It is enough to show that (3) implies (1). Suppose that I is an analytic P-ideal which is not an intersection of F_σ-ideals; we must show that there is a condition in $P = P(I)$ forcing an anticlique which is not covered by countably many ground model anticliques. Let $a \subset \omega$ be a set which belongs to every F_σ-ideal containing I, yet $a \notin I$. To simplify the notation, assume that $a = \omega$; otherwise, work under the condition $a^{<\omega} \in P$.

Let $M \prec H_\theta$ be a countable elementary submodel containing I. Let Y be the compact Polish space of all ultrafilters on $RO(P) \cap M$. Let $X = K(Y)$ and consider the open set $H \subset X^\omega$ consisting of all sequences $\langle K_n : n \in \omega \rangle \in X^\omega$ such that $\bigcap_n K_n = 0$. By compactness, the set H is open. For every condition $T \in P$, let $K_T = \{F \in Y : \{p \in RO(P) \cap M : p \geq T\} \subset F\}$. This is a compact subset of Y, therefore an element of $X = K(Y)$. Let $\dot{A} = \{K_T : T$ is a tree in the $P(I)$-generic filter$\}$. Clearly, this is a P-name for an H-anticlique. We will show that $\dot{A}$ is forced not to be covered by countably many H-anticliques in the ground model.

Suppose that $\{B_i : i \in \omega\}$ is a countable collection of H-anticliques and $T \in P$ is a condition. We will find a condition $S \leq T$ such that $K_S \notin \bigcup_i B_i$; this will complete the proof. By compactness, for every $i \in \omega$ there is an ultrafilter F_i on $RO(P) \cap M$ such that $F_i \in \bigcap B_i$. We will find the condition $S \leq T$ so that for every $i \in \omega$ there is $p_i \in F_i$ such that $1 - p_i \geq S$. Then, for every $i \in \omega$ $F_i \notin K_S$ and therefore $K_S \notin B_i$ as required.

The construction of the condition S starts with a small claim:

Claim 4.12. *For every tree* $U \in P$ *with trunk* t *and every* $j \in \omega$ *there is a tree* $V \leq U$ *with the same trunk such that*

(1) *there is a set* $c \subset \omega$ *such that* $V = \{s \in U : s$ *is compatible with* $t^\frown n\}$ *for every* $n \in c$;
(2) *for every* $i \in j$ *there is a condition* $q_i \in F_i$ *such that* $V \leq 1 - q_i$;

(3) *for every $i \in \omega$ there is a finite set u of immediate successors of t in the tree V and a condition $q_i \in F_i$ such that the tree V with the nodes in u erased is below $1 - q_i$.*

Proof. Finally, we will use the assumptions on the ideal I. By a result of Solecki [5], since I is an analytic P-ideal it is possible to find a lower semicontinuous submeasure μ on ω such that $I = \{b \subset \omega : \limsup_n \mu(b \setminus n) = 0\}$. Observe that for every $k \in \omega$ there is a partition of ω into finitely many singletons and finitely many pieces of μ-mass $< 2^{-k}$. If this failed, then the singletons together with sets of μ-mass $< 2^{-k}$ generate an F_σ-ideal which contains I as a subset and does not contain ω as an element, contradicting our assumptions on I. By the elementarity of the model M, such partitions exist in the model M as well.

Let $a = \{n \in \omega \colon t^\frown n \in U\}$. Let $\varepsilon = \limsup_n \mu(a \setminus n) > 0$. Let $\langle k_i : i \in \omega\rangle$ be a sequence of numbers such that $\sum_i 2^{-k_i} < \varepsilon/2$. The previous paragraph shows that there are sets $b_i \subset \omega$ in the model M such that each b_i is either a singleton or a set of μ-mass $< 2^{-k_i}$ such that either the Boolean value $q_i = \|$the generic element of ω^ω does not start with $t\|$ is in F_i, or the Boolean value $q_i = \|$the generic element of ω^ω starts with $t^\frown n$ for some $n \in b_i\|$ is in the ultrafilter F_i. It is now easy to find a set $c \subset a$ such that $\limsup_n \mu(c \setminus n) > \varepsilon/2$ such that for all $i \in j$, $b_i \cap c = 0$, and for every $i \in \omega$ $b_i \cap c$ is finite. The tree $V = \{s \in U : s$ is compatible with some $t^\frown n$ for some $n \in c\}$ clearly works as desired. □

Assume for simplicity that the trunk of the tree T is empty. A standard fusion argument using Claim 4.12 repeatedly yields a tree $S \leq T$ with empty trunk such that for every $i \in \omega$, there is a nonempty finite tree $u_i \subset S$ such that for every node $t \in u_i$ there is an element $q_i^t \in F_i$ such that the tree S_t obtained from S by restricting to t and erasing all immediate successors of t which are in u_i, is stronger than $1 - q_i^t$. Let $p_i = \prod_{t \in u_i} q_i^t$ and observe that S, p_i work as desired. □

Question 4.13. Does the conjunction of Y-properness and c.c.c. imply Y-c.c.?

Question 4.14. Suppose that I is a suitably definable σ-ideal on a Polish space X. Suppose that the quotient poset P_I of Borel I-positive sets ordered by inclusion is proper. Are the following equivalent?

(1) P_I is Y-proper;
(2) for every Polish compact space Y and every open set $H \subset Y^\omega$, every H-anticlique in the P_I extension is covered by countably many H-anticliques in the ground mode.

5. General treatment

Y-c.c. and Y-properness are preserved under a suitable notion of iteration, and there are suitable forcing axioms associated with them. The treatment is complicated enough to warrant a more general approach of which Y-c.c. and Y-properness are the most important instances.

Definition 5.1. A property $\Phi(F, B)$ of subsets F of complete Boolean algebras B is a *regularity property* if the following is provable in ZFC:

(1) (nontriviality) $\Phi(\{1\}, B)$ for every complete Boolean algebra B;
(2) (closure up) $\Phi(F, B) \to \Phi(F', B)$ whenever $F' = \{p \in B : \exists q \in F\ q \leq p\}$;

(3) (restriction) whenever $p \in B$ then $\Phi(F, B)$ implies $\Phi(F \cap (B \restriction p), B \restriction p)$, and $\Phi(F, B \restriction p)$ implies $\Phi(F, B)$. Here, $B \restriction p$ is the Boolean algebra $\{q \in B : q \leq p\}$ with the usual operations;
(4) (complete subalgebras) if B_0 is a complete subalgebra of B_1: for every $F \subset B_1$ $\Phi(F, B_1) \to \Phi(F \cap B_0, B_0)$ holds, and for every $F \subset B_0$ $\Phi(F, B_0) \to \Phi(F, B_1)$ holds;
(5) (iteration) if $\dot{B}_1$ is a B_0-name for a complete Boolean algebra, $F_0 \subset B_0$, $\dot{F}_1$ a name for a subset of B_1, $\Phi(F_0, B_0)$ and $1 \Vdash \Phi(\dot{F}_1, \dot{B}_1)$, then $\Phi(F_0 * \dot{F}_1, B_0 * \dot{B}_1)$ where

$$F_0 * \dot{F}_1 = \{\langle p_0, \dot{p}_1 \rangle \in B_0 * \dot{B}_1 : p_0 \wedge \|\dot{p}_1 \in \dot{F}_1\| \in F_0\}$$

.

If the Boolean algebra B is clear from the context, we write $\Phi(F)$ for $\Phi(F, B)$.

In the last item, we use the Boolean presentation of the two-step iteration. Let B_0 be a complete Boolean algebra and $\dot{B}_1$ a B_0-name for a complete Boolean algebra. Consider the poset of all pairs $\langle p_0, \dot{p}_1 \rangle$ such that $p_0 \in B_0$, $\dot{p}_1$ is a B_0-name for an element of $\dot{B}_1$, $p_0 \neq 0$ and $p_0 \Vdash \dot{p}_1 \neq 0$. The ordering is defined by $\langle q_0, q_1 \rangle \leq \langle p_0, \dot{p}_1 \rangle$ if $q_0 \leq p_0$ and $q_0 \Vdash \dot{q}_1 \leq \dot{p}_1$. It is not difficult to check that the separative quotient of this partial ordering (together with a zero element) is complete (admits arbitrary suprema and infima) and therefore forms a complete Boolean algebra which we will denote by $B_0 * \dot{B}_1$.

The central example of a regularity property studied in this paper is $\Phi(F) =$"F is a centered set". Other possibilities include $\Phi(F) =$"any two elements of F are compatible" or $\Phi(F) =$"for every collection $\{p_n : n \in \omega\} \subset F$ the Boolean value $\liminf_n p_n$ is nonzero". There are many other sensible possibilities.

The class of regularity properties is closed under countable conjunctions. The disjunctions are more slippery but also more rewarding. To treat them, we introduce an additional notion.

Definition 5.2. Let G be a set with a binary operation $*$. A property $\Phi(g, F, B)$ of subsets F of complete Boolean algebras B and elements $g \in G$ is a *G-regularity property* if for each $g \in G$, $\Phi(g, \cdot, \cdot)$ satisfies the demands (1-4) of Definition 5.1 and (5) is replaced with

(5) if $\dot{B}_1$ is a B_0-name for a complete Boolean algebra, $F_0 \subset B_0$, $\dot{F}_1$ a name for a subset of B_1, $\Phi(g_0, F_0, B_0)$ and $1 \Vdash \Phi(\check{g}_1, \dot{F}_1, \dot{B}_1)$, then $\Phi(g_0 * g_1, F_0 * \dot{F}_1, B_0 * \dot{B}_1)$.

A typical case appears when G is a countable semigroup. If G is clear from context, we omit it from the notation. It is clear that every regularity property is a G-regularity property for $G = \{1\}$ with the multiplication operation. Good nontrivial examples include $G =$the rationals in the interval $(0, 1]$ with multiplication, and $\Phi(\varepsilon, F, B) =$"there is a finitely additive probability measure μ on B such that $\mu(p) \geq \varepsilon$ for all $p \in F$". Another example studied by Steprans obtains when $G \subset \omega^\omega$ is a countable set closed under composition, with the composition operation, and $\Phi(g, F, B) =$"for every $n \in \omega$ and every collection of $g(n)$ many elements of F, there are n many elements in the collection with a common lower bound".

Definition 5.3. Suppose that $\langle G, * \rangle$ is a set with a binary operation. Suppose that Φ is a G-regularity property of subsets of complete Boolean algebras.

(1) A poset P is Φ-*c.c.* if for every condition $q \in P$ and every countable elementary submodel $M \prec H_\theta$ containing P, G there is an element $g \in G \cap M$ and a set $F \in M$ such that $\Phi(g, F)$ holds and F contains all elements of $RO(P) \cap M$ weaker than q.
(2) P is Φ-*proper* if for every countable elementary submodel $M \prec H_\theta$ containing P, G and every condition $p \in P \cap M$ there is a Φ-*master condition* $q \leq p$: this is a condition which is master for M and for every $r \leq q$, there is an element $g \in G \cap M$ and a set $F \in M$ such that $\Phi(g, F)$ holds and F contains all elements of $RO(P) \cap M$ weaker than q.

Clearly, Y-c.c. and Y-properness are special cases of Φ-c.c. and Φ-properness where $\Phi(F) =$"F is a centered set". Certain natural posets may satisfy other variations of Φ-c.c. For example, the random poset satisfies Φ-c.c. for $\Phi(F) =$"any two elements of F are compatible" or $\Phi(F) =$"for every collection $\{p_n : n \in \omega\} \subset F$ the Boolean value $\liminf_n p_n$ is nonzero".

There are many attractive arguments drawing abstract consequences from Φ-c.c. and Φ-properness for various regularity properties Φ. We will limit ourselves to several striking consequences of this kind.

Theorem 5.4. *Suppose that Φ is a regularity property such that $\Phi(F)$ implies that F contains no uncountable antichain. Then Φ-c.c. implies c.c.c.*

Proof. For contradiction, assume that P is a Φ-c.c. poset with an antichain A of size $\aleph_1$. Let $M \prec H_\theta$ be a countable elementary submodel containing P, A, and let $q \in A \setminus M$ be any element. Let $F \in M$ be a subset of $RO(P)$ such that $\Phi(F)$ holds and F contains all elements of $RO(P) \cap M$ weaker than r. Let I be the σ-ideal on A σ-generated by the sets $B \subset A$ such that $\sum B \notin F$.

Claim 5.5. *I is a nontrivial c.c.c. σ-ideal containing all singletons.*

Proof. For the nontriviality, use the elementarity of the model M. If $B_n \subset A$ for $n \in \omega$ are generating elements of the σ-ideal I in the model M, then $q \notin B_n$ for each n by the definitions, and so $q \notin \bigcup_n B_n$ and $\bigcup_n B_n \neq A$. Thus, no countable union of generating sets in the model M of the σ-ideal I covers all of A, and by the elementarity of the model M this is true even for generating sets in V.

If the σ-ideal I failed to be c.c.c. then there would be an uncountable collection C of pairwise disjoint I-positive sets. As the sets in C are pairwise disjoint and A is an antichain, the Boolean sums $\sum B$ for $B \in C$ are pairwise incompatible. They all must be elements of F by the definition of I. However, this contradicts the assumption on the regularity property Φ. □

However, by a classical theorem of Ulam [14], in ZFC there are no nontrivial c.c.c. σ-ideals on sets of size $\aleph_1$ which contain no singletons. This is a contradiction. □

Theorem 5.6. *Let Φ be a regularity property such that $\Phi(F)$ implies that F contains no infinite antichain. For every Φ-proper poset P, if $H \subset [X]^2$ is an open graph on a second countable space X, then every H-anticlique in the P-extension is covered by countably many anticliques in the ground model.*

Note that the statement "F contains no infinite antichains" in itself is not a regularity property as it does not satisfy the iteration clause of regularity.

Proof. Suppose that P is a Φ-proper poset and $H \subset [X]^2$ is an open graph on a second countable space. Let $\dot{A}$ be a P-name for an anticlique and let $F \subset RO(P)$ be a set satisfying Φ.

Claim 5.7. *The set $B(\dot{A}, F) = \{x \in X :$ for every open neighborhood $O \subset X$ of x, the Boolean value $\|\check{O} \cap \dot{A} \neq 0\|$ is in $F\}$ is a union of countably many H-anticliques.*

Proof. Remove all basic open neighborhoods O from the set $B(\dot{A}, F)$ such that $O \cap B(\dot{A}, F)$ is a union of countably many H-anticliques; it will be enough to show that the remainder B is empty. Suppose not; then for every basic open set $O \subset X$, the set $B \cap O$, if nonempty, is not an H-anticlique. This allows us to build by induction on $n \in \omega$ basic open sets $O_n, U_n \subset X$ such that

- $O_n \times U_n \subset H$;
- $O_{n+1}, U_{n+1} \subset U_n$;
- the sets $B \cap O_n$ and $B \cap U_n$ are both nonempty.

For each $n \in \omega$, let $p_n \in F$ be the Boolean value of $\|\check{O}_n \cap \dot{A} \neq 0\|$. By the assumption on Φ, there must be numbers $n \neq m$ such that the conditions p_n, p_m are compatible. Denote their lower bound by q. Then $q \Vdash \dot{A} \cap \check{O}_n \neq 0$ and $\dot{A} \cap \check{O}_m \neq 0$, which together with the fact that $O_n \times O_m \subset H$ contradicts the assumption that $\dot{A}$ is forced to be an H-anticlique. □

Now, let $p \in P$ be a condition, let $M \prec H_\theta$ be a countable elementary submodel containing $P, p, \dot{A}, H, X$. Let $q \leq p$ be a Φ-master condition for the model M. We claim that q forces $\dot{A}$ to be covered by the H-anticliques in the model M; this will complete the proof.

Suppose that this fails and let $r \leq q$ and $x \in X$ be a point which is not in any anticlique in the model M and yet $r \Vdash \check{x} \in \dot{A}$. Let $F \in M$ be a set satisfying Φ and containing all conditions $s \in RO(P) \cap M$ such that $s \geq r$. Then, for every basic open set $O \subset X$ containing x it is the case that $\|\check{O} \cap \dot{A} \neq 0\| \geq r$, and since the Boolean value is an element of the model M, it is the case that $\|\check{O} \cap \dot{A} \neq 0\| \in F$ and so $x \in B(F, \dot{A})$. The latter set is a union of H-anticliques in the model M as per the claim. This is a contradiction. □

Steprans [6]and Todorcevic [9, Theorem 7] produced for every number $k \geq 2$ a poset P_k which is σ-k-linked and yet adds an anticlique for an open hypergraph in dimension $k+1$ which is not covered by countably many anticliques in the ground model. Thus, the various finite dimensions of open hypergraphs do have significance. Once finitely additive measures enter the picture, all finite dimensions are well-behaved:

Theorem 5.8. *Suppose that Φ is a regularity property such that $\Phi(F)$ implies that there is a finitely additive probability measure μ on B and a real number $\varepsilon > 0$ such that $\forall p \in F\ \mu(p) > \varepsilon$. Then, for every $n \in \omega$, every second countable space X, and every open set $H \subset X^n$, every H-anticlique in Φ-proper extension is covered by countably many ground model H-anticliques.*

Proof. Let P be a Φ-proper poset and $\dot{A}$ a P-name for an H-anticlique. Let $F \subset RO(P)$ be a set with $\Phi(F)$. Let $B(\dot{A}, F) = \{x \in X :$ for every open neighborhood $O \subset X$ with $x \in O$, $\|\check{O} \cap \dot{A} \neq 0\| \in F\}$.

Claim 5.9. *$B(\dot{A}, F) \subset X$ is a union of countably many H-anticliques.*

Proof. First, remove from the set $B(\dot{A}, F)$ all open neighborhoods in which the set is the union of countably many anticliques. We claim that the remainder $B \subset X$ is empty; this will complete the proof of the claim.

Suppose for contradiction that $B \neq 0$. Note that for every open neighborhood $O \subset X$, if $O \cap B \neq 0$ then $O \cap B$ is not an H-anticlique. Let μ be a finitely additive probability measure on $RO(P)$ such that for some fixed $\varepsilon > 0$, $\mu(p) \geq \varepsilon$ for every condition $p \in F$. For every open set $O \subset X$, write $q(O) = \|\check{O} \cap \dot{A} \neq 0\|$. By induction on $m \in \omega$ build basic open sets $O^i_m : i \in n$ and numbers $0 \neq i_m \in n$ so that

- for every $i \in n$ it is the case that $B \cap O^i_m \neq 0$;
- $\prod_i O^i_m \subset H$;
- $O^i_{m+1} \subset O^{i_m}_m$;
- writing $q_m = q(O^0_m) - q(O^{i_m}_m)$, it is the case that $\mu(q_m) \geq \varepsilon/n$.

This is easy to do: at stage m, the set $B \cap O^{i_m}_m$ is nonempty and therefore not an anticlique, which makes it possible to find sets O^i_{m+1} for $i \in n$ satisfying the first three items. Now, since $\mu(q(O^0_m)) > \varepsilon$, if for every number $0 \neq i \in n$ it were the case that $\mu(q(O^0_m) - q(O^i_m)) < \varepsilon/n$, then the conjunction $\bigwedge_i q(O^i_m)$ would have positive μ-mass by the finite additivity of μ. This conjunction would force $\dot{A}$ to contain an H-edge, contradicting the initial assumptions.

In the end, the conditions q_m for $m \neq 0$ form an antichain and each of them has μ-mass at least ε/n, a contradiction with the finite additivity of the probability measure μ. □

The rest of the argument follows word by word the conclusion of the proof of Theorem 5.6. □

Theorem 5.10. *Suppose that Φ is a regularity property such that $\Phi(F)$ implies that F contains no infinite antichains. Suppose that P is a Φ-proper poset and κ is a cardinal. For every function $f \in \kappa^\kappa$ in the P-extension, if $f \restriction a$ is in the ground model for every countable ground model set $a \subset \kappa$, then f is in the ground model.*

Proof. We will start with an abstract claim. Let κ be an uncountable cardinal. A *coherent system* on κ is a collection S of partial countable functions on κ, closed under subsets, such that for every countable set $a \subset \kappa$ there is $g \in S$ with $\mathrm{dom}(g) = a$, and there is no infinite collection of pairwise incompatible functions in S.

Claim 5.11. *For every coherent system S on κ, the set $H = \{f \in \kappa^\kappa\colon$ for every countable set $a \subset \kappa$, $f \restriction a \in S\}$ is nonempty and finite.*

Proof. To see that the set H is nonempty, consider the sets $H_a = \{f \in \kappa^\kappa \colon f \restriction a \in S\}$ for every countable set $a \subset \kappa$. Intersection of any countable collection of such sets is nonempty by the assumptions on S. Let U be an ultrafilter on κ^κ containing all sets H_a for $a \subset \kappa$ countable. For each such set $a \subset \omega$, there are only finitely many functions in S with domain a, and so one of them, denoted by g_a, satisfies $\{f \in \kappa^\kappa \colon g_a \subset f\} \in U$. It is immediate that $\bigcup_a g_a \in H$.

To prove the finiteness of H, suppose for contradiction that f_n for $n \in \omega$ are pairwise distinct functions in H. Then, there is a countable set $a \subset \kappa$ such that the functions $f_n \restriction a$ for $n \in \omega$ are pairwise distinct. They all belong to the set S, contradicting the coherence assumption on S. □

Now suppose that P is a Φ-proper poset and $\dot{f}$ is a P-name for a function from κ to κ. Let $p \in P$ be a condition forcing $\dot{f} \restriction a \in V$ for every countable set $a \subset \kappa$; we must produce a function $e \in \kappa^\kappa$ and a stronger condition forcing $\check{e} = \dot{f}$. Let $M \prec H_\theta$ be a countable elementary submodel containing $P, p, \dot{f}$, and let $q \leq p$ be a Φ-master condition for M. Find a condition $r \leq q$ deciding all values of $\dot{f} \restriction M$, yielding a function $h \colon M \to \kappa$. We will find a function $e \in M \cap \kappa^\kappa$ such that $h \subset e$. Then, since r is a master condition for M and $r \Vdash \check{e} \restriction M = \dot{f} \restriction M (= \check{h})$, it must be the case that $r \Vdash \check{e} = \dot{f}$. This will complete the proof.

Towards the construction of the function e, let $F \subset RO(P)$ be an upwards closed set in the model M such that $\Phi(F)$ holds and F contains all elements of $RO(P) \cap M$ weaker than r. Let $S = \{g \colon g$ is a partial function from κ to κ with countable domain and the Boolean value $\|\check{g} \subset \dot{f}\|$ belongs to $F\}$. We claim that $S \in M$ is a coherent system. Closure of S under subsets is clear from the definitions. S contains no infinite set of pairwise incompatible functions since F contains no infinite antichain. For every countable set $a \in M$ the function $h \restriction a$ is in $M \cap S$ since $r \Vdash \dot{f} \restriction a = h \restriction a \in V$ and r is a master condition for the model M. By the elementarity of the model M, the coherence of the system S follows.

Now, let $H \in M$ be the finite set of functions from κ to κ obtained by the application of the claim to the coherent system S. We claim that the function h is a subset of one element of H. Indeed, if this was not the case, then there would be a finite set $c \subset \kappa \cap M$ such that $h \restriction c$ is not a subset of any function in the finite set H. Let $T = \{g \in S \colon g$ is a function compatible with $h \restriction c\}$. Just as in the previous paragraph, $T \in M$ is a coherent system, and there is a function $e \in \kappa^\kappa$ such that every restriction of e to a countable set is in T. This function must appear on the finite list H while $e \restriction c = h \restriction c$. Contradiction! $\square$

6. Iteration theorems

As with most forcing properties, the point of the properties introduced in the previous section is that they are preserved under suitable iterations and their associated forcing axioms can be forced with a poset in the same category.

Definition 6.1. Suppose that Φ is a G-regularity property of subsets of complete Boolean algebras.

(1) if κ is a cardinal then Φ-MA_κ is the statement that for every c.c.c. Φ-c.c. poset P and every list of open dense subsets of P of size κ there is a filter on P meeting them all;
(2) Φ-PFA is the statement that for every Φ-proper poset P and every list of $\aleph_1$ many open dense subsets of P there is a filter on P meeting them all.

In the important special case of $\Phi(F) =$"F is centered", we will write YMA_κ and YPFA for Φ-MA_κ and Φ-PFA.

Theorem 6.2. *Let Φ be a G-regularity property. Then the conjunction of c.c.c. and Φ-c.c. is preserved under*

(1) *restriction to a condition;*
(2) *complete subalgebras;*
(3) *the finite support iteration.*

Proof. The first two items follow easily from the subalgebra and restriction clauses of regularity. The two-step iteration part of (3) follows just as easily from the iteration clause of regularity. If P_0 has Φ-c.c. and $\dot{P}_1$ is a P-name such that $P_0 \Vdash \dot{P}_1$ has Φ-c.c., we must show that $P_0 * \dot{P}_1$ has Φ-c.c.

Let $M \prec H_\theta$ be a countable elementary submodel containing $P_0, \dot{P}_1$ and let $\langle q_0, \dot{q}_1 \rangle$ be an arbitrary condition in the iteration. We must find a set $F \in M$ on $RO(P_0) * RO(\dot{P}_1)$ and $g \in G \cap M$ such that $\Phi(g, F)$ holds and for every condition $\langle p_0, \dot{p}_1 \rangle \in RO(P_0) * RO(\dot{P}_1)$ in the model M, if $\langle p_0, \dot{p}_1 \rangle \geq \langle q_0, \dot{q}_1 \rangle$ then $\langle p_0, \dot{p}_1 \rangle \in F$. To this end, write $\dot{G}_0$ for the canonical P_0-name for its generic filter and $M[\dot{G}_0]$ for the P_0-name for the set $\{\tau/\dot{G}_0 : \tau \in M \text{ is a } P_0\text{-name}\}$. It is well known that $M[\dot{G}_0]$ is forced to be a countable elementary submodel of H_θ of the generic extension $V[\dot{G}_0]$ and its intersection with the ground model is equal to M. Strengthening q_0 if necessary and using Φ-c.c. of the poset P_1 in the extension, we may find a name $\dot{F}_1 \in M$ for a subset of $RO(\dot{P}_1)$ and $g_1 \in G \cap M$ such that $1 \Vdash \Phi(\check{g}_1, \dot{F}_1)$, and $q_0 \Vdash \{p \in RO(\dot{P}_1) \cap M[\dot{G}_0] \colon p \geq \dot{q}_1\} \subset \dot{F}_1$. Use the Φ-c.c. of P_0 to find some $g_0 \in G \cap M$ and $F_0 \in M$ such that $F_0 \subset RO(P_0)$, $\Phi(g_0, F_0)$ and $\{p \in RO(P_0) \cap M : p \geq q_0\} \subset F_0$. By the iteration clause of regularity, $\Phi(g_0 * g_1, F_0 * \dot{F}_1)$ holds. We claim that $F = F_0 * \dot{F}_1$ witnesses Φ-c.c. for the iteration.

Indeed, suppose that $\langle p_0, \dot{p}_1 \rangle \in M$ is a condition in the iteration weaker than $\langle q_0, \dot{q}_1 \rangle$. Thus, $q_0 \Vdash \dot{p}_1 \geq \dot{q}_1$, $\dot{p}_1 \in M[\dot{G}_0]$, and so $\dot{p}_1 \in \dot{F}_1$. The Boolean value $\|\dot{p}_1 \in \dot{F}_1\|$ is in the model M and it is weaker than q_0, so the conjunction $p_0 \wedge \|\dot{p}_1 \in \dot{F}_1\| \in M$ is still weaker than q_0 and so belongs to the set F_0. Thus, $\langle p_0, \dot{p}_1 \rangle \in F_0 * \dot{F}_1$ as desired.

The general proof proceeds by induction on β =the length of the iteration. The case β successor is handled by the two-step iteration case. Suppose that β is limit, M is a countable elementary submodel of H_θ, and q is any condition in the iteration. The domain of q is a finite subset of β; let $\alpha = \max(M \cap \mathrm{dom}(q))$. Write P for the whole iteration, P_0 for the initial segment of the iteration up to α inclusive, and $\dot{P}_1$ for the remainder of the iteration; thus, $\dot{P}_1$ is a P_0-name. The condition q can be viewed as a pair $\langle q_0, \dot{q}_1 \rangle$ where $q_0 \in P_0$ and $q_0 \Vdash \dot{q}_1 \in \dot{P}_1$. Since $\alpha \in \beta$, the induction hypothesis guarantees the existence of a subset $F_0 \in M$ of $RO(P_0)$ and an element $g \in M \cap G$ such that $\Phi(g, F_0)$ holds and for every condition $p \in RO(P_0)$ in the model M, weaker than q_0, belongs to the set F_0. Let $F \in M$ be the subset of $RO(P)$ consisting of pairs $\langle p_0, \dot{p}_1 \rangle \in RO(P_0) * RO(\dot{P}_1)$ where $p_0 \wedge \|\dot{p}_1 = 1\| \in F_0$. By the nontriviality and the finite iteration clauses of regularity, $\Phi(g * h, F, RO(P))$ holds for every $h \in G$; we claim that the set F works as desired. Suppose that $p \geq q$ is a condition in the model M in $RO(P)$; we must show that $p \in F$. The condition p can be viewed as a pair $\langle p_0, p_1 \rangle$ such that $p_0 \in RO(P_0)$ and $p_0 \Vdash \dot{p}_1 \in RO(\dot{P}_1)$. Since $p \geq q$, it is the case that $p_0 \geq q_0$ and $q_0 \Vdash \dot{p}_1 \geq \dot{q}_1$.

The important point is that the latter formula means that $q_0 \Vdash \dot{p}_1 = 1$. If this were not the case, by the c.c.c. of P_1 there would be a strengthening $r_0 \leq q_0$ and a condition $\dot{r}_1 \in M \cap P_1$ such that $r_0 \Vdash \dot{r}_1$ is incompatible with $\dot{p}_1$. Now, by c.c.c. of P_0, P_0 forces the domain of $\dot{r}_1$ to be a subset of M and therefore disjoint from $\mathrm{dom}(\dot{q}_1)$. Thus, $r_0 \Vdash \dot{r}_1, \dot{q}_1$ are compatible, contradicting the assumption that $q_0 \Vdash \dot{p}_1 \geq \dot{q}_1$.

Now, the Boolean value $\|\dot{p}_1 = 1\| \in RO(P_0)$ is an element of M and it is weaker than q_0. The same is true of p_0. Therefore, the conjunction $p_0 \wedge \|\dot{p}_1 = 1\|$ must belong to the set F_0, and so $\langle p_0, \dot{p}_1 \rangle \in F$ as desired. □

As an abstract consequence of Theorem 6.2, it is possible to force Martin's Axiom for c.c.c. Φ-c.c. posets with a Φ-c.c. poset. The following simple general theorem does not seem to appear in the literature:

Theorem 6.3. *Suppose that Ψ is a property of complete Boolean algebras which provably in ZFC implies c.c.c. and is preserved under complete subalgebras and the finite support iteration. Let κ be an uncountable regular cardinal and suppose that $\diamondsuit_{\mathtt{cof}(\kappa)\cap\kappa^+}$ holds. There is a complete Boolean algebra satisfying Ψ forcing MA_κ for posets satisfying Ψ.*

Corollary 6.4. *Let Φ be a G-regularity property of sets of Boolean algebras. Let κ be an uncountable regular cardinal and suppose that $\diamondsuit_{\mathtt{cof}(\kappa)\cap\kappa^+}$ holds. There is a c.c.c. Φ-c.c. poset forcing Φ-MA_κ.*

Proof. Let $\langle E_\alpha : \alpha \in \kappa^+\rangle$ be a diamond sequence for $\mathtt{cof}(\kappa)\cap\kappa^+$. This specifically means the following. Fix a wellordering $\prec$ of the set H_{κ^+} of ordertype κ^+. Each set E_α is of hereditary cardinality κ and whenever $A \subset H_{\kappa^+}$ is a set, then the set

$$\{\alpha \in \mathtt{cof}(\kappa)\cap\kappa^+ \colon E_\alpha = \{x \in H_{\kappa^+} \colon x \in A \text{ and the rank of } x \text{ in } \prec \text{ is less than } \alpha\}\}$$

is stationary.

In the following, for a poset P we will write $\Psi(P)$ for the statement $\Psi(RO(P))$. Consider the finite support iteration $R = \langle R_\alpha, \dot{Q}_\alpha \colon \alpha \in \kappa^+\rangle$ obtained by the following rule: if α is an ordinal such that E_α codes an R_α-name for a poset, and in the R_α-extension $\Psi(\dot{E}_\alpha)$ holds, then $\dot{Q}_\alpha = E_\alpha$. Otherwise, let $\dot{Q}_\alpha$ =the R_α-name for the trivial poset. We claim that the iteration works as required.

Suppose that in the R-extension, P is a poset, $\Psi(P)$ holds, and $\langle D_\beta \colon \beta \in \kappa\rangle$ are open dense subsets of it. We must produce a filter meeting them all. First of all, without loss of generality, we may assume that $\|P\| \leq \mathfrak{c} \leq \kappa^+$. If this were not the case, let $N \prec H_\theta$ be an elementary submodel of size $\mathfrak{c}$ containing $P, \tau, \langle D_\beta : \beta \in \kappa\rangle$ as elements, κ as a subset, and such that $N^\omega \subset N$. Then, $N \cap P$ is a regular subposet of P, therefore $\Psi(N \cap P)$ holds by the closure of Ψ under complete subalgebras, it has size $\leq \mathfrak{c}$ and all sets $D_\beta \cap N$ for $\beta \in \kappa$ are dense in it. If there is a filter $G \subset N \cap P$ meeting all the sets $D_\beta \cap N$ for $\beta \in \kappa$, then we are done.

Thus, without loss of generality assume that $\|P\| = \kappa^+$, $P \subset H_{\kappa^+}$ and use the c.c.c. of R find an R-name $\tau \subset H_{\kappa^+}$ for it so that $R \Vdash \Psi(\tau)$. Back in the ground model, find an elementary submodel $N \prec H_\theta$ of size κ containing $P, \tau, \langle D_\beta : \beta \in \kappa\rangle$ as elements, κ as a subset, such that $N^\omega \subset N$ and, writing $\alpha = N \cap \kappa^+$, it is the case that $\tau \cap N = E_\alpha$. We claim that $R_\alpha \Vdash E_\alpha$ is a poset satisfying Ψ, thus $\dot{Q}_\alpha = E_\alpha$, and the generic filter added by the α-th stage of the iteration generates a filter on P meeting all the dense subsets as required.

Let $G_\alpha \subset R_\alpha$ be a generic filter and for the remainder of the proof work in $V[G_\alpha]$. Let R^α be the remainder of the iteration, so $\Psi(R^\alpha)$ holds. Write $P_\alpha = E_\alpha/G_\alpha$; thus $R^\alpha \Vdash P_\alpha \subset P$. The elementarity of the model N has an important consequence:

Claim 6.5. *The map $\pi \colon P_\alpha \to R^\alpha * \dot{P}$ given by $\pi(p) = \langle 1, \check{p}\rangle$ is a regular embedding.*

Proof. We must verify that if $A \subset P_\alpha$ is a maximal antichain, then $\pi'' A \subset R^\alpha * \dot{P}$ is a maximal antichain as well, or equivalently $R^\alpha \Vdash \dot{A} \subset \dot{P}$ is maximal. To prove this, suppose that $G^\alpha \subset R^\alpha$ is a filter generic over the model $V[G_\alpha]$, and let $G \subset R$ be the concatenation of G_α and G^α. Then in $V[G]$ the following holds:

- $N[G]$ is an elementary submodel of $H_\theta[G]$;
- $P \cap N[G] = P_\alpha$;
- $(P_\alpha)^\omega \cap V[G_\alpha] \subset N[G]$.

For the last item, return to the ground model for a moment and observe that every R_α-name σ for an element of $(\dot{P}_\alpha)^\omega$ is at the same time an R-name for an element of $(\dot{P})^\omega$. At the same time, $N \cap R = R_\alpha$ and N is closed under countable sequences, therefore N contains σ as an element.

It follows that $A \in N[G]$. Since $A \subset P_\alpha$ is a maximal antichain and $P \cap N[G] = P_\alpha$, $N[G] \models A \subset P$ is a maximal antichain. Since $N[G]$ is elementary in $H_\theta[G]$, $A \subset P$ must be a maximal antichain as desired. □

Now, still arguing in the model $V[G_\alpha]$, both steps in the iteration $R^\alpha * \dot{P}$ satisfy Ψ and so does the iteration. P_α is a regular subposet of this iteration and therefore satisfies Ψ as well. Therefore, at stage α of the iteration the poset P_α is forced with, and the resulting filter on $P_\alpha \subset P$ meets all the open dense subsets on the list $\langle D_\beta : \beta \in \kappa \rangle$. □

Now, let us move to the proper variations. Φ-properness is not preserved under the countable support iteration. To provide a trivial example, consider the countable support iteration of an atomic poset with two atoms, of length ω_1. Clearly, each poset in the iteration is Y-proper, and the iteration is isomorphic to adding a subset of ω_1 with countable approximations. This poset is not Y-proper by Theorem 4.1(1). Even so, it is possible to force the forcing axiom for Y-proper posets with an Y-proper poset using the technology of [4]. This is the contents of the following theorem.

Theorem 6.6. *Suppose that Φ is a G-regularity property and there is a supercompact cardinal. Then there is a Φ-proper forcing P forcing Φ-PFA.*

Proof. We first verify the preservation of Φ-properness under two-step iteration. This follows immediately from the iteration clause of regularity:

Claim 6.7. *If P is Φ-proper and $P \Vdash \dot{Q}$ is Φ-proper, then $P * \dot{Q}$ is Φ-proper. If $M \prec H_\theta$ is a countable elementary submodel containing P, $\dot{Q}$, and $p \in P$ is a Φ-master condition for M in P and $p \Vdash \dot{q}$ is a Φ-master condition for $M[G]$ in $\dot{Q}$, then $\langle p, \dot{q} \rangle$ is a Φ-master condition for M in $P * \dot{Q}$.*

Let κ be an inaccessible cardinal and $f : \kappa \to V_\kappa$ be a function. Let $I \subset \kappa + 1$ be the set of all inaccessible cardinals β such that $\langle V_\beta, f \restriction \beta \rangle \prec \langle V_\kappa, f \rangle$; in particular, $\kappa \in I$.

For every ordinal $\beta \in I$ define the orders Q_β by $m \in Q_\beta$ if m is a finite $\in$-chain whose elements are either countable elementary submodels of V_β (the *countable nodes*) or sets V_δ for some $\delta \in I \cap \beta$ (the *transitive nodes*). Moreover, the chain m must be closed under intersections. The ordering is that of reverse inclusion. Observe that if $\delta \in \beta$ are elements of I then $Q_\delta \subset Q_\beta$.

By transfinite recursion on $\beta \in I$ we will define partial orders $\langle P_\beta, \leq_\beta \rangle$. The canonical names for their respective generic filters will be denoted by $\dot{G}_\beta$. The elements of P_β will be certain pairs $p = \langle m(p), w(p) \rangle$, where $m(p) \in Q_\beta$ and $w(p)$ is a function on $m(p)$. For such a pair p, if $\delta \in I$ is such that $V_\delta \in m(p)$, write $p \restriction \delta$ for the pair $\langle m(p) \cap V_\delta, w(p) \restriction V_\delta \rangle$. The posets P_β are defined by the following recursive formula.

A set p is an element of P_β if $p = \langle m(p), w(p)\rangle$ where $m(p) \in Q_\beta$ and $w(p)$ is a function with $\mathrm{dom}(w(p)) = m(p)$ such that for every transitive node $V_\delta \in m(p)$, $p \restriction \delta \in P_\delta$. Moreover, $w(p)(M)$ is equal to trash for all nodes $M \in m(p)$ except possibly some transitive nodes $M = V_\delta$ such that $f(\delta)$ is a P_δ-name, $P_\delta \Vdash f(\delta)$ is a Φ-proper forcing, $w(p)(V_\delta)$ is a P_δ-name for an element of $f(\delta)$, and for every countable node $N \in m(p)$ such that $\{P_\delta, f(\delta)\} \in N$, $p \restriction \delta \Vdash_{P_\delta}$ the condition $w(p)(V_\delta)$ is Φ-master for the model $N[\dot{G}_\delta]$.

Note that due to the closure of $m(p)$ under intersections and to the fact that the set I consists of inaccessible cardinals, it is sufficient to verify the last condition for all countable nodes N which are between V_δ and the next transitive node on $m(p)$.

The ordering is defined by $q \leq_\beta p$ if $m(q) \leq m(p)$ and for every transitive node $V_\delta \in m(p)$, if $w(p)(V_\delta) =$ trash then $w(q)(V_\delta) =$ trash, and if $w(q)(V_\delta) \neq$ trash then $q \restriction \delta \leq_\beta p \restriction \delta$ and $q \restriction \delta \Vdash m(q)(V_\delta) \leq m(p)(V_\delta)$.

Claim 6.8. *$\leq_\beta$ is a transitive relation.*

Proof. This is an elementary argument by transfinite induction on β. ☐

Suppose that δ, β are elements of I such that $\delta \in \beta$. Define p^0_δ to be the condition in P_β which is $\langle \{V_\delta\}, \{\langle V_\delta, \mathrm{trash}\rangle\}\rangle$. In the event that $f(\delta)$ happens to be a P_δ-name and $P_\delta \Vdash f(\delta)$ is Φ-proper, then also define p^1_δ to be the condition in P_β which is $\langle \{V_\delta\}, \{\langle V_\delta, 1_{f(\delta)}\rangle\}\rangle$.

Claim 6.9. *Suppose that $\delta, \beta \in I$ are ordinals such that $\delta \in \beta$, and $p \in P_\beta$ is a condition below p^0_δ or p^1_δ. Suppose that $q \in P_\delta$ and $q \leq_\delta p \restriction \delta$. Then $r = \langle m_q \cup m_p, w(q) \cup w(p) \setminus V_\delta\rangle$ is a condition in P_β and $r \leq_\beta p$.*

Proof. This is another elementary argument by transfinite induction on β. ☐

If $\delta \in I$ is an ordinal less than κ such that $f(\delta)$ is an P_δ-name for an Φ-proper forcing, we will write $P_{\delta+1}$ for the two-step iteration $P_\delta * f(\delta)$. For an ordinal $\beta > \delta$ and a condition $p \in P_\beta$ such that $V_\delta \in m(p)$ and $w(p)(V_\delta) \neq$ trash, we will write $p \restriction \delta + 1$ for the condition in $P_{\delta+1}$ which is the pair $\langle p \restriction \delta, w(p)(V_\delta)\rangle$. The following claim is now easy to show:

Claim 6.10. *Let δ, β be ordinals in I such that $\delta \in \beta$.*

1. *The conditions p^0_δ and p^1_δ both force the filter $G_\beta \cap P_\delta$ to be P_δ-generic;*
2. *if δ is such that $f(\delta)$ is an P_δ-name for a Φ-proper forcing, then $p^1_\delta \Vdash G_\beta \cap P_{\delta+1}$ is $P_{\delta+1}$-generic.*

Let $p, q \in P_\beta$ be conditions. Say that p, q are in *Δ-position* if there is a countable node $M \in m(p)$ such that the model M contains q as well as P_γ and $f(\gamma)$ for all $V_\gamma \in m(q)$ as elements, and writing V_δ for the largest transitive node on $m(p)$ below M, it is the case that $V_\delta \in m(q)$, $q \restriction \delta + 1$ is compatible with $p \restriction \delta + 1$ and all countable nodes of $m(p)$ between V_δ and M belong to $m(q)$. If there are no transitive nodes in $m(p)$ below M, then we require just that all countable nodes of $m(p)$ below M belong to $m(q)$.

Claim 6.11. *If p, q are in Δ-position then they are compatible in P_β.*

Proof. Let $M \in m(p)$ be the model witnessing the Δ-position of p, q. Let us treat the case that there is a largest transitive node on $m(p)$ below M, denote it by V_δ, and assume that $w(p)(V_\delta) \neq$ trash. Since $p \restriction \delta + 1$ and $q \restriction \delta + 1$ are compatible, there is a condition $r \in P_\delta$ below both $p \restriction \delta + 1$ and $q \restriction \delta + 1$, and a P_δ-name τ such that $r \Vdash_\delta \tau \leq w(p)(V_\delta), w(q)(V_\delta)$ in the poset $f(\delta)$. To construct the lower bound s of p, q, we must define $m(s)$ and $w(s)$.

Let $m(s) = m(r) \cup m(q) \cup m(p) \cup n$, where n is the set of all intersections of the form $N \cap V_\gamma$, where V_γ is a transitive node on $m(q)$ and $N = M$ or else M is one of the countable nodes on $m(p)$ above M such that there is no transitive node between M and N. First, we must verify that m is a condition in Q_δ. This is a mechanical checking of the clauses of the definition of Q_δ; we will outline only the nontrivial points in the argument. For the closure of $m(s)$ under intersection, the only nontrivial case is to check is that if $N_0 \in m(q) \setminus V_\delta$ and $N_1 \in m(p) \setminus V_\delta$ are countable nodes, then $N_0 \cap N_1 \in m(r)$. To see this, note that $N_0 \in M$ and so $N_0 \subset M$, $N_0 \cap N_1 = N_0 \cap (N_1 \cap M)$, the countable node $N_1 \cap M$ is in $m(p)$ by the closure of $m(p)$ under intersections, and there are two cases.

Case 1. Either $N_1 \cap M \in V_\delta$. Then $N_0 \cap N_1 = (N_0 \cap V_\delta) \cap (N_1 \cap M \cap V_\delta)$. Now, the model $N_0 \cap V_\delta \in m(q)$ by the closure of $m(q)$ under intersections, $N_1 \cap M \cap V_\delta \in m(p)$ by the closure of $m(p)$ under intersection, so both of them are in $m(r)$ since $r \leq p \restriction \delta, q \restriction \delta$, and so $N_0 \cap N_1 \in m(r)$ by the closure of $m(r)$ under intersections.

Case 2. Or $N_1 \cap M \notin V_\delta$. Then $N_1 \cap M$ must be one of the models on $m(p)$ between V_δ and M, or it may be equal to M itself. In the former case $N_1 \cap M \in m(q)$ as p, q are in Δ-position, and so $N_0 \cap N_1 = N_0 \cap N_1 \cap M \in m(q)$ by the closure of $m(q)$ under intersections. In the latter case, $N_0 \cap N_1 = N_0 \cap M = N_0 \in m(q)$, and we are finished.

To verify that $m(s)$ forms an $\in$-chain, first observe that $m(r) \cup m(q) \cup m(p)$ is a concatenation of three $\in$-chains ($m(r)$, $m(q) \setminus V_\delta$, and $m_p \setminus M$, since $m(r) \in V_\delta$ and $m(q) \in M$) and so an $\in$-chain. Now inspect the models in the set n. Let V_γ be a transitive node in $m(q)$ and K its predecessor in $m(q)$. Then $K, V_\gamma \in M$. Since the countable nodes between M and the next transitive node in $m(p)$ above M form an $\in$-chain and all contain M as an element, they also contain K, V_γ and so their intersections with V_γ form an $\in$-chain whose nodes all contain K and are contained in V_γ. This immediately implies that $m(s)$ forms an $\in$-chain.

The definition of $w(s)$ breaks into several cases, all of which except for one are trivial.

Case 1. For $V_\gamma \in m(r)$, let $w(s)(V_\gamma) = w(r)(V_\gamma)$.

Case 2. The value $w(s)(V_\delta)$ will be equal to τ. This condition is forced to be Φ-master for all the relevant models on $m(s)$ above V_δ: these models come either from $m(p)$ or $m(q)$ or from intersections with transitive nodes, and τ is stronger than both $w(p)(V_\delta)$ and $w(q)(V_\delta)$.

Case 3. Now suppose that γ is such that $\delta \in \gamma$ and $V_\gamma \in m(q)$, $f(\gamma)$ is a P_γ-name for an Φ-proper poset, and $w(q)(V_\gamma) \neq$ trash. To define $w(s)(V_\gamma)$, consider the set n_γ of all countable nodes in $m(s)$ below the next transitive node V_{γ^*} above V_γ (if such a node does not exist, just take all countable nodes on $m(s)$) which contain $P_\gamma, f(\gamma)$. Observe that this set is linearly ordered by $\in$, starts with (perhaps) some models on $m(q)$, after which comes $M \cap V_{\gamma^*}$ and then (perhaps) some other models. By the definition of P_β, the condition $q \restriction \gamma$ forces in P_γ that $w(q)(V_\gamma)$ is a Φ-master condition for $K[G_\gamma]$ for all models $K \in n_\gamma \cap m(q)$ and the poset $f(\gamma)$. Moreover, $P_\gamma, f(\gamma)$ and $w(q)(V_\gamma)$ all belong to the next model $M \cap V_{\gamma^*}$ on the set n_γ beyond the models from $m(q)$. Therefore, using the definition of Φ-master condition repeatedly, gradually strengthening the P_γ-name for the condition $w(q)(V_\gamma)$, it is possible to find a name $w(s)(V_\gamma)$ forced by $q \restriction \gamma$ to be Φ-master for all models $N[G_\gamma]$ where $N \in n_\gamma$.

Case 4. To define $w(s)(V_\gamma)$ for the transitive nodes V_γ on $m(s)$ above the model M, just let $w(s)(V_\gamma) = w(p)(V_\gamma)$.

It is not difficult to verify now that $s = \langle m(s), w(s)\rangle$ is a condition, and it is a lower bound of p, q in P_δ. □

For every countable elementary submodel $M \prec H_\theta$ containing κ, f and an ordinal $\beta \in I$ let $p_M \in P_\beta$ be the unique condition with $m(p) = \{M \cap V_\beta\}$.

Claim 6.12. *Let $\beta \in I$ be an ordinal. The poset P_β is Φ-proper, and for every countable elementary submodel $M \prec H_\theta$ containing the ordinal β as well as κ, f, the condition p_M is a Φ-master condition for M.*

Proof. This is proved by induction on $\beta \in I$. Suppose that $\beta \in I$ is an ordinal below which the statement has been verified, and let $M \prec H_\theta$ be a countable elementary submodel containing β.

To verify that p_M is a master condition, suppose that $p \leq p_M$ is an arbitrary condition and $D \in M$ is an open dense subset of P_β; we must produce a condition $q \in D \cap M$ compatible with p. Strengthening p if necessary, we may assume that $p \in D$. For definiteness assume that there are some transitive nodes in $m(p)$ below $M \cap V_\beta$, and let V_δ denote the largest one of them. For definiteness assume that $w(p)(V_\delta) \neq$ trash, the other cases are simpler.

By the closure of $m(p)$ under intersections, $V_\delta \cap M \in m(p)$ holds, and by the induction hypothesis, $p \restriction \delta$ is a master condition for the model M in the poset P_δ. By the definition of the poset P_β, $p \restriction \delta \Vdash w(p)(V_\delta)$ is a master condition for $M[\dot{G}_\delta]$. Therefore, $p \restriction \delta + 1$ is a master condition for M in the poset $P_{\delta+1}$. Now, $p \restriction \delta + 1$ forces in $P_{\delta+1}$ that there is a condition $q \in D$ such that $q \restriction \delta + 1$ is in the generic filter $G_{\delta+1}$ and $m(q)$ contains all countable nodes of $m(p)$ between V_δ and M. This is clear since $q = p$ will work. Since $p \restriction \delta + 1$ is M-master, it forces that there must be such a condition q in the model M. In other words, there must be a condition $q \in M \cap D$ such that $p \restriction \delta + 1$ and $q \restriction \delta + 1$ are compatible and $m(q)$ contains all countable nodes of $m(p)$ between V_δ and M. But then, p, q are in Δ-position and therefore compatible. The proof that p_M is a master condition for the model M is complete.

To verify that p_M is a Φ-master condition, suppose that $p \in P_\beta$ is a condition below p_M; we must find an element $g \in G \cap M$ and a set $F \in M$ on $RO(P_\beta)$ such that $\Phi(g, F, RO(P_\beta))$ holds and such that for every condition $q \in M \cap RO(P)$, if $q \geq p$ then $q \in F$. For definiteness assume that there are some transitive nodes in $m(p)$ below $M \cap V_\beta$, and let V_δ denote the largest one of them. For definiteness, also assume that $w(p)(V_\delta) \neq$ trash, the other cases are simpler.

Let $\bar{p} = \langle m(\bar{p}), w(\bar{p})\rangle$ be the condition defined in the following way: $m(\bar{p})$ contains V_δ, all the countable nodes of $m(p)$ between V_δ and M, and the intersections of these nodes with V_δ. The map $w(\bar{p})$ returns only one nontrivial value, at V_δ, where it indicates the sum of all conditions in $f(\delta)$ which are Φ-master for all models on $m(\bar{p})$ containing P_δ and $f(\delta)$. It is clear that $\bar{p} \in M$ is a condition weaker than p. By the restriction clause of regularity, it will be enough to find the requested set F in $RO(P_\beta \restriction \bar{p})$.

By Claim 6.10, the algebra $A = RO(P_{\delta+1} \restriction \bar{p})$ can be naturally viewed as a complete subalgebra of $B = RO(P_\beta \restriction \bar{p})$. By the induction hypothesis applied at δ

and the two-step iteration Claim 6.7, the condition $p \restriction \delta+1$ is Φ-master for M and $P_{\delta+1}$. Thus, there are a set $F_0 \subset A$ and an element $g \in G$ in the model M such that $\Phi(g, F_0, A)$ holds and F_0 contains all elements of $A \cap M$ weaker than $p \restriction \delta+1$. We will show that the set F, obtained as the upwards closure of F_0 in the algebra B, has the requested properties.

Certainly $\Phi(g, F, B)$ holds by the subalgebra and closure clauses of regularity, and $F \in M$. We must verify that if $b \in B \cap M$ is weaker than p then $b \in F$. To this end, consider the lower projection function $\mathrm{proj} : B \to A$ defined by $\mathrm{proj}(b) = \sum\{a \in A\colon a \leq b\} \leq b$. We claim that if $b \in B \cap M$ is weaker than p, then $\mathrm{proj}(b) \in A \cap M$ is weaker than $p \restriction \delta + 1$. This will complete the proof as then $\mathrm{proj}(b) \leq b$ must be an element of F_0 and so $b \in F$.

Suppose for contradiction that $p \restriction \delta + 1$ is not stronger than $\mathrm{proj}(b)$. Then $p \restriction \delta+1$ must be compatible in B with $1-b$ by the definition of projection. Since $p \restriction \delta + 1$ is a master condition for M by the first part of the proof of the claim, this means that there must be a condition $q \leq \bar{p}$ in the poset P_β and the model M such that $q \restriction \delta + 1$ is compatible with $p \restriction \delta + 1$, and q is below $1 - b$. But then, p and q are in Δ-position, as $m(q)$ contains all nodes on $m(\bar{p})$ and so all countable nodes on $m(p)$ between V_δ and M. The conditions p, q are therefore compatible by Claim 6.11. Their common lower bound will be below both p and $1-b$, contradicting the assumption that $p \leq b$. □

Now, suppose that κ is a supercompact cardinal and $f\colon \kappa \to V_\kappa$ is the Laver prediction function. Let $P = P_\kappa$ be the Φ-proper forcing obtained from the function f using the scheme above. A routine argument now shows that P forces Φ-PFA to hold. □

The iteration theorem allows us to finally prove some interesting consistency results.

Theorem 6.13. *YPFA implies*

(1) *PID;*
(2) *there are only five cofinal types of directed posets of size* $\aleph_1$*;*
(3) *all Aronszajn trees are special;*
(4) $\mathfrak{c} = \aleph_2$.

YPFA does not imply

(5) *OCA;*
(6) *c.c.c. is productive.*

Items (4) and (6) are due to Todorcevic.

Proof. YPFA implies PID since the PID posets are Y-proper by Theorem 4.6. YPFA implies the five cofinal types statement since the posets used for it are ideal-based [16] and the ideal-based posets are Y-proper by Theorem 4.4. Todorcevic remarked that the classification of cofinal types of size $\aleph_1$ is a consequence of the conjunction of PID and $\mathfrak{p} > \aleph_1$, which are both consequences of PFA.

The proof of $\mathfrak{c} = \aleph_2$ follows the PFA argument with a small change. PID implies that $\mathfrak{b} \leq \aleph_2$ [13], and so PFA implies $\mathfrak{b} = \aleph_2$. Similarly to the oldest proof that PFA implies $\mathfrak{c} = \aleph_2$, the argument is concluded by showing that $\mathfrak{c} = \mathfrak{b}$. This is quite involved and we only outline the main points. Fix a modulo finite increasing, unbounded sequence $\vec{y} = \langle y_\alpha\colon \alpha \in \omega_1\rangle$. For every $x \in 2^\omega$, consider a poset P_x

which is the iteration $P^0_x * \dot{P}^1_x$. The first step of the iteration is the $\in$-collapse of $\aleph_2$ to $\aleph_1$; the main point is that it is Y-proper and preserves unbounded sequences. The second step of the iteration uses $\vec{y}$ and the fact that it remains unbounded to code the point $x \in 2^\omega$ into a closed unbounded subset of ω_2^V via a c.c.c. poset; the main point is that it is in fact even Y-c.c. Thus, the iteration P_x is Y-proper. An application of YPFA to the poset P_x yields coding of the point x by an ordinal $< \omega_2$, proving that $\mathfrak{c} = \aleph_2$. Details (except for showing that $\dot{P}^1_x$ is forced to be Y-c.c.) can be found in [2, Theorem 3.16].

For (5) and (6), Todorcevic supplied an argument using entangled linear orders. Suppose that there is a supercompact cardinal and the continuum hypothesis holds. Then, there is a set A of reals of size $\aleph_1$ which forms an entangled linear ordering [8, Theorem 1]. Note that entangledness is a statement about nonexistence of uncountable anticliques in a certain graph on A^n and the graph in the case that A is a set of reals is open. Force YPFA with a Y-proper ordering. In the resulting model, A is still entangled, thus OCA fails and also by [8, Theorem 6] c.c.c. is not productive. □

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