Asymptotic analysis in thermodynamics of viscous fluids

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Thermal systems in equilibrium:

State variables: ϱ, ϑ

Thermodynamic functions: internal energy $e = e(\varrho, \vartheta)$, pressure $p = p(\varrho, \vartheta)$, entropy $s = s(\varrho, \vartheta)$

- the entropy s can be viewed as an increasing function of the total energy e,

$$\frac{\partial s}{\partial e} = \frac{1}{\vartheta} > 0$$

- maximization of the total entropy

$$S = \int \varrho s \, \mathrm{d}x$$

  over the set of all allowable states of the system yields the equilibrium state provided the system is mechanically and thermally insulated
- (**Third law of thermodynamics**) the entropy tends to zero when the absolute temperature tends to zero
- the entropy remains constant in those processes, where the material responds *elastically*

Gibbs' equation and thermodynamic stability:

Gibbs' equation:

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\left(\frac{1}{\varrho}\right)$$

Hypothesis of thermodynamic stability:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0$$

$$\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

for any ϱ, $\vartheta > 0$

Balance laws:

Balance law (weak form):

$$\int_0^T \int_\Omega \Big(d(t,x)\partial_t \varphi(t,x) + \mathbf{F}(t,x)\cdot\nabla_x \varphi(t,x) \Big) \ \mathrm{d}x \ \mathrm{d}t + < s; \varphi >$$

$$= - \int_\Omega d_0(x)\varphi(0,x) \ \mathrm{d}x + \int_0^T \int_{\partial\Omega} F_b(x)\varphi(t,x) \ \mathrm{d}S_x \ \mathrm{d}t$$

for any test function $\varphi \in C^\infty_c([0,T) \times \overline{\Omega})$

Balance law (strong form):

$$\partial_t d + \mathrm{div}_x F = s \text{ in } (0,T) \times \Omega, \ d(0,\cdot) = d_0, \ \mathbf{F} \cdot \mathbf{n}|_{\partial\Omega} = F_b$$

Navier-Stokes-Fourier system:

Equation of continuity:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation:

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

Entropy equation:

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Boundary conditions:

Complete slip:
$\mathbf{u} \cdot \mathbf{n}\|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}\|_{\partial\Omega} = 0$
No-slip:
$\mathbf{u}\|_{\partial\Omega} = 0$
No energy flux:
$\mathbf{q} \cdot \mathbf{n}\|_{\partial\Omega} = 0$

Diffusion flux, transport coefficients:

Newton's rheological law:

$$\mathbb{S} = \mu\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I},$$

with the *shear viscosity coefficient* μ and the *bulk viscosity coefficient* η

Fourier's law:

$$\mathbf{q} = -\kappa \nabla_x \vartheta,$$

where κ is the *heat conductivity coefficient*

Energetically closed systems

- $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, $s = s(\varrho, \vartheta)$ are given functions satisfying Gibbs' equation and hypothesis of thermodynamic stability
- the state of the fluid at a given instant $t \in (0, T)$ and a spatial position $x \in \Omega \subset R^3$ is determined through the state variables $\varrho = \varrho(t, x)$, $\vartheta = \vartheta(t, x)$, and $\mathbf{u} = \mathbf{u}(t, x)$. The density ϱ is a non-negative measurable function, the absolute temperature ϑ is a measurable function satisfying $\vartheta(t, x) > 0$ for a.a. $(t, x) \in (0, T) \times \Omega$
- the total mass is a constant of motion,

$$M(t) = \int_\Omega \varrho(t, \cdot) \ \mathrm{d}x = \int_\Omega \varrho_0 \ \mathrm{d}x = M_0 \text{ for a.a. } t \in (0, T),$$

and so is the total energy

$$E(t) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x$$

$$= \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) - \varrho_0 F \right) \ \mathrm{d}x \text{ for a.a. } t \in (0, T)$$

- the time evolution of the system is governed by the following system of equations (integral identities):

CONSERVATION OF MASS (RENORMALIZED):

$$\int_0^T \int_\Omega \Big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi + \Big(b(\varrho) - b'(\varrho)\varrho\Big)\mathrm{div}_x\mathbf{u}\varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$= -\int_\Omega b(\varrho_0)\varphi(0,\cdot) \ \mathrm{d}x$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, for any b, $b' \in C_c^\infty[0,\infty)$, and also for $b(\varrho) = \varrho$

Balance of momentum (weak):

$$\int_0^T \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi + p(\varrho, \vartheta) \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$\int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x \varphi - \varrho \nabla_x F \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \ \mathrm{d}x$$

for any test function $\varphi \in C^\infty_c([0,T) \times \Omega; R^3)$

If the complete slip boundary conditions are imposed, the space of admissible test functions must be extended to $C^\infty_c([0,T) \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$

Entropy balance (weak):

$$\int_0^T \int_\Omega \left(\varrho s(\varrho, \vartheta) \partial_t \varphi + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q} \cdot \nabla_x \varphi}{\vartheta} \right) \mathrm{d}x \ \mathrm{d}t + < \sigma; \varphi >$$

$$= - \int_\Omega \varrho_0 s(\varrho_0, \vartheta_0) \varphi(0, \cdot) \ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0, T) \times \overline{\Omega}; R^3)$, where the *entropy production rate* $\sigma \in \mathcal{M}^+([0, T] \times \overline{\Omega})$ satisfies

$$\sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

- the viscous stress $\mathbb{S}$ is determined by Newton's rheological law, the heat flux $\mathbf{q}$ satisfies Fourier's law

Existence of global-in-time solutions: Hypotheses:

[H1] the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ satisfy:

$$\varrho_0, \vartheta_0 \in L^\infty(\Omega),\ \mathbf{u}_0 \in L^\infty(\Omega; R^3),$$

$$\varrho_0(x) \geq 0, \vartheta(x) > 0 \text{ for a.a. } x \in \Omega$$

[H2] The potential of the driving force F belongs to $W^{1,\infty}(\Omega)$

[H3] the pressure $p = p(\varrho, \vartheta)$ is given by

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,\ a > 0,$$

where

$$P \in C^1[0, \infty),\ P(0) = 0,\ P'(Z) > 0 \text{ for all } Z \geq 0,$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} \leq c \text{ for all } Z > 0,$$

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0$$

the specific internal energy e obeys

$$e(\varrho, \vartheta) = \frac{3}{2} \frac{\vartheta^{5/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho},$$

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho},$$

with

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}$$

[H4] the transport coefficients μ, η, and κ are continuously differentiable functions of the temperature ϑ satisfying

$$\mu \in W^{1,\infty}[0,\infty),\ 0 < \underline{\mu}(1 + \vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^\alpha),$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta^\alpha),$$

where

$$1/2 \leq \alpha \leq 1;$$

and

$$0 < \underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3)$$

Global-in-time existence theorem:

Theorem

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}, \nu > 0$. Suppose that the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ satisfy hypothesis **[H1]** *and that the driving force potential F obeys* **[H2]**. *Furthermore, let the thermodynamic functions p, e, and s be as in* **[H3]**, *while the transport coefficients μ, η, and κ satisfy* **[H4]**.
Then the Navier-Stokes-Fourier system admits a weak solution ϱ, ϑ, and $\mathbf{u}$ *belonging to the class:*

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega)),\ \vartheta \in L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)),$$

$$\mathbf{u} \in L^2(0,T;W^{1,q}(\Omega;R^3)),\ q = \frac{8}{5-\alpha}.$$

A priori estimates:

Total dissipation balance:

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta) - \varrho F \right) (\tau, \cdot) \ \mathrm{d}x + \overline{\vartheta} \sigma \left[[0, \tau] \times \overline{\Omega} \right]$$

$$= \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) - \overline{\vartheta} \varrho_0 s(\varrho_0, \vartheta_0) - \varrho_0 F \right) \ \mathrm{d}x$$

Existence theory: A priori bounds:

Energy bounds:

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega)),\ \vartheta \in L^\infty(0,T;L^4(\Omega))$$
$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega;R^3))$$

Dissipation estimates:

$$\vartheta \in L^2(0,T;W^{1,2}(\Omega)),\ \mathbf{u} \in L^2((0,T;W^{1,2}(\Omega;R^3))$$

Pressure estimates:

$$p(\varrho,\vartheta) \in L^q((0,T)\times\Omega)),\ q>1$$

COMPACTNESS OF TEMPERATURE:

Monotonicity of the entropy:

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \Big) (\vartheta_\varepsilon - \vartheta) \ \mathrm{d}x \ \mathrm{d}t \geq 0$$

Entropy equation:

$$\overline{\varrho s(\varrho, \vartheta) \vartheta} = \overline{\varrho s(\varrho, \vartheta)} \ \vartheta$$

Renormalized continuity equation:

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho) \mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

Young measure identity

$$\nu_{t,x}[b(\varrho) h(\vartheta)] = \nu_{t,x}[b(\varrho)] \ \nu_{t,x}[h(\vartheta)]$$

Compactness of density:

Oscillations defect measure:

$$\sup_{k\geq 1}\left[\limsup_{\varepsilon\to 0}\int_0^T\int_\Omega |T_k(\varrho_\varepsilon)-T_k(\varrho)|^\gamma \ \mathrm{d}x \ \mathrm{d}t\right]<\infty,\ \gamma>8/3$$

$$T_k(z)=\min\{z,k\}$$

Renormalized equation:

$$\int_\Omega \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho) \right) (\tau, \cdot) \ \mathrm{d}x$$

$$+ \int_0^\tau \int_\Omega \left(\overline{T_k(\varrho)\mathrm{div}_x \mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \int_\Omega \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho) \right) (0, \cdot) \ \mathrm{d}x +$$

$$\int_0^\tau \int_\Omega \left(T_k(\varrho)\mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} \right) \ \mathrm{d}x \ \mathrm{d}t$$

Result of Lions on the effective viscous pressure:

$$\overline{\mathcal{R} : [\mathbb{S}] T_k(\varrho)} - \overline{\mathcal{R} : [\mathbb{S}]}\ \overline{T_k(\varrho)} = \overline{p(\varrho) T_k(\varrho)} - \overline{p(\varrho)}\ \overline{T_k(\varrho)} \geq 0$$

where

$$\mathcal{R} : [\mathbb{S}] = \partial_{x_i} \Delta^{-1} \partial_{x_j} \left[\mu(\vartheta) \left(\partial_{x_i} u_j + \partial_{x_j} u_i - \frac{2}{3} \mathrm{div}_x \mathbf{u} \delta_{i,j} \right) \right]$$

Compactness of commutators:

$$\| \mathcal{R} : [\mu \mathcal{U}] - \mu \mathcal{R} : \mathcal{U} \|_{W^{\alpha,p}} \leq c \|\mathcal{U}\|_{L^2} \|\mu\|_{W^{1,2}}, \ \alpha > 0, \ p > 1$$

$$\mathcal{R} : \mathcal{U} = \frac{4}{3} \mathrm{div}_x \mathbf{u}$$

Equilibrium states:

- equilibrium solutions minimize the entropy production;
- equilibrium solutions maximize the total entropy of the system in the class of all admissible states;
- all solutions to the evolutionary system driven by a conservative time-independent external force tend to an equilibrium for large time.

Total dissipation balance:

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \overline{\vartheta}\varrho s(\varrho,\vartheta) - \varrho F \right)(\tau,\cdot)\ \mathrm{d}x + \overline{\vartheta}\sigma\left[[0,\tau]\times\overline{\Omega}\right]$$

$$= \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) - \overline{\vartheta}\varrho_0 s(\varrho_0,\vartheta_0) - \varrho_0 F \right)\ \mathrm{d}x$$

Static states:

$$\nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) = \tilde{\varrho} \nabla_x F, \ \ \tilde{\varrho} \geq 0, \ \ \tilde{\vartheta} = \mathrm{const} > 0 \text{ in } \Omega,$$

$$\int_\Omega \tilde{\varrho} \ \mathrm{d}x = M_0, \int_\Omega \left(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \overline{\vartheta} \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \right) \ \mathrm{d}x = D_\infty [\overline{\vartheta}]$$

Positivity of the static density distribution:

[P]

$$\lim_{\varrho \to 0} \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0 \text{ for any fixed } \vartheta > 0.$$

Theorem

Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, and s are continuously differentiable in $(0,\infty)^2$, and that they satisfy Gibbs' equation , hypothesis of thermodynamic stability, together with condition [P]. Let $F \in W^{1,\infty}(\Omega)$. Then for given constants $M_0 > 0$, E_0, there is at most one solution $\tilde{\varrho}$, $\tilde{\vartheta}$ of static problem in the class of locally Lipschitz functions subjected to the constraints

$$\int_\Omega \tilde{\varrho} \, \mathrm{d}x = M_0, \ \int_\Omega \Big(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \Big) \, \mathrm{d}x = E_0. \tag{1}$$

In addition, $\tilde{\varrho}$ is strictly positive in Ω, and, moreover,

$$\int_\Omega \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) \, \mathrm{d}x \geq \int_\Omega \varrho s(\varrho, \vartheta) \, \mathrm{d}x$$

for any couple $\varrho \geq 0$, $\vartheta > 0$ of measurable functions satisfying (1).

CONSERVATIVE SYSTEMS, ATTRACTORS:

- $\Omega \subset R^3$ a bounded Lipschitz domain
- the structural hypotheses [H1] - [H4], with [P], are satisfied
- the (initial) values of the total mass M_0, the energy E_0, and the entropy S_0 are given

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For any $\varepsilon > 0$ there exists $T = T(\varepsilon)$ such that

$$\left\{ \begin{array}{c} \|(\varrho \mathbf{u})(t, \cdot)\|_{L^{5/4}(\Omega;R^3)} \leq \varepsilon, \\ \\ \|\varrho(t, \cdot) - \tilde{\varrho}\|_{L^{5/3}(\Omega)} \leq \varepsilon, \\ \\ \|\vartheta(t, \cdot) - \overline{\vartheta}\|_{L^4(\Omega)} \leq \varepsilon \end{array} \right\} \text{for a.a. } t > T(\varepsilon)$$

for any weak solution $\{\varrho, \mathbf{u}, \vartheta\}$ of the Navier-Stokes-Fourier system defined on $(0, \infty) \times \Omega$ and satisfying

$$\left\{ \begin{array}{c} \int_\Omega \varrho(t,\cdot)\ \mathrm{d}x > M_0, \\ \\ \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F\right)(t,\cdot)\ \mathrm{d}x < E_0, \\ \\ \mathrm{ess}\liminf_{t\to 0} \int_\Omega \varrho s(\varrho,\vartheta)(t,\cdot)(t,0)\ \mathrm{d}x > S_0, \end{array} \right\}$$

where $\tilde{\varrho}$, $\overline{\vartheta}$ is a solution of the static problem determined uniquely by the condition

$$\int_\Omega \tilde{\varrho}\ \mathrm{d}x = \int_\Omega \varrho\ \mathrm{d}x,$$

$$\int_\Omega \left(\tilde{\varrho} e(\tilde{\varrho},\overline{\vartheta}) - \tilde{\varrho} F\right)\ \mathrm{d}x = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F\right)\ \mathrm{d}x$$

Systems driven by a non-conservative force:

Theorem

Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Under the hypotheses [H1] - [H4], [P], let $\{\varrho, \vartheta, \mathbf{u}\}$ be a weak solution of the Navier-Stokes-Fourier system driven by an external force $\mathbf{f} = \mathbf{f}(x)$ on the time interval $[T_0, \infty)$, where $\mathbf{f} \not\equiv \nabla_x F$.

Then

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) (t, \cdot) \ \mathrm{d}x \to \infty \text{ as } t \to \infty.$$

Theorem

Assume that $\mathbf{f} = \mathbf{f}(t,x)$, $\mathbf{f} \in L^\infty((0,T) \times \Omega; R^3)$.
The either

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right)(t, \cdot) \ \mathrm{d}x \to \infty \text{ as } t \to \infty$$

or

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right)(t, \cdot) \ \mathrm{d}x \leq E_\infty \text{ for a.a. } t > T_0$$

for a certain constant E_∞. *Moreover, in the latter case, each sequence* $\tau_n \to \infty$ *contains a subsequence (not relabeled) such that*

$$\mathbf{f}(\tau_n + \cdot, \cdot) \to \nabla_x F \text{ weakly-(*) in } L^\infty((0,1) \times \Omega; R^3)$$

for a certain $F = F(x)$, $F \in W^{1,\infty}(\Omega)$ *that, in general, may depend on the choice of* $\{\tau_n\}_{n=1}^\infty$.

Highly oscillating driving forces

- $\Omega \subset R^3$ a bounded (Lipschitz) domain
- $$\mathbf{f} = \omega(t^\beta)\mathbf{w}(x),\ \beta > 2$$
$$\omega \in L^\infty(0,\infty),\ \omega \neq 0,\ \sup_{\tau>0} \left| \int_0^\tau \omega(t)\ \mathrm{d}t \right| < \infty$$

$$\varrho\mathbf{u}(t,\cdot) \to 0 \text{ in } L^p(\Omega; R^3)$$

$$\varrho(t,\cdot) \to \overline{\varrho} \text{ in } L^p(\Omega),\ M_0 = \overline{\varrho}|\Omega|$$

$$\vartheta(t,\cdot) \to \overline{\vartheta} \text{ in } L^p(\Omega)$$

as $t \to \infty$

Scaled Navier-Stokes-Fourier system:

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0,$$

$$\mathrm{Sr}\ \partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2} \nabla_x p = \frac{1}{\mathrm{Re}} \mathrm{div}_x \mathbb{S} + \frac{1}{\mathrm{Fr}^2} \varrho \nabla_x F,$$

$$\mathrm{Sr}\ \partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) + \frac{1}{\mathrm{Pe}} \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma,$$

together with

$$\mathrm{Sr}\ \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{\mathrm{Ma}^2}{2} \varrho |\mathbf{u}|^2 + \varrho e - \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \varrho F \right) \mathrm{d}x = 0,$$

$$\sigma \geq \frac{1}{\vartheta} \left(\frac{\mathrm{Ma}^2}{\mathrm{Re}} \mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}} \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Boundary conditions:

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}=0,\ [\mathbb{S}\mathbf{n}]\times\mathbf{n}|_{\partial\Omega}=0,\ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega}=0$$

Characteristic numbers:

△ Symbol	△ Definition	△ Name
Sr	$L_{\rm ref}/(T_{\rm ref}U_{\rm ref})$	Strouhal number
Ma	$U_{\rm ref}/\sqrt{p_{\rm ref}/\varrho_{\rm ref}}$	Mach number
Re	$\varrho_{\rm ref}U_{\rm ref}L_{\rm ref}/\mu_{\rm ref}$	Reynolds number
Fr	$U_{\rm ref}/\sqrt{L_{\rm ref}f_{\rm ref}}$	Froude number
Pe	$p_{\rm ref}L_{\rm ref}U_{\rm ref}/(\vartheta_{\rm ref}\kappa_{\rm ref})$	Péclet number

Low Mach number limit on "large" domains:

Scaled Navier-Stokes-Fourier system:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \sigma_\varepsilon$$

with the total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\varepsilon} \left(\frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) (t, \cdot) \, \mathrm{d}x = 0$$

Newton's rheological law:

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathbb{I} \mathrm{div}_x \mathbf{u} \right) + \eta(\vartheta) \mathbb{I} \ \mathrm{div}_x \mathbf{u},$$

Fourier's law:

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta,$$

Entropy production rate:

$$\sigma_\varepsilon \geq \frac{1}{\vartheta} \left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta} |\nabla_x \vartheta|^2 \right) \geq 0.$$

Conservative boundary conditions:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$$

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$$

Ill-prepared initial data:

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^1_{0,\varepsilon}, \ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^1_{0,\varepsilon}$$

$$\overline{\varrho}, \ \overline{\vartheta} > 0, \ \int_{\Omega_\varepsilon} \varrho^1_{0,\varepsilon} \ {\rm d}x = \int_{\Omega_\varepsilon} \vartheta^1_{0,\varepsilon} \ {\rm d}x = 0 \text{ for all } \varepsilon > 0$$

$$\{\varrho^1_{0,\varepsilon}\}_{\varepsilon>0}, \ \{\vartheta^1_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^2 \cap L^\infty(\Omega_\varepsilon)$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}$$

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2 \cap L^\infty(\Omega_\varepsilon; R^3)$$

Spatial domains:

$\Omega \subset R^3$ is an unbounded domain with a compact smooth boundary $\partial\Omega$

$$\Omega_\varepsilon = B_{r(\varepsilon)} \cap \Omega$$

where $B_{r(\varepsilon)}$ is a ball centered at zero with a radius $r(\varepsilon)$, with

$$\lim_{\varepsilon \to 0} \varepsilon r(\varepsilon) = \infty$$

Target system:

$$\varrho_\varepsilon \to \overline{\varrho},\ \vartheta_\varepsilon \to \overline{\vartheta} \ \text{strongly in } L^p$$

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ \text{weakly in } L^2$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \ \text{weakly in } L^p$$

$$\mathrm{div}_x \mathbf{U} = 0$$

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U})\Big) + \nabla_x \Pi = \mathrm{div}_x(\mu(\overline{\vartheta})\nabla_x \mathbf{U})$$

$$\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta})\Big(\partial_t \Theta + \mathrm{div}_x(\Theta \mathbf{U})\Big) - \mathrm{div}_x(\kappa(\overline{\vartheta})\nabla_x \vartheta) = 0$$

Stability of static equilibria in the low Mach number limit

Total dissipation balance:

$$\int_{\Omega_\varepsilon} \left(\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \Big[H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho_\varepsilon - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big] \right)(\tau, \cdot) \, \mathrm{d}x$$

$$+ \frac{\overline{\vartheta}}{\varepsilon^2} \sigma_\varepsilon \Big[[0, \tau] \times \overline{\Omega}_\varepsilon \Big] =$$

$$\int_{\Omega_\varepsilon} \left(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \Big[H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho_{0,\varepsilon} - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big] \right) \, \mathrm{d}x$$

Helmholtz function:

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \overline{\vartheta} \varrho s(\varrho, \vartheta)$$

- $\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})$ is a strictly convex function
- $\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta)$ is decreasing if $\vartheta < \overline{\vartheta}$ and increasing whenever $\vartheta > \overline{\vartheta}$ for any fixed ϱ

Coercivity of Helmholtz function:

For any

$$0 < \underline{\varrho} < \tilde{\varrho} < \overline{\varrho}$$

there exists a positive constant $\Lambda = \Lambda(\underline{\varrho}, \overline{\varrho}, \overline{\vartheta})$ such that

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \tilde{\varrho}) \frac{\partial H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} - H_{\overline{\vartheta}}(\tilde{\varrho}, \overline{\vartheta})$$

$$\geq \Lambda \left\{ \begin{array}{l} |\varrho - \tilde{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2 \text{ if } \underline{\varrho} < \varrho < \overline{\varrho},\ \overline{\vartheta}/2 < \vartheta < 2\overline{\vartheta}, \\ \\ \varrho e(\varrho, \vartheta) + \overline{\vartheta} |s(\varrho, \vartheta)| + 1 \text{ otherwise} \end{array} \right.$$

Uniform bounds for $\varepsilon \to 0$

$$h = [h]_{\rm ess} + [h]_{\rm res},\ [h]_{\rm ess} = \Psi(\varrho_\varepsilon, \vartheta_\varepsilon) h,\ [h]_{\rm res} = \Big(1 - \Psi(\varrho_\varepsilon, \vartheta_\varepsilon) \Big) h$$

$$\Psi \in C^\infty_c(0,\infty)^2,\ 0 \leq \Psi \leq 1,$$

$\Psi \equiv 1$ in an open neighborhood of the point $[\overline{\varrho}, \overline{\vartheta}]$.

Uniform bounds:

$$\mathrm{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\mathrm{ess}} \right\|_{L^2(\Omega_\varepsilon)} \leq c$$

$$\mathrm{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\mathrm{res}} \right\|_{L^{5/4}(\Omega_\varepsilon)} \leq c$$

$$\mathrm{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\mathrm{ess}} \right\|_{L^2(\Omega_\varepsilon)} \leq c$$

$$\mathrm{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right]_{\mathrm{ess}} \right\|_{L^4(\Omega_\varepsilon)} \leq c$$

$$\mathrm{ess}\sup_{t\in(0,T)} \left\| \sqrt{\varrho}\mathbf{u} \right\|_{L^2(\Omega_\varepsilon;R^3)} \leq c$$

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega})} \leq \varepsilon^2 c$$

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}\Omega_\varepsilon;R^3)} \, \mathrm{d}t \leq c$$

$$\int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)} \, \mathrm{d}t \leq c$$

LIGHTHILL'S ACOUSTIC EQUATION:

"time lifting" Σ_ε of the measure σ_ε:

$$< \Sigma_\varepsilon ; \varphi > = < \sigma_\varepsilon ; I[\varphi] >$$

$$I[\varphi](t,x) = \int_0^t \varphi(z,x) \ {\rm d}z \ \text{for any} \ \varphi \in L^1(0,T; C(\overline{\Omega}_\varepsilon))$$

Lighthill's equation:

$$\varepsilon \partial_t Z_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{F}^1_\varepsilon,$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \omega \nabla_x Z_\varepsilon = \varepsilon \Big(\mathrm{div}_x \mathbb{F}^2_\varepsilon + \nabla_x F^3_\varepsilon + \frac{A}{\varepsilon^2 \omega} \nabla_x \Sigma_\varepsilon \Big),$$

supplemented with the homogeneous Neumann boundary conditions

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0$$

where

$$Z_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega} \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) + \frac{A}{\varepsilon \omega} \Sigma_\varepsilon, \ \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

$$\mathbf{F}^1_\varepsilon = \frac{A}{\omega} \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) \mathbf{u}_\varepsilon + \frac{A}{\omega} \frac{\kappa \nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon}$$

$$\mathbb{F}^2_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$$

$$F^3_\varepsilon = \omega \left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2} \right) + A \varrho_\varepsilon \left(\frac{s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^2} \right) - \left(\frac{p(\varrho_\varepsilon, \vartheta_\varepsilon) - p(\overline{\varrho}, \overline{\vartheta})}{\varepsilon^2} \right)$$

Acoustic potential:

Neumann Laplacean:

$$\Delta_N,\ \Delta_N[v] = \Delta v,\ \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0,\ v(x) \to 0 \text{ as } |x| \to \infty$$

$$\mathcal{D}(\Delta_N) = \{ w \in L^2(\Omega) \mid w \in W^{2,2}(\Omega),\ \nabla_x w \cdot \mathbf{n}|_{\partial\Omega} = 0 \}$$

Limiting absorption principle:

$$\sup_{\lambda \in C, 0 < \alpha \leq \mathrm{Re}[\lambda] \leq \beta < \infty,\ \mathrm{Im}[\lambda] \neq 0} \left\| \mathcal{V} \circ (-\Delta_N - \lambda)^{-1} \circ \mathcal{V} \right\|_{\mathcal{L}[L^2(\Omega); L^2(\Omega)]} \leq c_{\alpha,\beta}$$

$$\mathcal{V}(x) = (1 + |x|^2)^{-\frac{s}{2}},\ s > 1$$

Acoustic potential:

$$\Phi_\varepsilon = \Delta_N^{-1}[\mathrm{div}_x \mathbf{V}_\varepsilon],$$

$$\varepsilon \partial_t Z_\varepsilon + \Delta_N \Phi_\varepsilon = \varepsilon \mathrm{div}_x \mathbf{F}^1_\varepsilon,$$

$$\varepsilon \partial_t \Phi_\varepsilon + \omega Z_\varepsilon = \varepsilon \Delta_N^{-1} \mathrm{div}_x \mathrm{div}_x \mathbb{F}^2_\varepsilon.$$

Duhamel's formula:

$$\Phi_\varepsilon(t, \cdot)$$

$$= \exp\left(\pm \mathrm{i}\frac{t}{\varepsilon}\sqrt{-\Delta_N}\right)\left[\Delta_N[h^1_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[h^2_\varepsilon] \pm \mathrm{i}\left(\Delta_N[h^3_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[h^4_\varepsilon]\right)\right]$$

$$+ \int_0^t \exp\left(\pm \mathrm{i}\frac{t-s}{\varepsilon}\sqrt{-\Delta_N}\right)\left[\Delta_N[H^1_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[H^2_\varepsilon]\right.$$

$$\left.\pm \mathrm{i}\left(\Delta_N[H^3_\varepsilon] + \frac{1}{\sqrt{-\Delta_N}}[H^4_\varepsilon]\right)\right] \ \mathrm{d}s$$

with

$$\{h^i_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega),$$

$$\{H^i_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^2((0,T)\times\Omega)$$

A result of Kato:

Theorem

Let A be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin R,\ v \in \mathcal{D}(A^*),\ \|v\|_X = 1} \|A \circ R_H[\lambda] \circ A^*[v]\|_X < \infty.$$

Then

$$\sup_{w \in X,\ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \|A \exp(-\mathrm{i}tH)[w]\|_X^2 \ \mathrm{d}t \leq \Gamma^2.$$

Application of Kato's theorenm:

$$X = L^2(\Omega),\ H = \sqrt{-\Delta_N},\ A[v] = \varphi G(-\Delta_N)[v],\ v \in X$$

$G \in C_c^\infty(0,\infty),\ \varphi \in C_c^\infty(\Omega)$ are given functions