INSTITUTE OF MATHEMATICS

Weak Fraïssé categories

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Abstract

We develop the theory of weak Fraïssé categories, where the crucial concept is the weak amalgamation property, discovered relatively recently in model theory. We show that, in a suitable framework, every weak Fraïssé category has its unique *limit*, a special object in a bigger category, characterized by certain variant of injectivity. This significantly extends the known theory of Fraïssé limits.

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1 Introduction

We develop category-theoretic framework for the theory of limits of weak Fraïssé classes. Fraïssé theory belongs to the folklore of model theory, however actually it can be easily formulated in pure category theory. The crucial point is the notion of *amalgamation*, saying that two embeddings of a fixed object can be joined by further embeddings into a single one. More precisely, for every two arrows f, g with the same domain there should exist compatible arrows f', g' with the same codomain, such that $f' \circ f = g' \circ g$. A significant relaxing of the amalgamation property, called the *weak amalgamation property* has been discovered by Ivanov [4] and later independently by Kechris and Rosendal [5] during their study of generic automorphisms in model theory. It turns out that the weak amalgamation property is sufficient for constructing special unique objects satisfying certain variant of homogeneity. We show how do it in pure category theory. We partially rely on the concepts and results of [7]. Our goal is to obtain a general result on the existence of special (called *generic*) objects that are characterized up to isomorphisms in terms of certain injectivity property.

2 Preliminaries

We recall some basic definitions concerning categories. Let $\mathfrak{K}$ be a category. The class of $\mathfrak{K}$-objects will be denoted by $\mathrm{Obj}(\mathfrak{K})$. Given $a, b \in \mathrm{Obj}(\mathfrak{K})$, the set of all arrows from a to b will be denoted by $\mathfrak{K}(a, b)$. The identity of a $\mathfrak{K}$-object a will be denoted by id_a. We will use the letter $\mathfrak{K}$ to denote the class of all $\mathfrak{K}$-arrows. In other words,

$$\mathfrak{K} = \bigcup_{a,b \in \mathrm{Obj}(\mathfrak{K})} \mathfrak{K}(a, b).$$

One of the axioms of a category says that $\mathfrak{K}(a, b) \cap \mathfrak{K}(a', b') = \emptyset$ whenever $\langle a, b \rangle \neq \langle a', b' \rangle$. Thus, given $f \in \mathfrak{K}$, there are uniquely determined objects a, b such that $f \in \mathfrak{K}(a, b)$. In this case a is called the *domain* of f, denoted by $\mathrm{dom}(f)$, while b is called the *co-domain* of f, denoted by $\mathrm{cod}(f)$. The composition of arrows f and g will be denoted by $f \circ g$. The composition makes sense if and only if $\mathrm{dom}(f) = \mathrm{cod}(g)$. By a *sequence* in a category $\mathfrak{K}$ we mean a covariant functor $\vec{x} \colon \omega \to \mathfrak{K}$ and we denote by x_n the object $\vec{x}(n)$ and by x_n^m the bonding arrow from x_n to x_m (i.e., $x_n^m = \vec{x}(\langle n, m \rangle)$), where $n \leq m$. Recall that ω is treated here as a poset category, therefore $\langle n, m \rangle$ is the unique arrow from n to m provided that $m \geq n$.

Following [7], we say that a subcategory $\mathfrak{S} \subseteq \mathfrak{K}$ is *dominating* if the following conditions are satisfied.

(C) For every $x \in \mathrm{Obj}(\mathfrak{K})$ there is $f \in \mathfrak{K}$ such that $\mathrm{dom}(f) = x$ and $\mathrm{cod}(f) \in \mathrm{Obj}(\mathfrak{S})$.

(D) For every $y \in \mathrm{Obj}(\mathfrak{S})$, for every $\mathfrak{K}$-arrow $f \colon y \to z$ there is a $\mathfrak{K}$-arrow $g \colon z \to u$ such that $g \circ f \in \mathfrak{S}$ (in particular, $u \in \mathrm{Obj}(\mathfrak{S})$).

$$\begin{array}{ccc}
y & \xrightarrow{\;g'\;} & w \\
{\scriptstyle g}\big\uparrow & & \big\uparrow{\scriptstyle f'} \\
z' & \xrightarrow{\;f\;} & x
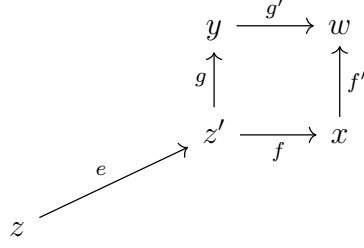\end{array}$$

$$z \xrightarrow{\;e\;} z'$$

Figure 1: Cofinal amalgamation

A subcategory $\mathfrak{S}$ satisfying condition (C) is called *cofinal*. We shall later need a weakening of domination. Namely, we say that $\mathfrak{S} \subseteq \mathfrak{K}$ is *weakly dominating* if (D) is replaced by

(W) For every $y \in \mathrm{Obj}(\mathfrak{S})$ there exists $j\colon y \to y'$ in $\mathfrak{S}$ such that for every $\mathfrak{K}$-arrow $f\colon y' \to z$ there is a $\mathfrak{K}$-arrow $g\colon z \to u$ satisfying $g \circ f \circ j \in \mathfrak{S}$.

The somewhat technical concept of weak domination will be more clear after we define weak amalgamations and formulate the first results. Note that for a full subcategory condition (D) follows from (C), therefore being weakly dominating is the same as being dominating.

For undefined notions concerning category theory we refer to Mac Lane's monograph [9].

3   Weak amalgamations

Let $\mathfrak{K}$ be a fixed category. We shall say that $\mathfrak{K}$ has the *amalgamation property* at $z \in \mathrm{Obj}(\mathfrak{K})$ if for every $\mathfrak{K}$-arrows $f\colon z \to x$, $g\colon z \to y$ there exist $\mathfrak{K}$-arrows $f'\colon x \to w$, $g'\colon y \to w$ satisfying $f' \circ f = g' \circ g$. Recall that $\mathfrak{K}$ has the *amalgamation property* (briefly: AP) if it has the amalgamation property at every $z \in \mathrm{Obj}(\mathfrak{K})$. A natural weakening is as follows. Namely, we say that $\mathfrak{K}$ has the *cofinal amalgamation property* (briefly: CAP) if for every $z \in \mathrm{Obj}(\mathfrak{K})$ there exists a $\mathfrak{K}$-arrow $e\colon z \to z'$ such that $\mathfrak{K}$ has the amalgamation property at z' (see Fig. 1).

Proposition 3.1. *A category has the cofinal amalgamation property if and only if it has a dominating subcategory with the amalgamation property.*

Proof. Assume $\mathfrak{K}$ has the CAP and let $\mathfrak{K}_0$ be the full subcategory of $\mathfrak{K}$ such that

$$\mathrm{Obj}(\mathfrak{K}_0) = \{z \in \mathrm{Obj}(\mathfrak{K})\colon \mathfrak{K} \text{ has the AP at } z\}.$$

We check that $\mathfrak{K}_0$ dominates $\mathfrak{K}$. CAP says that $\mathfrak{K}_0$ is cofinal in $\mathfrak{K}$, that is, (C) holds. As $\mathfrak{K}_0$ is full, (D) follows from (C).

Now suppose that $\mathfrak{S}$ is a dominating subcategory of $\mathfrak{K}$ and $\mathfrak{S}$ has the AP. Fix $z \in \mathrm{Obj}(\mathfrak{K})$ and using (C) choose a $\mathfrak{K}$-arrow $e\colon z \to u$ such that $u \in \mathrm{Obj}(\mathfrak{S})$. Fix

$\mathfrak{K}$-arrows $f\colon u \to x$, $g\colon u \to y$. Using (D), find $\mathfrak{K}$-arrows $f'\colon x \to x'$, $g'\colon y \to y'$ such that $f' \circ f, g' \circ g \in \mathfrak{S}$. Applying the AP, find $\mathfrak{S}$-arrows $f''\colon x' \to w$, $g''\colon y' \to w$ such that $f'' \circ f' \circ f = g'' \circ g' \circ g$. This shows that $\mathfrak{K}$ has the AP at u. □

The above proposition shows that, from the category-theoretic point of view, cofinal AP is not much different from AP, as long as we agree to restrict attention to subcategories. Below is a significant and important weakening of the cofinal AP. In model theory, it was first considered by Ivanov [4], later by Kechris and Rosendal [5], and very recently by Krawczyk and the author [6].

Let $\mathfrak{K}$ be a category. We say that $\mathfrak{K}$ has the *weak amalgamation property* (briefly: WAP)[1] if for every $z \in \mathrm{Obj}(\mathfrak{K})$ there exists a $\mathfrak{K}$-arrow $e\colon z \to z'$ such that for every $\mathfrak{K}$-arrows $f\colon z' \to x$, $g\colon z' \to y$ there are $\mathfrak{K}$-arrows $f'\colon x \to w$, $g'\colon y \to w$ satisfying

$$f' \circ f \circ e = g' \circ g \circ e.$$

In other words, the square in the diagram shown in Fig. 1 may not be commutative. The arrow e above will be called *amalgamable* in $\mathfrak{K}$. Thus, $\mathfrak{K}$ has the WAP if for every $\mathfrak{K}$-object z there exists an amalgamable $\mathfrak{K}$-arrow with domain z. Note also that saying "$\mathfrak{K}$ has the AP at $z \in \mathrm{Obj}(\mathfrak{K})$" is precisely the same as saying "id_z is amalgamable in $\mathfrak{K}$".

Lemma 3.2. *Let $e \in \mathfrak{K}$ be an amalgamable arrow. Then $i \circ e$ and $e \circ j$ are amalgamable for every compatible arrows $i, j \in \mathfrak{K}$.*

Proof. Assume $e\colon z \to u$ is amalgamable, $i\colon u \to v$, and fix $f\colon v \to x$, $g\colon v \to y$. Then $f \circ i\colon u \to x$ and $g \circ i\colon u \to y$ and therefore there are $f'\colon x \to w$ and $g'\colon y \to w$ such that $f' \circ f \circ i \circ e = g' \circ g \circ i \circ e$. This shows that $i \circ e$ is amalgamable. It is clear that $e \circ j$ is amalgamable as long as e is. □

Proposition 3.3. *Let $\mathfrak{K}$ be a category. The following properties are equivalent.*

(a) *$\mathfrak{K}$ has the weak amalgamation property.*

(b) *Every cofinal full subcategory of $\mathfrak{K}$ has the weak amalgamation property.*

(c) *$\mathfrak{K}$ has a full cofinal subcategory with the weak amalgamation property.*

(d) *$\mathfrak{K}$ is weakly dominated by a subcategory with the weak amalgamation property.*

Proof. (a) $\implies$ (b) Let $\mathfrak{S}$ be cofinal and full in $\mathfrak{K}$, and fix $z \in \mathrm{Obj}(\mathfrak{S})$. Find an amalgamable $\mathfrak{K}$-arrow $e\colon z \to v$. Using domination, we may find a $\mathfrak{K}$-arrow $i\colon v \to u$ such that $i \circ e \in \mathfrak{S}$. By Lemma 3.2, $i \circ e$ is amalgamable in $\mathfrak{K}$. We need to show that it is amalgamable in $\mathfrak{S}$. For this aim, fix $\mathfrak{S}$-arrows $f\colon v \to x$, $g\colon v \to y$. Applying the WAP, we find $\mathfrak{K}$-arrows $f'\colon x \to w$, $g'\colon y \to w$ such that $f' \circ f \circ i \circ e = g' \circ g \circ i \circ e$.

Finally, using domination again, find a $\mathfrak{K}$-arrow $j\colon w \to w'$ such that $w' \in \mathrm{Obj}(\mathfrak{S})$. Then $j \circ f'$ and $j \circ g'$ are $\mathfrak{S}$-arrows, because $\mathfrak{S}$ is a full subcategory of $\mathfrak{K}$ (this is the only place where we use fullness). Finally, we have $(j \circ f') \circ f \circ (i \circ e) = (j \circ g') \circ g \circ (i \circ e)$.

(b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) Obvious.

(d) $\Longrightarrow$ (a) Let $\mathfrak{S}$ be weakly dominating in $\mathfrak{K}$ and assume $\mathfrak{S}$ has the WAP. Fix $z \in \mathrm{Obj}(\mathfrak{K})$. First, find a $\mathfrak{K}$-arrow $i\colon z \to u$ with $u \in \mathrm{Obj}(\mathfrak{S})$. Now find an $\mathfrak{S}$-arrow $e\colon u \to v$ that is amalgamable in $\mathfrak{S}$. Let $j\colon v \to v'$ be an $\mathfrak{S}$ satisfying the assertion of (W). By Lemma 3.2, it suffices to show that $j \circ e$ is amalgamable in $\mathfrak{K}$. Fix $\mathfrak{K}$-arrows $f\colon v \to x$, $g\colon v \to y$. Using domination, find $\mathfrak{K}$-arrows $f'\colon x \to x'$, $g'\colon y \to y'$ such that $f' \circ f \circ j \in \mathfrak{S}$ and $g' \circ g \circ j \in \mathfrak{S}$. Using the fact that e is amalgamable in $\mathfrak{S}$, we find $\mathfrak{S}$-arrows $f''\colon x' \to w$ and $g''\colon y' \to w$ satisfying

$$ f'' \circ (f' \circ f \circ j) \circ e = g'' \circ (g' \circ g \circ j) \circ e. $$

Thus $j \circ e$ is amalgamable in $\mathfrak{K}$. $\qquad\square$

Example showing that fullness is relevant?

4 Weak Fraïssé sequences

We now define the crucial concept of this note. Let $\mathfrak{K}$ be a fixed category. A sequence $\vec{u}\colon \omega \to \mathfrak{K}$ will be called a *weak Fraïssé sequence* if the following conditions are satisfied.

(G1) For every $x \in \mathrm{Obj}(\mathfrak{K})$ there is n such that $\mathfrak{K}(x, u_n) \neq \emptyset$.

(G2) For every $n \in \omega$ there exists $m \geq n$ such that for every $\mathfrak{K}$-arrow $f\colon u_m \to y$ there are $k \geq m$ and a $\mathfrak{K}$-arrow $g\colon y \to u_k$ satisfying $g \circ f \circ u_n^m = u_n^k$.

$$ \cdots \longrightarrow u_n \xrightarrow{\;\;u_n^m\;\;} u_m \xrightarrow{\;\;u_m^k\;\;} u_k \longrightarrow \cdots $$

with $f\colon u_m \to y$ and $g\colon y \to u_k$.

Condition (G1) simply says that the image of $\vec{u}$ is cofinal in $\mathfrak{K}$. Condition (G2) looks a bit technical, although it is actually strictly connected with the weak amalgamation property:

Lemma 4.1. *Every category with a weak Fraïssé sequence is directed and has the weak amalgamation property.*

Proof. Let $\vec{u}$ be a weak Fraïssé sequence in $\mathfrak{K}$. Condition (G1) clearly implies that $\mathfrak{K}$ is directed, as every two $\mathfrak{K}$-objects have arrows into a single u_n for n big enough.

Fix $n \in \omega$ and let $m \geq n$ be as in (G2). We claim that u_n^m is amalgamable in $\mathfrak{K}$. Indeed, if $f_0\colon u_m \to x_0$ and $f_1\colon u_m \to x_1$ are $\mathfrak{K}$-arrows then there are $k_0, k_1 \geq m$

and $\mathfrak{K}$-arrows $g_0 \colon x_0 \to u_{k_0}$, $g_1 \colon x_1 \to u_{k_1}$ such that $g_i \circ f_i \circ u_n^m = u_n^{k_i}$ for $i = 0, 1$. Let $k \geq \max\{k_0, k_1\}$. Then

$$(u_{k_0}^k \circ g_0) \circ f_0 \circ u_n^m = u_n^k = (u_{k_1}^k \circ g_1) \circ f_1 \circ u_n^m.$$

Now, if $z \in \mathrm{Obj}(\mathfrak{K})$ and $e \colon z \to u_n$ is a $\mathfrak{K}$-arrow (which exists by (G1)), then $u_n^m \circ e$ is amalgamable, by Lemma 3.2. $\qquad\square$

Lemma 4.2. *A category with a weak Fraïssé sequence is weakly dominated by a countable subcategory, namely, the subcategory generated by the image of a weak Fraïssé sequence.*

Proof. Assume $\vec{u} \colon \omega \to \mathfrak{K}$ is a weak Fraïssé sequence in a category $\mathfrak{K}$. Let $\mathfrak{S}$ be the subcategory generated by the image of $\vec{u}$. By (G1), $\mathfrak{S}$ is cofinal in $\mathfrak{K}$. Fix $x \in \mathrm{Obj}(\mathfrak{K})$ and let $e \colon x \to z$ be a $\mathfrak{K}$-arrow, where $z \in \mathrm{Obj}(\mathfrak{S})$. Then $z = u_n$ for some $n \in \omega$. Let $m > n$ be as in condition (G2). Then, by the proof of Lemma 4.1, u_n^m is amalgamable in $\mathfrak{K}$. Thus, condition (W) is satisfied. $\qquad\square$

Lemma 4.3. *Assume $\mathfrak{S} \subseteq \mathfrak{K}$ is weakly dominating and $\vec{u} \colon \omega \to \mathfrak{S}$ is a weak Fraïssé sequence in $\mathfrak{S}$. Then $\vec{u}$ is a weak Fraïssé sequence in $\mathfrak{K}$.*

Proof. It is clear that the image of $\vec{u}$ is cofinal in $\mathfrak{K}$. It remains to check (G2).

Fix n and let $m \geq n$ be such that (G2) holds in $\mathfrak{S}$, namely:

(1) For every $\mathfrak{S}$-arrow $f \colon u_m \to y$ there are $k \geq m$ and an $\mathfrak{S}$-arrow $g \colon y \to u_k$ such that $g \circ f \circ u_n^m = u_n^k$.

Let $e \colon u_m \to a$ be such that (W) holds, namely:

(2) For every $\mathfrak{K}$-arrow $f \colon a \to x$ there is a $\mathfrak{K}$-arrow $g \colon x \to y$ such that $g \circ f \circ e \in \mathfrak{S}$.

Applying (1), find $k \geq m$ and $i \colon a \to u_k$ such that $i \circ e \circ u_n^m = u_n^k$.

Fix a $\mathfrak{K}$-arrow $f \colon u_k \to x$. Then $f \circ i \colon a \to x$, therefore applying (2) we can find a $\mathfrak{K}$-arrow $g \colon x \to y$ such that $h := g \circ f \circ i \circ e \in \mathfrak{S}$. Applying (1) to the $\mathfrak{S}$-arrow h, we find $\ell \geq k$ and an $\mathfrak{S}$ arrow $j \colon y \to u_\ell$ such that $j \circ h \circ u_n^m = u_n^\ell$. Finally, we have

$$u_n^\ell = j \circ h \circ u_n^m = j \circ g \circ f \circ i \circ e \circ u_n^m = (j \circ g) \circ f \circ u_n^k,$$

which shows (G2). $\qquad\square$

The concept of a weak Fraïssé sequence is (as the name suggests) a natural generalization of the notion of a *Fraïssé sequence* from [7], where it is required that $m = n$ in condition (G2). On the other hand, we have:

Proposition 4.4. *Assume $\mathfrak{K}$ has the amalgamation property and $\vec{u}$ is a weak Fraïssé sequence in $\mathfrak{K}$. Then $\vec{u}$ is a Fraïssé sequence in $\mathfrak{K}$.*

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Proof. Fix n and a $\mathfrak{K}$-arrow $f\colon u_n \to y$. Let $m \geq n$ be such that (G2) holds. Using the AP, find $f'\colon y \to w$ and $g\colon u_m \to w$ such that $f' \circ f = g \circ u_n^m$. Using (G2), find $h\colon w \to u_k$ with $k \geq m$ such that $h \circ g \circ u_n^m = u_n^k$. Finally, $h \circ f' \circ f = u_n^k$. □

The property of being a Fraïssé sequence is not stable under isomorphisms of sequences, unless the category in question has the AP. It turns out that the property of being weak Fraïssé is stable.

Proposition 4.5. *Assume $\vec{u}$, $\vec{v}$ are isomorphic sequences in $\mathfrak{K}$. If $\vec{u}$ is weak Fraïssé then so is $\vec{v}$.*

Proof. Let $\vec{p}\colon \vec{u} \to \vec{v}$ and $\vec{q}\colon \vec{v} \to \vec{u}$ be arrows of sequences whose compositions are equivalent to the identities. Assume $\vec{u}$ is weak Fraïssé. Obviously, $\vec{v}$ satisfies (G1). It remains to check that $\vec{v}$ satisfies (G2).

Fix $n \in \omega$ and let $n' \geq n$ be such that $q_n\colon v_n \to u_{n'}$. Let $m \geq n'$ be such that (G2) holds for $\vec{u}$, namely, for every $f\colon u_m \to x$ there are $k \geq m$ and $g\colon x \to u_k$ satisfying $g \circ f \circ u_{n'}^m = u_{n'}^k$. Let $m' \geq m$ be such that $p_m\colon u_m \to v_{m'}$. Then $m' \geq n$. We claim that m' is "suitable" for condition (G2) concerning the sequence $\vec{v}$. For this aim, fix a $\mathfrak{K}$-arrow $f\colon v_{m'} \to y$. Applying (G2) to the sequence $\vec{u}$ and to the arrow $f \circ p_m$, we obtain $k \geq m$ and a $\mathfrak{K}$-arrow $g\colon y \to u_k$ satisfying

$$g \circ f \circ p_m \circ u_{n'}^m = u_{n'}^k.$$

Let $k' \geq k$ be such that $p_k\colon u_k \to v_{k'}$. Note that

$$p_m \circ u_{n'}^m \circ q_n = v_n^{m'} \qquad \text{and} \qquad p_k \circ u_{n'}^k \circ q_n = v_n^{k'},$$

because the composition of $\vec{p}$ with $\vec{q}$ is equivalent to the identity of $\vec{v}$, as shown in the following diagram.
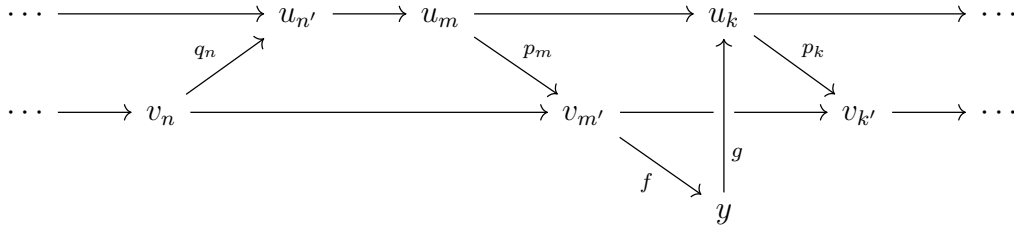
Thus

$$(p_k \circ g) \circ f \circ v_n^{m'} = p_k \circ g \circ f \circ p_m \circ u_{n'}^m \circ q_n = p_k \circ u_{n'}^k \circ q_n = v_n^{k'}.$$

This shows (G2) and completes the proof. □

We shall later see that two sequences that are weak Fraïssé in the same category are necessarily isomorphic. It remains to show their existence. We shall say that $\mathfrak{K}$ is a *weak Fraïssé category*, if it is directed, has the weak amalgamation property, and is weakly dominated by a countable subcategory.

Theorem 4.6. *Let $\mathfrak{K}$ be a category. The following properties are equivalent:*

(a) $\mathfrak{K}$ *is a weak Fraïssé category.*

(b) *There exists a weak Fraïssé sequence in $\mathfrak{K}$.*

Proof. Implication (b) $\Longrightarrow$ (a) is the content of Lemmas 4.1 and 4.2. It remains to show (a) $\Longrightarrow$ (b). By Lemma 4.3, we may assume that $\mathfrak{K}$ itself is countable.

Instead of constructing a weak Fraïssé sequence, we shall use the following simple claim, known in set theory as the Rasiowa-Sikorski Lemma:

Claim 4.7. *Let $\langle P, \leq \rangle$ be a partially ordered set and let $\mathscr{D}$ be a countable family of cofinal subsets of P. Then there exists a sequence $p_0 \leq p_1 \leq p_2 \leq \cdots$ in P such that $D \cap \{p_n \colon n \in \omega\} \neq \emptyset$ for every $D \in \mathscr{D}$.*

Let $\mathfrak{K}^{<\omega}$ denote the set of all finite sequences in $\mathfrak{K}$, that is, all covariant functors from $n = \{0, 1, \ldots, n-1\}$ into $\mathfrak{K}$, where $n \in \omega$ is arbitrary. We shall use the same convention as for infinite sequences, namely, if $\vec{x} \colon n \to \mathfrak{K}$ then we shall write x_i instead of $x(i)$ and x_i^j instead of $x(\langle i, j \rangle)$. Given $\vec{a}, \vec{b} \in \mathfrak{K}^{<\omega}$, define $\vec{a} \leq \vec{b}$ if $\vec{b}$ extends $\vec{a}$. Clearly, $\langle \mathfrak{K}^{<\omega}, \leq \rangle$ is a partially ordered set. An increasing sequence in $\mathfrak{K}^{<\omega}$ gives rise to an infinite sequence in $\mathfrak{K}$, as long as it does not stabilize. Let P be the subset of $\mathfrak{K}^{<\omega}$ consisting of all sequences $\vec{x} \colon n \to \mathfrak{K}$ such that x_i^j is amalgamable in $\mathfrak{K}$ whenever $i < j$. We shall work in the partially ordered set $\langle P, \leq \rangle$.

Given $x \in \mathrm{Obj}(\mathfrak{K})$, define $\mathscr{U}_x$ to be the set of all $\vec{x} \in P$ such that there is a $\mathfrak{K}$-arrow from x to x_i for some $i < \mathrm{dom}(\vec{x})$. As $\mathfrak{K}$ is directed and has the weak AP, $\mathscr{U}_x$ is cofinal in $\langle \mathfrak{K}^{<\omega}, \leq \rangle$. This follows from the fact that every $\mathfrak{K}$-arrow can be prolonged to an amalgamable one (see Lemma 3.2).

Fix $n \in \omega$ and $f \in \mathfrak{K}$. Define $\mathscr{V}_{n,f}$ to be the set of all $\vec{x} \in P$ such that $n + 1 \in \mathrm{dom}(x)$ and the following implication holds:

(∗) If $x_{n+1} = \mathrm{dom}(f)$ then there are $k > n$ and $g \in \mathfrak{K}$ such that $g \circ f \circ x_n^{n+1} = x_n^m$.

We check that $\mathscr{V}_{n,f}$ is cofinal in P. Fix $\vec{a} \in P$. First, we extend $\vec{a}$ by using amalgamable arrows so that $n + 1 < \mathrm{dom}(\vec{a})$. Now if $a_{n+1} \neq \mathrm{dom}(f)$ then already $\vec{a} \in \mathscr{V}_{n,f}$, so suppose $a_{n+1} = \mathrm{dom}(f)$. Let $k = \mathrm{dom}(\vec{a})$ and assume $f \colon a_{n+1} \to y$. Knowing that a_n^{n+1} is amalgamable, we can find $\mathfrak{K}$-arrows $g \colon y \to w$, $h \colon a_{k-1} \to w$ such that $g \circ f \circ a_n^{n+1} = h \circ a_{n+1}^{k-1} \circ a_n^{n+1}$. Extend $\vec{a}$ by adding the arrow h on the top, so that (∗) holds. The extended sequence is a member of $\mathscr{V}_{n,f}$. This shows that $\mathscr{V}_{n,f}$ is cofinal in $\langle P, \leq \rangle$.

Finally, observe that a sequence $\vec{p_0} \leq \vec{p_1} \leq \vec{p_2} \leq \cdots$ satisfying the assertion of the Rasiowa-Sikorski Lemma (with $\mathscr{D}$ consisting of all possible $\mathscr{U}_x$ and $\mathscr{V}_{f,n}$) yields a weak Fraïssé sequence in $\mathfrak{K}$. This completes the proof. $\qquad\square$

A weak Fraïssé sequence $\vec{u}$ is *normalized* if for every n condition (G2) holds with $m = n + 1$. More precisely, for every n, for every arrow $f \colon u_{n+1} \to y$ there are $k > n$ and and an arrow $g \colon y \to u_k$ such that $g \circ f \circ u_n^{n+1} = u_n^k$. The sequence obtained in the proof above is normalized. Clearly, every weak Fraïssé sequence

contains a subsequence that is normalized. In a normalized weak Fraïssé sequence all non-identity bonding arrows are amalgamable. It turns out that the converse is true as well:

Lemma 4.8. *Let $\vec{u}$ be a weak Fraïssé sequence in $\mathfrak{K}$ such that u_n^{n+1} is amalgamable for every $n \in \omega$. Then $\vec{u}$ is normalized.*

Proof. Fix a $\mathfrak{K}$-arrow $f\colon u_{n+1} \to y$. Let $m > n+1$ be as in condition (G2) applied to $n+1$ instead of n. Using the fact that u_n^{n+1} is amalgamable, we find $\mathfrak{K}$-arrows $h\colon y \to z$ and $f'\colon u_m \to z$ such that $h \circ f \circ u_n^{n+1} = f' \circ u_{n+1}^m \circ u_n^{n+1}$. Using (G2), we find $k \geq m$ and a $\mathfrak{K}$-arrow $g'\colon z \to u_k$ satisfying $g' \circ f' \circ u_{n+1}^m = u_{n+1}^k$. Let $g := g' \circ h$. Then

$$g \circ f \circ u_n^{n+1} = g' \circ h \circ f \circ u_n^{n+1} = g' \circ f' \circ u_{n+1}^m \circ u_n^{n+1} = u_{n+1}^k \circ u_n^{n+1} = u_n^k,$$

showing that $\vec{u}$ is normalized. $\qquad\square$

The following fact will be essential for proving a variant of homogeneity of generic objects.

Lemma 4.9. *Assume $\vec{u}$, $\vec{v}$ are normalized weak Fraïssé sequences in $\mathfrak{K}$ and $f\colon u_1 \to v_1$ is a $\mathfrak{K}$-arrow. Then there exists an isomorphism of sequences $\vec{h}\colon \vec{u} \to \vec{v}$ extending $f \circ u_0^1$.*

Proof. We construct the following (not necessarily commutative!) diagram

$$
\begin{array}{ccccccccc}
u_{k_1} & \longrightarrow & u_{k_1+1} & \longrightarrow & u_{k_2} & \longrightarrow & u_{k_2+1} & \longrightarrow & u_{k_3} & \longrightarrow & \cdots \\
& f_1 \downarrow & & g_1 \uparrow & & f_2 \downarrow & & g_2 \uparrow & \\
v_0 & \longrightarrow & v_{\ell_1} & \longrightarrow & v_{\ell_1+1} & \longrightarrow & v_{\ell_2} & \longrightarrow & v_{\ell_2+1} & \longrightarrow & \cdots
\end{array}
$$

in which $k_1 = 0$, $\ell_1 = 1$, and $f_1 = f$. Furthermore,

(1) $g_i \circ v_{\ell_i}^{\ell_i+1} \circ f_i \circ u_{k_i}^{k_i+1} = u_{k_i}^{k_{i+1}}$,

(2) $f_{j+1} \circ u_{k_{j+1}}^{k_{j+1}+1} \circ g_j \circ v_{\ell_j}^{v_{\ell_j}+1} = v_{\ell_j}^{\ell_{j+1}}$

holds for all $i, j \in \omega$. The construction is possible, because both sequences are normalized weak Fraïssé, and hence (1), (2) are straightforward applications of the normalized variant of (G2). Define

$$h_i = f_i \circ u_{k_i}^{k_i+1} \qquad \text{and} \qquad q_j = g_j \circ v_{\ell_j}^{\ell_j+1}.$$

Equations (1) and (2) give $q_i \circ h_i = u_{k_i}^{k_{i+1}}$ and $h_{j+1} \circ q_j = v_{\ell_j}^{\ell_{j+1}}$ for $i, j \in \omega$. Thus $\vec{h} = \{h_n\}_{n \in \omega}$ is an isomorphism from $\vec{u}$ to $\vec{v}$ and it extends $h_1 = f \circ u_0^1$. $\qquad\square$

Corollary 4.10. *A category may have, up to isomorphisms, at most one weak Fraïssé sequence.*

Proof. Let $\vec{u}$, $\vec{v}$ be weak Fraïssé in $\mathfrak{K}$. Replacing them by subsequences, we may assume that they are normalized. By (G1), there exists a $\mathfrak{K}$-arrow $f\colon u_1 \to v_k$ for some k. Further refining $\vec{v}$, we may assume $k = 1$. Now Lemma 4.9 yields an isomorphism from $\vec{u}$ to $\vec{v}$. $\qquad\square$

We finish this section by proving the following weakening of cofinality (in model theory usually called *universality*).

Lemma 4.11. *Let $\vec{u}$ be a weak Fraïssé sequence in $\mathfrak{K}$ and let $\vec{x}$ be a sequence in $\mathfrak{K}$ such that x_n^{n+1} is amalgamable in $\mathfrak{K}$ for every $n \in \omega$. Then there exists a $\sigma\mathfrak{K}$-arrow $\vec{e}\colon \vec{x} \to \vec{u}$.*

Proof. For simplicity, we assume that the sequence $\vec{u}$ is normalized. We construct inductively $\mathfrak{K}$-arrows $e_n\colon x_n \to u_{s(n)}$ so that the following conditions are satisfied.

(1) $u_{s(n)}^{s(n+1)} \circ e_n = e_{n+1} \circ x_n^{n+1}$.

(2) $e_n = e_n' \circ x_n^{n+2}$ for some $\mathfrak{K}$-arrow $e_n'\colon x_{n+2} \to u_{s(n)}$.

We start with $e_0 = e_0' \circ x_0^2$, where e_0' is an arbitrary $\mathfrak{K}$-arrow from x_{n+2} into some $u_{s(0)}$, which exists by (G1). Suppose $e_0, \ldots, e_n$ have been constructed. Let $f\colon x_{n+3} \to w$ and $g\colon u_{s(n)+1} \to w$ be $\mathfrak{K}$-arrows such that

$$f \circ x_{n+2}^{n+3} \circ x_{n+1}^{n+2} = g \circ u_{s(n)}^{s(n)+1} \circ e_n' \circ x_{n+1}^{n+2}.$$

This is possible, because x_{n+1}^{n+2} is amalgamable in $\mathfrak{K}$. Using (G2) and the fact that $\vec{u}$ is normalized, we find a $\mathfrak{K}$-arrow $h\colon w \to u_{s(n+1)}$, with $s(n+1) > s(n)$, such that $h \circ g \circ u_{s(n)}^{s(n)+1} = u_{s(n)}^{s(n+1)}$. Define $e_{n+1}' := h \circ f$ and $e_{n+1} := e_{n+1}' \circ x_{n+1}^{n+3}$. Then

$$e_{n+1} \circ x_n^{n+1} = h \circ f \circ x_{n+1}^{n+3} \circ x_n^{n+1} = h \circ g \circ u_{s(n)}^{s(n)+1} \circ e_n' \circ x_{n+1}^{n+2} \circ x_n^{n+1} = u_{s(n)}^{s(n+1)} \circ e_n.$$

It follows that the construction can be carried out, obtaining a $\sigma\mathfrak{K}$-arrow $\vec{e}\colon \vec{x} \to \vec{u}$ with $\vec{e} = \{e_n\}_{n\in\omega}$. $\qquad\square$

5 Generic objects

The previous section was somewhat technical, as we were working in the rather abstract category of sequences. We now prepare the setup suitable for investigating generic objects. For obvious reasons, they could be called *limits* of weak Fraïssé categories. Another possible name would be *weak Fraïssé limit*, however, in our opinion this would be inappropriate, because we only relax the axioms of Fraïssé, showing that the *limit* is still unique and may only have somewhat weaker properties. After all, a weak Fraïssé category may contain a weakly dominating Fraïssé subcategory, having the same limit. In any case, we shall avoid the word *limit*, adapting the terminology from set-theoretic forcing, calling the limit of a weak Fraïssé sequence a *generic object*. The formal definition (stated below) does not use sequences.

In this section $\mathfrak{K}$ will denote, as before, a fixed category. Now we also assume that $\mathfrak{L} \supseteq \mathfrak{K}$ is a bigger category such that $\mathfrak{K}$ is full in $\mathfrak{L}$ and the following conditions are satisfied:

(L0) All $\mathfrak{L}$-arrows are monic.

(L1) Every sequence in $\mathfrak{K}$ has co-limit in $\mathfrak{L}$ and every $\mathfrak{L}$-object is the co-limit of some sequence in $\mathfrak{K}$.

(L2) Every $\mathfrak{K}$-object is ω-small in $\mathfrak{L}$.

Recall that an object x is ω-*small* in $\mathfrak{L}$ if for every $Y = \lim \vec{y}$, where $\vec{y}$ is a sequence in $\mathfrak{L}$, for every $\mathfrak{L}$-arrow $f \colon x \to Y$ there are n and an $\mathfrak{L}$-arrow $f' \colon x \to y_n$ such that $f = y_n^\infty \circ f'$, where y_n^∞ denotes the nth arrow from the co-limiting co-cone. We will actually need this property for sequences $\vec{y}$ in $\mathfrak{K}$ only. We shall use the following convention: The $\mathfrak{L}$-objects and $\mathfrak{L}$-arrows will be denoted by capital letters, while the $\mathfrak{K}$-objects and arrows will be denoted by small letters.

Typical examples of pairs $\langle \mathfrak{K}, \mathfrak{L} \rangle$ satisfying (L0)–(L2) come from model theory: $\mathfrak{K}$ could be any class of finite structures of a fixed first order language while $\mathfrak{L}$ should be the class of all structures isomorphic to the unions of countable chains of $\mathfrak{K}$-objects. The arrows in both categories are typically all embeddings.

It turns out that for every category $\mathfrak{K}$ in which all arrows are monic, the sequence category $\sigma \mathfrak{K}$ can play the role of $\mathfrak{L}$, however in applications one usually has in mind a more concrete and natural category satisfying (L0)–(L2).

We say that $U \in \mathrm{Obj}(\mathfrak{L})$ is *weakly $\mathfrak{K}$-injective* if for every $\mathfrak{L}$-arrow $e \colon a \to U$ there exists a $\mathfrak{K}$-arrow $i \colon a \to b$ such that for every $\mathfrak{K}$-arrow $f \colon b \to y$ there is an $\mathfrak{L}$-arrow $g \colon y \to U$ satisfying $g \circ f \circ i = e$, as shown in the following diagram.

$$
\begin{array}{ccc}
a & \xrightarrow{\ i\ } b & \xrightarrow{\ f\ } y \\
{\scriptstyle e}\downarrow & & \\
U & & \\
\end{array}
\quad {\scriptstyle g}
$$

We say that $U \in \mathrm{Obj}(\mathfrak{L})$ is *generic* over $\mathfrak{K}$ (or $\mathfrak{K}$-*generic*) if the following conditions are satisfied.

(U) Every $\mathfrak{K}$-object has an $\mathfrak{L}$-arrow into U (in other words: $\mathfrak{L}(x, U) \neq \emptyset$ for every $x \in \mathrm{Obj}(\mathfrak{K})$).

(WI) U is weakly $\mathfrak{K}$-injective.

As one can expect, this concept is strictly related to weak Fraïssé sequences. Recall that we always assume (L0)–(L2).

Theorem 5.1. *Let $U = \lim \vec{u}$, where $\vec{u}$ is a sequence in $\mathfrak{K}$. Then U is $\mathfrak{K}$-generic if and only if $\vec{u}$ is a weak Fraïssé sequence in $\mathfrak{K}$.*

Proof. Assume first that U is generic over $\mathfrak{K}$. Condition (U) combined with (L2) shows that the sequence $\vec{u}$ satisfies (G1). In order to check (G2), fix $n \in \omega$ and apply the weak $\mathfrak{K}$-injectivity of U to the arrow $u_n^\infty \colon u_n \to U$. We obtain a $\mathfrak{K}$-arrow $i \colon u_n \to b$ such that for every $\mathfrak{L}$-arrow $f \colon b \to y$ there is an $\mathfrak{L}$-arrow $g \colon y \to U$ satisfying $g \circ f \circ i = u_n^\infty$. Taking $f = \mathrm{id}_b$, we obtain an $\mathfrak{L}$-arrow $j \colon b \to U$ such that

$$j \circ i = u_n^\infty.$$

Applying (L2), we get $m > n$ and a $\mathfrak{K}$-arrow $k \colon b \to u_m$ such that $j = u_m^\infty \circ k$. Thus

$$u_m^\infty \circ u_n^m = u_n^\infty = j \circ i = u_m^\infty \circ k \circ i.$$

By (L0), u_m^∞ is a monic, therefore

$$k \circ i = u_n^m.$$

We claim that m is a witness for (G2). Fix a $\mathfrak{K}$-arrow $f \colon u_m \to y$. Applying weak $\mathfrak{K}$-injectivity to the arrow $f \circ k$, we find $g \colon y \to U$ such that

$$g \circ f \circ k \circ i = u_n^\infty.$$

Using (L2), we find $\ell > m$ and an $\mathfrak{L}$-arrow $g' \colon y \to u_\ell$ such that $g = u_\ell^\infty \circ g'$. Now we have

$$u_\ell^\infty \circ g' \circ f \circ u_n^m = u_\ell^\infty \circ g' \circ f \circ k \circ i = g \circ f \circ k \circ i = u_n^\infty = u_\ell^\infty \circ u_n^\ell.$$

As u_ℓ^∞ is a monic, we conclude that $g' \circ f \circ u_n^m = u_n^\ell$, showing (G2).

Now suppose that $\vec{u}$ is a weak Fraïssé sequence in $\mathfrak{K}$. Then (G1) implies that U satisfies (U). It remains to show that U is weakly $\mathfrak{K}$-injective. Fix $e \colon a \to U$. Using (L2), find n and a $\mathfrak{K}$-arrow $e' \colon a \to u_n$ such that $e = u_n^\infty \circ e'$. Let $m > n$ be such that the assertion of (G2) holds. Define $i = u_n^m \circ e'$. Fix a $\mathfrak{K}$-arrow $f \colon u_m \to y$. There are $k \geq m$ and a $\mathfrak{K}$-arrow $g' \colon y \to u_k$ such that $g' \circ f \circ u_n^m = u_n^k$. Let $g = u_k^\infty \circ g'$. Then

$$g \circ f \circ i = u_k^\infty \circ g' \circ f \circ u_n^m \circ e' = u_k^\infty \circ u_n^k \circ e' = u_n^\infty \circ e' = e.$$

Thus, i witnesses the weak $\mathfrak{K}$-injectivity of U. $\qquad\square$

Recall that $\mathfrak{K}$ has a weak Fraïssé sequence if and only if it is a weak Fraïssé category, i.e., it is directed, has the weak amalgamation property, and is weakly dominated by a countable subcategory.

Corollary 5.2. *A $\mathfrak{K}$-generic object exists if and only if $\mathfrak{K}$ is a weak Fraïssé category.*

Proof. If $\mathfrak{K}$ is a weak Fraïssé category then it has a weak Fraïssé sequence, whose co-limit in $\mathfrak{L}$ is a $\mathfrak{K}$-generic object by Theorem 5.1. Conversely, if U is a $\mathfrak{K}$-generic object then, by (L1), $U = \lim \vec{u}$ for some sequence $\vec{u}$ in $\mathfrak{K}$. The sequence $\vec{u}$ is weak Fraïssé in $\mathfrak{K}$ by Theorem 5.1. Hence $\mathfrak{K}$ is a weak Fraïssé category by Theorem 4.6. $\qquad\square$

Corollary 5.3. *A $\mathfrak{K}$-generic object, if exists, is unique up to isomorphism.*

Proof. Suppose U, V are $\mathfrak{K}$-generic. By (L1), $U = \lim \vec{u}$, $V = \lim \vec{v}$, where $\vec{u}$, $\vec{v}$ are sequences in $\mathfrak{K}$. By Theorem 5.1, both $\vec{u}$ and $\vec{v}$ are weak Fraïssé in $\mathfrak{K}$. By Corollary 4.10, there exists an isomorphism from $\vec{u}$ to $\vec{v}$ in the category of sequences. This leads to an isomorphism between U and V. $\qquad\square$

Corollary 5.4. *Let U be a $\mathfrak{K}$-generic object. If $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $\mathfrak{K}$ such that each bonding arrow x_n^{n+1} is amalgamable in $\mathfrak{K}$, then there exists an $\mathfrak{L}$-arrow from X to U.*

Proof. Knowing that $U = \lim \vec{u}$, where $\vec{u}$ is a weak Fraïssé sequence in $\mathfrak{K}$, it suffices to apply Lemma 4.11. $\qquad\square$

We now turn to the question of homogeneity.

Theorem 5.5. *Let U be a $\mathfrak{K}$-generic object and let $e\colon a \to b$ be an amalgamable arrow in $\mathfrak{K}$. Then for every $\mathfrak{L}$-arrows $i\colon b \to U$, $j\colon b \to U$ there exists an automorphism $h\colon U \to U$ satisfying $h \circ i \circ e = j \circ e$.*

This is illustrated in the following diagram in which the triangle is not necessarily commutative.

Proof. Assume $U = \lim \vec{u}$, where $\vec{u}$ is a normalized weak Fraïssé sequence in $\mathfrak{K}$. By (L2), there are $k, \ell \in \omega$ such that $i = u_k^\infty \circ i'$ and $j = u_\ell^\infty \circ j'$. We may assume that $\ell > 0$, replacing j' by $u_\ell^{\ell+1} \circ j'$, if necessary. We may also assume that i' is amalgamable, replacing it by $u_k^{k+1} \circ i'$ (and increasing k), if necessary. Now

$$a \xrightarrow{\ e\ } b \xrightarrow{\ i'\ } u_k \xrightarrow{\ u_k^{k+1}\ } u_{k+1} \longrightarrow \cdots$$

and

$$u_{\ell-1} \xrightarrow{\ u_{\ell-1}^{\ell}\ } u_\ell \xrightarrow{\ u_\ell^{\ell+1}\ } u_{\ell+1} \longrightarrow \cdots$$

are normalized weak Fraïssé sequences and $j'\colon b \to u_\ell$ is a $\mathfrak{K}$-arrow. By Lemma 4.9, there is an isomorphism of sequences $\vec{h}$ extending $j' \circ e$. This leads to an isomorphism $h\colon U \to U$ satisfying $h \circ i \circ e = j \circ e$. $\qquad\square$

Note that if id_a is amalgamable in $\mathfrak{K}$ then this is indeed homogeneity (with respect to a). In particular, if $\mathfrak{K}$ has the amalgamation property then the $\mathfrak{K}$-generic object is *homogeneous*, that is, for every $\mathfrak{L}$-arrows $a\colon i \to U$, $j\colon a \to U$ with $a \in \mathrm{Obj}(\mathfrak{K})$ there exists an automorphism $h\colon U \to U$ satisfying $h \circ i = j$. In general, the property of U described in Theorem 5.5 can be called *weak homogeneity*. We will elaborate this in the next section.

6 Weak homogeneity

In the classical (model-theoretic) Fraïssé theory, an important feature is that the Fraïssé class can be reconstructed from its limit U, simply as the class of all finitely generated substructures (called the *age* of U). Actually, a countably generated model U is the Fraïssé limit of its age $\mathfrak{K}$ if and only if U is homogeneous with respect to $\mathfrak{K}$, in the sense described above, where $\mathfrak{K}$ is treated as a category with embeddings. That is why a Fraïssé class is always assumed to be hereditary (i.e., closed under finitely generated substructures). This cannot be formulated in category theory, however it becomes in some sense irrelevant, as we can always work in the category of *all* finitely generated structures of a fixed language, or in a selected (usually full) subcategory. On the other hand, we can consider subcategories of a fixed category $\mathfrak{K}$ and define the concept of being hereditary with respect to $\mathfrak{K}$. By this way we can talk about objects that are generic relative to a subcategory of $\mathfrak{K}$. We can also look at homogeneity and its weakening in a broader setting.

   We continue using the framework from the previous section, namely, we assume that $\mathfrak{K} \subseteq \mathfrak{L}$ is a pair of categories satisfying (L0)–(L2). Given a class of objects $\mathscr{F} \subseteq \mathrm{Obj}(\mathfrak{K})$, we can say that it is *hereditary* in $\mathfrak{K}$ if for every $x \in \mathscr{F}$, for every $\mathfrak{K}$-arrow $f\colon y \to x$ we have that $y \in \mathscr{F}$. Thus, the notion of being hereditary strongly depends on the category $\mathfrak{K}$ we are working with (the bigger category $\mathfrak{L}$ plays no role here). Actually, it is more convenient, and within the philosophy of category theory, to define this concept for arbitrary subcategories (note that a class of objects may be viewed as a subcategory in which the arrows are identities only). Namely, we say that a subcategory $\mathfrak{S}$ of $\mathfrak{K}$ is *hereditary* if for every compatible $\mathfrak{K}$-arrows f, g the following equivalence holds:

$$f \circ g \in \mathfrak{S} \Longleftrightarrow f \in \mathfrak{S}.$$

Note that a hereditary subcategory $\mathfrak{S}$ is necessarily full. Indeed, if $f\colon a \to b$ is such that $b \in \mathrm{Obj}(\mathfrak{S})$ then $\mathrm{id}_b \in \mathfrak{S}$, therefore $f = \mathrm{id}_b \circ f \in \mathfrak{S}$. It is very easy to check that a family of objects $\mathscr{F}$ is hereditary if and only if the full subcategory $\mathfrak{S}$ with $\mathrm{Obj}(\mathfrak{S}) = \mathscr{F}$ is hereditary (as a subcategory). Conversely, if $\mathfrak{S}$ is a hereditary subcategory of $\mathfrak{K}$, then $\mathrm{Obj}(\mathfrak{S})$ is a hereditary class.

   Natural examples of hereditary subcategories of $\mathfrak{K}$ are of the form

$$\mathfrak{K}_V := \{f \in \mathfrak{K}\colon \mathfrak{L}(\mathrm{cod}(f), V) \neq \emptyset\},$$

where $V \in \mathrm{Obj}(\mathfrak{L})$. One could call $\mathfrak{K}_V$ the *age* of V relative to $\mathfrak{K}$. It is natural to ask when $\mathfrak{K}_V$ is a weak Fraïssé category and when V is its "limit". The answer is given below.

   Fix $V \in \mathrm{Obj}(\mathfrak{L})$. We say that V is *weakly homogeneous* if for every $\mathfrak{L}$-arrow $f\colon a \to V$ with $a \in \mathrm{Obj}(\mathfrak{K})$ there exist a $\mathfrak{K}$-arrow $e\colon a \to b$ and an $\mathfrak{L}$-arrow $i\colon b \to V$ such that $f = i \circ e$ and for every $\mathfrak{L}$-arrow $j\colon b \to V$ there is an automorphism $h\colon V \to V$ satisfying $f = h \circ j \circ e$. This is shown in the following diagram in which,

again, the triangle may not be commutative.
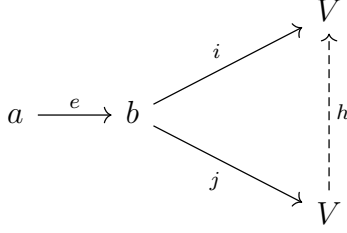
Note that in this case, if $j'\colon b \to V$ is another $\mathfrak{L}$-arrow, then there exists an automorphism $h'\colon V \to V$ such that $f = h' \circ j' \circ e$. Thus, $k := h^{-1} \circ h'$ is an automorphism of V satisfying $k \circ j' \circ e = j \circ e$. This, by Theorem 5.5, shows that the $\mathfrak{K}$-generic object is weakly homogeneous. The following result says that weakly homogeneous objects are generic with respect to their age.

Theorem 6.1. *Let $V \in \mathrm{Obj}(\mathfrak{L})$ and let $\mathfrak{S} := \mathfrak{K}_V$ be the age of V, as defined above. The following conditions are equivalent.*

(a) *V is weakly homogeneous.*

(b) *V is weakly $\mathfrak{S}$-injective.*

(c) *$\mathfrak{S}$ is a weak Fraïssé category and V is $\mathfrak{S}$-generic.*

Proof. (a) $\Longrightarrow$ (b) Fix an $\mathfrak{L}$-arrow $f\colon a \to V$. Let $i\colon b \to V$ and $e\colon a \to b$ be as in the definition of weak homogeneity. Fix an arbitrary $\mathfrak{S}$-arrow $g\colon b \to y$. There exists an $\mathfrak{L}$-arrow $k\colon y \to V$. Apply the weak homogeneity to $j := k \circ g$. By this way we obtain an automorphism $h\colon V \to V$ satisfying $h \circ k \circ g \circ e = f$.

(b) $\Longrightarrow$ (c) By definition, V is $\mathfrak{S}$-generic, as it is $\mathfrak{S}$-cofinal. Corollary 5.2 says that $\mathfrak{S}$ is a weak Fraïssé category (formally, one should replace $\mathfrak{L}$ by a suitable subcategory, so that (L1) will hold).

(c) $\Longrightarrow$ (a) Trivial, by the comment after the definition of weak homogeneity. $\square$

7   The Banach-Mazur game

In this section we explore connections between generic objects and a natural infinite game which is a generalization of the classical Banach-Mazur game in topology.

We fix a category $\mathfrak{K}$. The *Banach-Mazur game* played on $\mathfrak{K}$ is described as follows. There are two players: *Eve* and *Odd*. Eve starts by choosing $a_0 \in \mathrm{Obj}(\mathfrak{K})$. Then Odd chooses $a_1 \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_0^1\colon a_0 \to a_1$. More generally, after Odd's move finishing with an object a_{2k-1}, Eve chooses $a_{2k} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k-1}^{2k}\colon a_{2k-1} \to a_{2k}$. Next, Odd chooses $a_{2k+1} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k}^{2k+1}\colon a_{2k} \to a_{2k+1}$. Thus, the result of the play is a sequence

$$\vec{a}\colon \omega \to \mathfrak{K}.$$

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Of course, one needs to add the objective of the game, namely, a condition under which one of the players wins. So, let us assume that $\mathfrak{K}$ is a subcategory of a bigger category $\mathfrak{L}$, so that *some* sequences in $\mathfrak{K}$ have co-limits in $\mathfrak{L}$. For the moment, we do not need to assume neither of the conditions (L0)–(L2). Now choose a family $\mathscr{W} \subseteq \mathrm{Obj}(\mathfrak{L})$. We define the game $\mathrm{BM}\,(\mathfrak{K}, \mathscr{W})$ with the rules described above, adding the statement that Odd *wins the game* if and only if the co-limit of the resulting sequence $\vec{a}$ is isomorphic to a member of $\mathscr{W}$. So, Eve wins if either the sequence $\vec{a}$ has no co-limit in $\mathfrak{L}$ or its co-limit is isomorphic to none of the members of $\mathscr{W}$.

We are particularly interested in the case $\mathscr{W} = \{W\}$ for some $W \in \mathrm{Obj}(\mathfrak{L})$, where the game $\mathrm{BM}\,(\mathfrak{K}, \mathscr{W})$ will be denoted simply by $\mathrm{BM}\,(\mathfrak{K}, W)$. Before we turn to it, we discuss some basic properties of the Banach-Mazur game.

Recall that a *strategy* of Odd is a function Σ assigning to each finite sequence $\vec{s}\colon n \to \mathfrak{K}$ of odd length a $\mathfrak{K}$-arrow $\Sigma(\vec{s})\colon s_{n-1} \to s$, called *Odd's response* to $\vec{s}$. We say that *Odd plays according to* Σ if the resulting sequence $\vec{a}$ satisfies $a_n^{n+1} = \Sigma(\vec{a} \restriction n)$ for every odd $n \in \omega$. Odd's strategy Σ is *winning* in $\mathrm{BM}\,(\mathfrak{K}, \mathscr{W})$ if $\lim \vec{a}$ is isomorphic to a member of $\mathscr{W}$ whenever Odd plays according to Σ, no matter how Eve plays. These concepts are defined for Eve analogously. A strategy Σ of Eve is defined on sequences of even length, including the empty sequence, where $\Sigma(\emptyset)$ is simply a $\mathfrak{K}$-object a_0, the starting point of a play according to Σ.

Theorem 7.1. *Let $\mathfrak{K} \subseteq \mathfrak{L}$ be two categories and let $\mathscr{W} \subseteq \mathrm{Obj}(\mathfrak{L})$. Let $\mathfrak{S}$ be a weakly dominating subcategory of $\mathfrak{K}$. Then Odd has a winning strategy in $\mathrm{BM}\,(\mathfrak{K}, \mathscr{W})$ if and only if he has a winning strategy in $\mathrm{BM}\,(\mathfrak{S}, \mathscr{W})$. The same applies to Eve.*

Proof. Let Σ be Odd's winning strategy in $\mathrm{BM}\,(\mathfrak{K}, \mathscr{W})$. We describe his winning strategy in $\mathrm{BM}\,(\mathfrak{S}, \mathscr{W})$. We denote the resulting sequence of a play in $\mathrm{BM}\,(\mathfrak{S}, \mathscr{W})$ by $\vec{s}$. So, suppose Eve started with $s_0 \in \mathrm{Obj}(\mathfrak{S})$. Odd first chooses an $\mathfrak{S}$-arrow $i_0\colon s_0 \to a_0$ so that condition (W) of the definition of weak domination holds, namely, for every $\mathfrak{K}$-arrow $f\colon a_0 \to x$ there is a $\mathfrak{K}$-arrow $g\colon x \to t$ such that $g \circ f \circ i_0 \in \mathfrak{S}$. Let $a_0^1 = \Sigma(a_0)$, so $a_0^1\colon a_0 \to a_1$ with $a_1 \in \mathrm{Obj}(\mathfrak{K})$. Using (W), Odd finds a $\mathfrak{K}$-arrow $j_0\colon a_1 \to s_1$ and he responds with $s_0^1 := j_0 \circ a_0^1 \circ i_0$. In general, the strategy is described in the following commutative diagram.

$$
\begin{array}{ccccccccc}
s_0 & \longrightarrow & s_1 & \longrightarrow & \cdots & \longrightarrow & s_{2n} & \longrightarrow & s_{2n+1} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle i_0} & & \uparrow{\scriptstyle j_0} & & & & \downarrow{\scriptstyle i_n} & & \uparrow{\scriptstyle j_n} & & \\
a_0 & \xrightarrow[a_0^1]{} & a_1 & & & & a_{2n} & \xrightarrow[a_{2n}^{2n+1}]{} & a_{2n+1} & &
\end{array}
$$

Namely, when Eve finishes with s_{2n}, Odd first chooses a suitable $\mathfrak{S}$-arrow $i_n\colon s_{2n} \to a_{2n}$ realizing the weak domination. Next, he uses Σ to find a $\mathfrak{K}$-arrow $a_{2n}^{2n+1}\colon a_{2n} \to a_{2n+1}$. Specifically, a_{2n}^{2n+1} is Odd's response to the sequence $a_0 \to a_1 \to \cdots \to a_{2n}$ in which the arrows are suitable compositions of those from the diagram above. Odd responds with $s_{2n}^{2n+1} := j_n \circ f_n \circ i_n$, where j_n comes from the weak domination of $\mathfrak{S}$ (condition (W)). This is a winning strategy, because the resulting sequence $\vec{s}$ is

isomorphic to the sequence $\vec{a}$, where

$$a_{2k+1}^{2k+2} = i_{k+1} \circ s_{2k+1}^{2k+2} \circ j_k$$

for every $k \in \omega$; this sequence is the result of a play of $\mathrm{BM}\,(\mathfrak{K}, \mathscr{W})$ in which Odd was using strategy Σ.

Now suppose Odd has a winning strategy Σ in $\mathrm{BM}\,(\mathfrak{S}, \mathscr{W})$. Playing the game $\mathrm{BM}\,(\mathfrak{K}, \mathscr{W})$, assume Eve started with $a_0 \in \mathrm{Obj}(\mathfrak{K})$. Odd first uses (C) to find an arrow $i_0 \colon a_0 \to s_0$ with $s_0 \in \mathrm{Obj}(\mathfrak{S})$. Next, he takes the arrow $s_0^1 \colon s_0 \to s_1$ according to Σ. Specifically, $s_0^1 = \Sigma(s_0)$. He responds with $a_0^1 := j_0 \circ s_0^1 \circ i_0$, where $j_0 \colon s_1 \to a_1$ is an $\mathfrak{S}$ from condition (W), namely, for every $\mathfrak{K}$-arrow $f \colon a_1 \to x$ there is a $\mathfrak{K}$-arrow $g \colon x \to s$ satisfying $g \circ f \circ j_0 \in \mathfrak{S}$. In general, the strategy described in the following commutative diagram.

Here, i_n comes from condition (W), namely, $i_n \circ a_{2n-1}^{2n} \circ j_{n-1} \in \mathfrak{S}$. Furthermore, $s_{2n}^{2n+1} = \Sigma(\vec{v})$, where $\vec{v}$ is the sequence $s_0 \to s_1 \to s_2 \to \cdots \to s_{2n}$ obtained from the diagram above (note that all its arrows are in $\mathfrak{S}$). Finally, j_n is such that the assertion of (W) holds, that is, for every $\mathfrak{K}$-arrow $f \colon a_{2n+1} \to x$ there is a $\mathfrak{K}$-arrow $g \colon x \to t$ such that $g \circ f \circ j_n \in \mathfrak{S}$. Odd's response is $a_{2n}^{2n+1} := j_n \circ s_{2n}^{2n+1} \circ i_n$. This strategy is winning in $\mathrm{BM}\,(\mathfrak{K}, \mathscr{W})$, because the resulting sequence $\vec{a}$ is isomorphic to the sequence $\vec{s}$ in which

$$s_{2k+1}^{2k+2} = i_{k+1} \circ a_{2k+1}^{2k+2} \circ j_k \in \mathfrak{S}$$

for every $k \in \omega$. The sequence $\vec{s}$ results from a play of $\mathrm{BM}\,(\mathfrak{S}, \mathscr{W})$ in which Odd was using his winning strategy Σ.

The case of Eve's winning strategies is almost the same, as the rules are identical for both players, except Eve's first move. $\qquad\square$

Theorem 7.2. *Assume $\{\mathscr{W}_n\}_{n\in\omega}$ is such that each $\mathscr{W}_n \subseteq \mathrm{Obj}(\mathfrak{L})$ is closed under isomorphisms and Odd has a winning strategy in $\mathrm{BM}\,(\mathfrak{K}, \mathscr{W}_n)$ for each $n \in \omega$. Then Odd has a winning strategy in*

$$\mathrm{BM}\left(\mathfrak{K}, \bigcap_{n\in\omega} \mathscr{W}_n\right).$$

In particular, $\bigcap_{n\in\omega} \mathscr{W}_n \neq \emptyset$.

Proof. Let Σ_n denote Odd's winning strategy in $\mathrm{BM}\,(\mathfrak{K}, \mathscr{W}_n)$. Let $\{I_n\}_{n\in\omega}$ be a partition of all even natural numbers into infinite sets. Let $J_n = I_n \cup \{i + 1 \colon i \in I_n\}$.

Given a finite sequence $\vec{s}$ whose length n is odd, let k be such that $n - 1 \in I_k$ and define
$$\Sigma(\vec{s}) = \Sigma_k(\vec{s} \upharpoonright (J_k \cap n)).$$
We claim that Σ is a winning strategy of Odd in the game $\mathrm{BM}\left(\mathfrak{K}, \bigcap_{n \in \omega} \mathscr{W}_n\right)$.

Indeed, suppose $\vec{a}$ is the result of a play in which Odd has been using strategy Σ. Then $\vec{a} \upharpoonright J_k$ is a sequence resulting from another play in which Odd was using strategy Σ_k. Thus $\lim \vec{a} = \lim(\vec{a} \upharpoonright J_k) \in \mathscr{W}_k$. Hence $\lim \vec{a} \in \bigcap_{n \in \omega} \mathscr{W}_n$. □

We now turn to the case where $\mathscr{W}$ is the isomorphism class of a single object. As the reader may guess, generic objects play a significant role here. In the next result we do not assume (L0)–(L2).

Theorem 7.3. *Let $\mathfrak{K} \subseteq \mathfrak{L}$ and assume that $\vec{u}$ is a weak Fraïssé sequence in $\mathfrak{K}$ with $U = \lim \vec{u}$ in $\mathfrak{L}$. Then Odd has a winning strategy in $\mathrm{BM}\,(\mathfrak{K}, U)$.*

Proof. We may assume that the sequence $\vec{u}$ is normalized. Odd's strategy is as follows. Suppose $a_0 \in \mathrm{Obj}(\mathfrak{K})$ is Eve's first move. Using (G1), Odd finds $k \in \omega$ together with a $\mathfrak{K}$-arrow $f_0 \colon a_0 \to u_k$. His response is $a_0^1 := u_k^{k+1} \circ f_0$. In particular, $a_1 = u_{k+1}$.

In general, suppose a_{2n-1}^{2n} was the nth move of Eve. Assume inductively that $a_{2n-1} = u_{\ell+1}$ and $a_{2n-2}^{2n-1} = u_\ell^{\ell+1} \circ f_{n-1}$ for some $\mathfrak{K}$-arrow f_{n-1}. Using (G2), Odd finds $m > \ell + 1$ together with a $\mathfrak{K}$-arrow $f_n \colon a_{2n} \to u_m$ satisfying
$$u_\ell^m = f_n \circ a_{2n-1}^{2n} \circ u_\ell^{\ell+1}.$$
Odd's response is $a_{2n}^{2n+1} := u_m^{m+1} \circ f_n$. In particular, $a_{2n+1} = u_{m+1}$. The strategy is shown in the following diagram.

$$\cdots \longrightarrow u_\ell \longrightarrow u_{\ell+1} \qquad\qquad u_m \longrightarrow u_{m+1} \longrightarrow \cdots$$

(diagram with arrows a_{2n-1}^{2n} from $u_{\ell+1}$ down to a_{2n}, and f_n from a_{2n} up to u_m)

It is clear that the resulting sequence $\vec{a}$ is isomorphic to $\vec{u}$, therefore $\lim \vec{a} = U$. □

The proof above is somewhat similar to that of Theorem 7.1. In fact, if the sequence $\vec{u}$ is one-to-one (that is, $u_n \neq u_m$ for $n \neq m$) then one can use Theorem 7.1 to play the game in the image of $\vec{u}$, where Odd's winning strategy is obvious.

Our goal is reversing Theorem 7.3, extending the results of Krawczyk and the author [6]. We start with a technical lemma. Recall that a category $\mathfrak{C}$ is *locally countable* if $\mathfrak{C}(x, y)$ is a countable set for every $\mathfrak{C}$-objects x, y.

Lemma 7.4. *Assume $\mathfrak{K} \subseteq \mathfrak{L}$ are two categories satisfying (L0)–(L2), $\mathfrak{K}$ is locally countable, $V \in \mathrm{Obj}(\mathfrak{L})$, and suppose $e \colon a \to V$ is an $\mathfrak{L}$-arrow with $a \in \mathrm{Obj}(\mathfrak{K})$ satisfying the following condition.*

($\times$) *For every $\mathfrak{K}$-arrow $f\colon a \to b$ there exists a $\mathfrak{K}$-arrow $f'\colon b \to b'$ such that for every $\mathfrak{L}$-arrow $i\colon b' \to V$ it holds that $e \neq i \circ f' \circ f$.*

Then Eve has a winning strategy in $\mathrm{BM}\,(\mathfrak{K}, V)$.

Proof. Eve's strategy is as follows. She starts with $a_0 := a$. At step $n > 0$, Eve chooses a $\mathfrak{K}$-arrow $f_n\colon a \to a_{2n-1}$ and responds with $a_{2n-1}^{2n} := f'$, where f' comes from condition ($\times$) applied to $f := f_n$. Thus

(†) $$(\forall\, i \in \mathfrak{L}(a_{2n}, V))\;\; e \neq i \circ a_{2n-1}^{2n} \circ f_n.$$

Of course, this strategy depends on the choice of the sequence $\{f_n\}_{n>0}$. We show that a suitable choice makes Eve's strategy winning. Namely, she needs to take care of all $\mathfrak{K}$-arrows from a into the sequence $\vec{a}$. More precisely, the following condition should be satisfied.

(‡) $$(\forall\, k > 0)(\forall\, g \in \mathfrak{K}(a, a_k))(\exists\, n > k)\;\; f_n = a_k^{2n-1} \circ g.$$

In order to achieve (‡), we use the fact that $\mathfrak{K}$ is locally countable. Specifically, for each $k > 0$, for each $g \in \mathfrak{K}(a, a_k)$ we inductively choose an integer $\varphi(k, g) > k$ in such a way that $\varphi(k', g') \neq \varphi(k, g)$ whenever $\langle k, g \rangle \neq \langle k', g' \rangle$. This is possible, because for a fixed k there are only countably many possibilities for g (we may first partition ω into infinite sets B_k and make sure that $\varphi(k, g) \in B_k$ for every g). We set $f_n := a_k^{2n-1} \circ g$ whenever $n = \varphi(k, g)$.

Now let $A = \lim \vec{a} \in \mathrm{Obj}(\mathfrak{L})$ and suppose that $h\colon V \to A$ is an isomorphism in $\mathfrak{L}$. Using (L2), we find a $\mathfrak{K}$-arrow $g\colon a \to a_k$ such that $h \circ e = a_k^\infty \circ g$, where a_k^∞ is part of the co-limiting co-cone. By (‡), there is $n > k$ such that $f_n = a_k^{2n-1} \circ g$. Consider $i := h^{-1} \circ a_{2n}^\infty$. We have

$$i \circ a_{2n-1}^{2n} \circ f_n = h^{-1} \circ a_{2n}^\infty \circ a_{2n-1}^{2n} \circ a_k^{2n-1} \circ g = h^{-1} \circ a_k^\infty \circ g = h^{-1} \circ h \circ e = e,$$

contradicting (†). This shows that Eve wins when using the strategy above. □

We are ready to prove the main result of this section.

Theorem 7.5. *Assume $\mathfrak{K} \subseteq \mathfrak{L}$ satisfy (L0)–(L2) and $\mathfrak{K}$ is locally countable. Given an $\mathfrak{L}$-object V, the following properties are equivalent.*

(a) *V is $\mathfrak{K}$-generic (in particular, $\mathfrak{K}$ is a weak Fraïssé category).*

(b) *Odd has a winning strategy in* $\mathrm{BM}\,(\mathfrak{K}, V)$.

(c) *Eve does not have a winning strategy in* $\mathrm{BM}\,(\mathfrak{K}, V)$.

Proof. (a) $\Longrightarrow$ (b) By (L1), $V = \lim \vec{v}$ for a sequence $\vec{v}$ in $\mathfrak{K}$. By Theorem 5.1, this sequence is weak Fraïssé in $\mathfrak{K}$. Thus (b) follows from Theorem 7.3.

(b) $\Longrightarrow$ (c) Obvious.

(c) $\implies$ (a) First, note that V satisfies (U), since if $x \in \mathrm{Obj}(\mathfrak{K})$ is such that $\mathfrak{L}(x, V) = \emptyset$ then Eve would have an obvious winning strategy, starting the game with x. Thus, supposing V is not $\mathfrak{K}$-generic, we deduce that it is not weakly $\mathfrak{K}$-injective. Hence, there exists $e \colon a \to V$ with $a \in \mathrm{Obj}(\mathfrak{K})$ such that for every $\mathfrak{K}$-arrow $f \colon a \to b$ there is a $\mathfrak{K}$-arrow $f' \colon b \to y$ such that no $\mathfrak{L}$-arrow $j \colon y \to V$ satisfies $j \circ f' \circ f = e$. This is precisely condition ($\times$) of Lemma 7.4, contradicting (c). $\qquad\square$

8 Applications

We briefly discuss how the results of previous sections can be interpreted in concrete categories of models and other structures.

First of all, $\mathfrak{K}$ could be a fixed category of finitely generated models of a fixed first-order language while $\mathfrak{L}$ could be the category of all models representable as unions of countable chains in $\mathrm{Obj}(\mathfrak{K})$. In both cases it is natural to consider embeddings as arrows. It is clear that conditions (L0)–(L2) are satisfied. In this setting, our results in Sections 5, 6, and in particular Theorem 6.1, are extensions of the classical results of Fraïssé [2]. Specifically, if $\mathrm{Obj}(\mathfrak{K})$ is countable up to isomorphisms and all the models in $\mathrm{Obj}(\mathfrak{K})$ are countable, then the joint embedding property together with the weak amalgamation property imply the existence of a unique weakly $\mathfrak{K}$-injective model in $\mathfrak{L}$ that might be called the *limit* of $\mathrm{Obj}(\mathfrak{K})$. Note that the property of being hereditary is ignored. The main reason is that the weak AP is stable under taking the hereditary closure. Recall that the *joint embedding property* is simply the property of being directed with respect to embeddings. If the models in $\mathfrak{K}$ are uncountable (this may happen if the language is uncountable) then we cannot deduce that $\mathfrak{K}$ is locally countable, and indeed $\mathfrak{K}$ might not be weakly dominated by a countable subcategory. Summarizing, a class $\mathfrak{M}$ of countable finitely generated models is called a *weak Fraïssé class* if it has the joint embedding property and the weak amalgamation property. Once this happens, it is a weak Fraïssé category (with embeddings as the class of arrows). This has already been discussed in the recent work [6], also in the context of the Banach-Mazur game. Our Theorem 7.5 in the special case of models summarizes the main results of [6]. Recall that if $\mathfrak{M}$ is a weak Fraïssé class then so is its hereditary closure, while if $\mathfrak{M}$ has the AP then its hereditary closure may fail the AP.

8.1 Projective weak Fraïssé theory

Following Irwin & Solecki [3] we say that a class of finite nonempty models $\mathfrak{K}$ is a *projective Fraïssé class* if it contains countably many types and satisfies the following two conditions:

(1) For every $X, Y \in \mathfrak{K}$ there exists $Z \in \mathfrak{K}$ having proper epimorphisms onto X and Y.

(2) Given proper epimorphisms $f\colon X \to Z$, $g\colon Y \to Z$ with $X, Y, Z \in \mathfrak{K}$, there exist $W \in \mathfrak{K}$ and proper epimorphisms $f'\colon W \to X$, $g'\colon W \to Y$ such that $f \circ f' = g \circ g'$.

Here, a mapping $f\colon A \to B$ is a *proper epimorphism* if it is a surjective homomorphism and satisfies for every n-ary relation R (in the language of the models from $\mathfrak{K}$) the condition

$$R^B(y_1, \ldots, y_n) \iff (\exists\, x_1, \ldots, x_n \in A)\, R^A(x_1, \ldots, x_n) \text{ and } (\forall\, i \le n)\, y_i = f(x_i).$$

It is clear that declaring arrows between $A, B \in \mathfrak{K}$ to be proper epimorphisms from B onto A, we obtain a Fraïssé category. It is also clear how to change condition (2) above, in order to obtain the projective *weak* amalgamation property. Of course, the category $\mathfrak{L} \supseteq \mathfrak{K}$ should consist of all inverse limits of sequences in $\mathfrak{K}$, treated as compact topological spaces with continuous epimorphisms. It is easy to check that conditions (L0)–(L3) are fulfilled.

As a very concrete example, we may consider $\mathfrak{K}$ to be the class of all finite nonempty sets with no extra structure. Then $\mathfrak{L}$ should be the class of all compact 0-dimensional metrizable spaces. Obviously, $\mathfrak{K}$ is a Fraïssé category and its limit is the Cantor set. A much more interesting example (leading to an exciting topological object, called the *pseudo-arc*) is contained in [3].

8.2 Droste & Göbel theory

The paper [1] by Droste & Göbel is the first treatment of model-theoretic Fraïssé limits from the category-theoretic perspective. Roughly speaking, they work in a category $\mathfrak{L}$ having the property that λ-small objects are co-dense and there are not too many of them. Under certain natural conditions, $\mathfrak{L}$ contains a special object which is the Fraïssé limit of the subcategory of all λ-small objects. In our case, $\lambda = \omega$, however we do not require that $\mathfrak{K} \subseteq \mathfrak{L}$ consists of *all* ω-small objects. We actually gave necessary and sufficient conditions for the existence of a $\mathfrak{K}$-generic object, assuming conditions (L0)–(L3) only, which are weaker than those of [1]. The results of Droste & Göbel can be easily extended to the case where the amalgamation property is replaced by its weak version.

8.3 Uncountable weak Fraïssé theory

There is nothing unusual or surprising in extending the theory of generic objects and weak Fraïssé sequences to the uncountable setting, namely, working in a category $\mathfrak{K}$ closed under co-limits of sequences of length $< \kappa$, where κ is an uncountable regular cardinal. Under certain circumstances, there exists a (unique up to isomorphism) weak Fraïssé sequence of length κ leading to a generic object in a larger category. In fact, it suffices to combine the results of Section 4 above with [7, Section 3]. We leave the details to interested readers.

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