New lower bounds on eigenvalues by conforming finite elements

Tomáš Vejchodský (vejchod@math.cas.cz)

Institute of Mathematics
Czech Academy of Sciences

Joint work with I. Šebestová (University of Concepción, Chile)

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Outline

- Classical Weinstein's and Kato's bounds
- Weinstein's and Kato's bounds in weak setting
- Numerical examples

Lower bounds on eigenvalues

Old problem:
Temple 1928, Weinstein 1937, Kato 1949, Lehmann 1949, 1950, ...

Many results: M.G. Armentano, G. Barrenechea, H. Behnke, C. Carstensen, R.G. Duran, D. Galistl, J. Gedicke, F. Goerisch, L. Grubišić, Jun Hu, J.R. Kuttler, Y.A. Kuznetsov, Fubiao Lin, Qun Lin, Xuefeng Liu, M. Plum, S.I. Repin, V.G. Sigillito, Hehu Xie, Yidu Yang, Zhimin Zhang, ... *many others*

Weinstein's bounds

Eigenvalue problem: Find $u_i \in D(A) \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$:

$$Au_i = \lambda_i u_i$$

Setting:

- V ... Hilbert space
- $A : D(A) \to V$ linear, symmetric operator
- $\{u_i\}$ form ON basis in V
- $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$

Properties:

- $\|v\|^2 = \sum_{j=1}^{\infty} \langle v, u_j \rangle^2$
- $\langle Av, v \rangle = \sum_{j=1}^{\infty} \lambda_j \langle v, u_j \rangle^2$
- $\|Av\|^2 = \sum_{j=1}^{\infty} \lambda_j^2 \langle v, u_j \rangle^2$

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Theorem 1 (Weinstein 1937):
Let $u_* \in D(A) \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary.
Let $\varepsilon = \|Au_* - \lambda_* u_*\| / \|u_*\|$.
Then there exists λ_n such that $\lambda_* - \varepsilon \leq \lambda_n \leq \lambda_* + \varepsilon$.

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Then there exists λ_n such that $\lambda_* - \varepsilon \leq \lambda_n \leq \lambda_* + \varepsilon$.

Proof: $$\|Au_* - \lambda_* u_*\|^2 = \sum_{j=1}^{\infty} \langle Au_* - \lambda_* u_*, u_j \rangle^2$$
$$= \sum_{j=1}^{\infty} |\lambda_j - \lambda_*|^2 \langle u_*, u_j \rangle^2 \geq \min_j |\lambda_j - \lambda_*|^2 \|u_*\|^2$$

Thus,

$$|\lambda_n - \lambda_*| = \min_j |\lambda_j - \lambda_*| \leq \frac{\|Au_* - \lambda_* u_*\|}{\|u_*\|} = \varepsilon$$

Kato's bounds

Theorem 2 (Kato 1949): Let $u_* \in D(A) \setminus \{0\}$ be arbitrary and $\lambda_* = \langle Au_*, u_* \rangle / \langle u_*, u_* \rangle$.
Let $\varepsilon = \|Au_* - \lambda_* u_*\| / \|u_*\|$ and $\nu \in \mathbb{R}$ satisfy

$$\lambda_{n-1} \le \lambda_* < \nu \le \lambda_{n+1} \quad \text{for some } n.$$

Then $\lambda_* - \dfrac{\varepsilon^2}{\nu - \lambda_*} \le \lambda_n$.

Kato's bounds

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$$\lambda_{n-1} \le \lambda_* < \nu \le \lambda_{n+1} \quad \text{for some } n.$$

Then $\lambda_* - \dfrac{\varepsilon^2}{\nu - \lambda_*} \le \lambda_n$.

Proof: We have $(\lambda_j - \lambda_n)(\lambda_j - \nu) \ge 0$ for all $j = 1, 2, \dots$.

$$0 \le \sum_{j=1}^{\infty} (\lambda_j - \lambda_n)(\lambda_j - \nu) \langle u_*, u_j \rangle^2 = \sum_{j=1}^{\infty} (\lambda_j^2 - (\lambda_n + \nu)\lambda_j + \lambda_n \nu) \langle u_*, u_j \rangle^2 =$$

$$\|Au_*\|^2 - (\lambda_n + \nu) \langle Au_*, u_* \rangle + \lambda_n \nu \|u_*\|^2 = \left(\varepsilon^2 + \lambda_*^2 - (\lambda_n + \nu)\lambda_* + \lambda_n \nu\right) \|u_*\|^2$$

because $\|Au_*\|^2 = (\varepsilon^2 + \lambda_*^2)\|u_*\|^2$ and $\langle Au_*, u_* \rangle = \lambda_* \|u_*\|^2$. □

Kato's bound for multiple eigenvalues

Theorem 3 (Kato 1949):

- Let $1 \le n \le s$.
- Let $V_* \subset V$, $\dim V_* = s - n + 1$ and $u_{*,i} \in V_*$, $\lambda_{*,i} \in \mathbb{R}$ satisfy

$$\langle Au_{*,i}, v_* \rangle = \lambda_{*,i} \langle u_{*,i}, v_* \rangle \quad \forall v_* \in V_*, \quad \|u_{*,i}\| = 1, \quad i = n, \dots, s.$$

- Let $\lambda_{s-1} \le \lambda_{*,s} < \nu \le \lambda_{s+1}$.
- Let $\mu_{\max} > 0$ be the maximal eigenvalues of $E v_k = \mu_k U v_k$, where $E_{ij} = \langle Au_{*,i} - \lambda_{*,i} u_{*,i}, Au_{*,j} - \lambda_{*,j} u_{*,j} \rangle$ and $U = \operatorname{diag}(\nu - \lambda_{*,n}, \dots, \nu - \lambda_{*,s})$.

Then $\lambda_{*,n} - \mu_{\max} \le \lambda_n$.

Corollary: If $\varepsilon_i = \|Au_{*,i} - \lambda_{*,i} u_{*,i}\|$ then

$$\lambda_{*,n} - \sum_{i=n}^{s} \frac{\varepsilon_i^2}{\nu - \lambda_{*,i}} \le \lambda_n.$$

Weak form – general setting

Eigenvalue problem: Find $u_i \in V \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$:

$$a(u_i, v) = \lambda_i b(u_i, v) \quad \forall v \in V.$$

Properties:

- $0 < \lambda_1 \leq \lambda_2 \leq \cdots$
- $b(u_i, u_j) = \delta_{ij}$
- $\|v\|_b^2 = \sum_{j=1}^{\infty} |b(v, u_j)|^2$
- $\|v\|_a^2 = \sum_{j=1}^{\infty} \lambda_j |b(v, u_j)|^2$

Weinstein's bound in the weak form

Theorem 4: Let $u_* \in V \setminus \{0\}$ and $\lambda_* \in \mathbb{R}$ be arbitrary and $w \in V$ be given by

$$a(w, v) = a(u_*, v) - \lambda_* b(u_*, v) \quad \forall v \in V.$$

Then

$$\min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \leq \frac{\|w\|_a^2}{\|u_*\|_b^2}.$$

Proof:

$$\|w\|_a^2 = \sum_{j=1}^{\infty} \lambda_j |b(w, u_j)|^2 = \sum_{j=1}^{\infty} \frac{|a(w, u_j)|^2}{\lambda_j}$$

$$= \sum_{j=1}^{\infty} \frac{|a(u_*, u_j) - \lambda_* b(u_*, u_j)|^2}{\lambda_j} = \sum_{j=1}^{\infty} \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} |b(u_*, u_j)|^2$$

Thus,

$$\|w\|_a^2 \geq \min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \sum_{j=1}^{\infty} |b(u_*, u_j)|^2$$

Weinstein's bound in the weak form

Corollary 2: If

$$\sqrt{\lambda_{n-1}\lambda_n} \le \lambda_* \le \sqrt{\lambda_n \lambda_{n+1}}$$

and

$$\|w\|_a \le \eta$$

then

$$\ell_n \le \lambda_n,$$

where $\ell_n = \dfrac{1}{4\|u_*\|_b^2}\left(-\eta + \sqrt{\eta^2 + 4\lambda_* \|u_*\|_b^2}\right)^2.$

Proof: Clearly,

$$\frac{(\lambda_n - \lambda_*)^2}{\lambda_n} = \min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \le \frac{\|w\|_a^2}{\|u_*\|_b^2} \le \frac{\eta^2}{\|u_*\|_b^2}$$

and solve for λ_n. □

Kato's bound in the weak form

Theorem 5: Let $u_* \in V \setminus \{0\}$ be arbitrary and let $\lambda_* = \|u_*\|_a^2 / \|u_*\|_b^2$. Let there be $\nu \in \mathbb{R}$ such that

$$\lambda_{n-1} \leq \lambda_* < \nu \leq \lambda_{n+1}$$

for a fixed index n. Let $\|w\|_a \leq \eta$. Then

$$L_n \leq \lambda_n,$$

where

$$L_n = \lambda_* \left(1 + \frac{\nu}{\lambda_*(\nu - \lambda_*)} \frac{\eta^2}{\|u_*\|_b^2} \right)^{-1}.$$

. . . for multiple eigenvalues

Theorem 6:

- Let $0 < n \le s$.
- Let $V_* \subset V$, $\dim V_* = s - n + 1$, and
- $\lambda_{*,i} > 0$, $u_{*,i} \in V_*$, $\|u_{*,i}\|_b = 1$, $i = n, \dots, s$, satisfy

$$a(u_{*,i}, v_*) = \lambda_{*,i} b(u_{*,i}, v_*) \quad \forall v_* \in V_*,$$

- Let $\lambda_{s-1} \le \lambda_{*,s} < \nu \le \lambda_{s+1}$.
- Let $w_i \in V$ be given by

$$a(w_i, v) = a(u_{*,i}, v) - \lambda_{*,i} b(u_{*,i}, v) \quad \forall v \in V$$

and let $\|w_i\|_a \le \eta_i$ for all $i = n, \dots, s$.

Then $L_n \le \lambda_n$, where $L_n = \lambda_{*,n} \left(1 + \nu \lambda_{*,n} \sum_{i=n}^{s} \frac{\eta_i^2}{\lambda_{*,i}^2 (\nu - \lambda_{*,i})} \right)^{-1}$.

Complementary upper bound on the residual

Theorem 7: Let $V = H_0^1(\Omega)$, $a(u,v) = (\nabla u, \nabla v)$, and $b(u,v) = (u,v)$. Let $w \in V$ satisfy

$$a(w,v) = a(u_*,v) - \lambda_* b(u_*,v) \quad \forall v \in V.$$

Let $\mathbf{q} \in \mathbf{H}(\operatorname{div},\Omega)$ be such that $-\operatorname{div}\mathbf{q} = \lambda_* u_*$ then

$$\|\nabla w\|_{L^2(\Omega)} \leq \eta = \|\nabla u_* - \mathbf{q}\|_{L^2(\Omega)}.$$

Proof: Let $v \in H_0^1(\Omega)$, then

$$\begin{aligned} a(w,v) &= (\nabla u_*, \nabla v) - \lambda_*(u_*,v) - (\operatorname{div}\mathbf{q}, v) - (\mathbf{q}, \nabla v) \\ &= (\nabla u_* - \mathbf{q}, \nabla v) - (\lambda_* u_* + \operatorname{div}\mathbf{q}, v) \\ &\leq \|\nabla u_* - \mathbf{q}\| \|\nabla v\| \end{aligned}$$

□

[Synge 1957], [Haslinger, Hlaváček 1976], [Křížek, Hlaváček 1984], [Neittaanmäki, Repin 2004], [Braess 2007], ...

Flux reconstruction

- FEM eigenpairs: $\Lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h$, $\|u_{h,n}\|_{L^2(\Omega)} = 1$, $n = 1, \dots, s$
- Flux reconstruction: $\mathbf{q}_{h,n} = \sum_{\mathbf{z}\in\mathcal{N}_h} \mathbf{q}_{\mathbf{z},n}$ [Braess, Schöberl 2006]
- Local mixed FEM: $\mathbf{q}_{\mathbf{z},n} \in \mathbf{W}_{\mathbf{z}}$, $d_{\mathbf{z},n} \in P_1^*(\mathcal{T}_{\mathbf{z}})$

$$
\begin{aligned}
(\mathbf{q}_{\mathbf{z},n}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} - (d_{\mathbf{z},n}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} &= (\psi_{\mathbf{z}} \nabla u_{h,n}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} && \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}} \\
-(\operatorname{div} \mathbf{q}_{\mathbf{z},n}, \varphi_h)_{\omega_{\mathbf{z}}} &= (r_{\mathbf{z},n}, \varphi_h)_{\omega_{\mathbf{z}}} && \forall \varphi_h \in P_1^*(\mathcal{T}_{\mathbf{z}})
\end{aligned}
$$

where

- $\omega_{\mathbf{z}}$ is the patch of elements around vertex $\mathbf{z} \in \mathcal{N}_h$
- $\mathcal{T}_{\mathbf{z}}$ is the set of elements in $\omega_{\mathbf{z}}$
- $\mathbf{W}_{\mathbf{z}} = \{\mathbf{w}_h \in \mathbf{H}(\operatorname{div}, \omega_{\mathbf{z}}) : \mathbf{w}_h|_K \in \mathbf{RT}_1(K)\ \forall K \in \mathcal{T}_{\mathbf{z}}$ and $\mathbf{w}_h \cdot \mathbf{n}_{\omega_{\mathbf{z}}} = 0$ on $\Gamma_{\omega_{\mathbf{z}}}^{\text{ext}}\}$
- $P_1^*(\mathcal{T}_{\mathbf{z}}) = \begin{cases} \{v_h \in P_1(\mathcal{T}_{\mathbf{z}}) : \int_{\omega_{\mathbf{z}}} v_h \,\mathrm{d}x = 0\} & \text{for } \mathbf{z} \in \mathcal{N}_h \setminus \partial\Omega \\ P_1(\mathcal{T}_{\mathbf{z}}) & \text{for } \mathbf{z} \in \mathcal{N}_h \cap \partial\Omega \end{cases}$
- $r_{\mathbf{z},n} = \Lambda_{h,n} \psi_{\mathbf{z}} u_{h,n} - \nabla \psi_{\mathbf{z}} \cdot \nabla u_{h,n}$

Flux reconstruction

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(\mathbf{q}_{\mathbf{z},n}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} - (d_{\mathbf{z},n}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} &= (\psi_{\mathbf{z}} \nabla u_{h,n}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} && \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}} \\
-(\operatorname{div} \mathbf{q}_{\mathbf{z},n}, \varphi_h)_{\omega_{\mathbf{z}}} &= (r_{\mathbf{z},n}, \varphi_h)_{\omega_{\mathbf{z}}} && \forall \varphi_h \in P_1^*(\mathcal{T}_{\mathbf{z}})
\end{aligned}
$$

- Error estimator: $\eta_n = \|\nabla u_{h,n} - \mathbf{q}_{h,n}\|_{L^2(\Omega)}$
- Weinstein's bound: $\ell_n = \left(-\eta_n + \sqrt{\eta_n^2 + 4\Lambda_{h,n}}\right)^2 / 4$

  provided $\Lambda_{h,n} \le \sqrt{\lambda_n \lambda_{n+1}}$.
- Kato's bound: $L_n = \Lambda_{h,n} \left(1 + \nu \Lambda_{h,n} \sum_{i=n}^{s} \frac{\eta_i^2}{\Lambda_{h,i}^2 (\nu - \Lambda_{h,i})}\right)^{-1}$

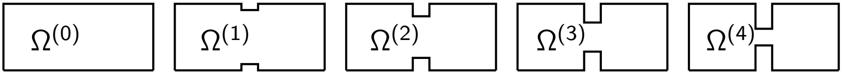
  provided $\Lambda_{h,s} < \nu \le \lambda_{s+1}$.

Guaranteed ν by homotopy method [Plum 1990, 1991]

$$\begin{aligned} -\Delta u_i &= \lambda_i u_i \quad && \text{in } \Omega \\ u_i &= 0 && \text{on } \partial\Omega \end{aligned}$$

- Let $\Omega = \Omega^{(m)} \subset \Omega^{(m-1)} \subset \cdots \subset \Omega^{(1)} \subset \Omega^{(0)}$.
- Let exact eigenvalues are known on $\Omega^{(0)}$.
- Courant minimax principle $\Rightarrow \lambda_n^{(k-1)} \leq \lambda_n^{(k)}$, $k = 1, 2, \ldots, m$.

Example:

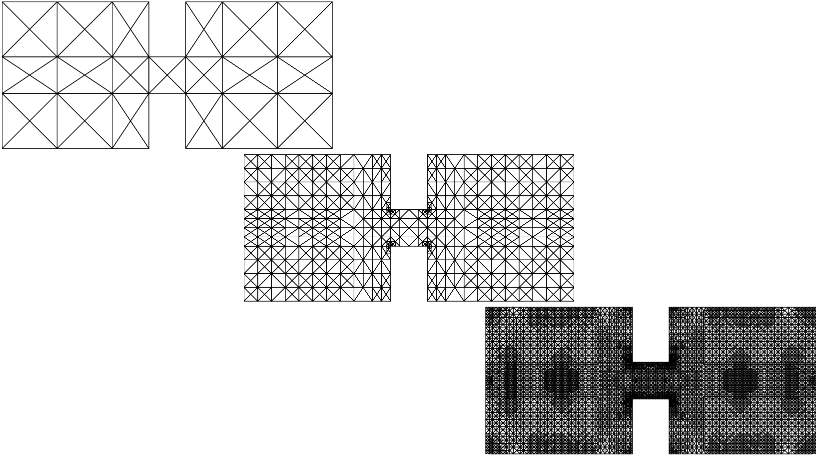


$\lambda_{11}^{(0)} \doteq 8.938$ | $\nu = \lambda_{11}^{(0)}$, $L_{10}^{(1)} \doteq 8.185$ | $\nu = L_{10}^{(1)}$, $L_8^{(2)} \doteq 7.399$ | $\nu = L_8^{(2)}$, $L_7^{(3)} \doteq 7.080$ | $\nu = L_7^{(3)}$

Example: Dumbbell – convergence

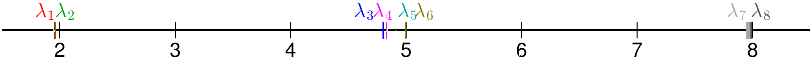
$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega = \text{dumbbell}$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

Adaptively refined meshes:



Example: Dumbbell – convergence

Spectrum:



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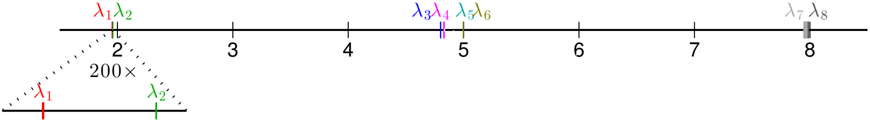

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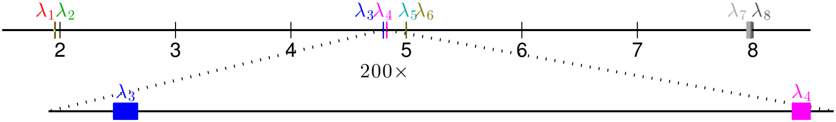

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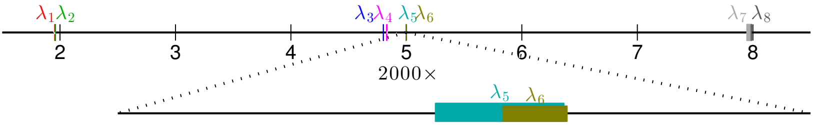

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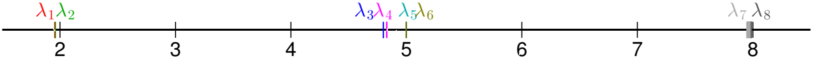
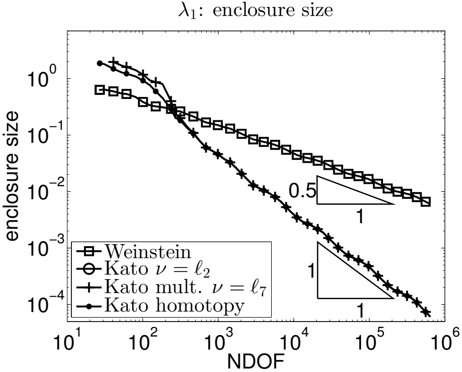
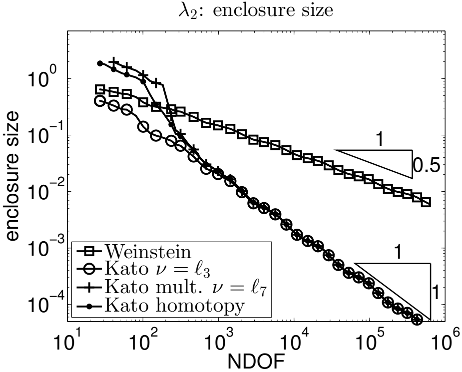


Eigenvalue enclosure sizes:





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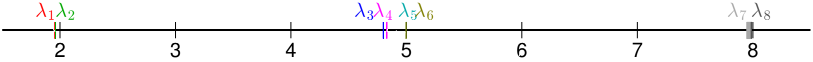

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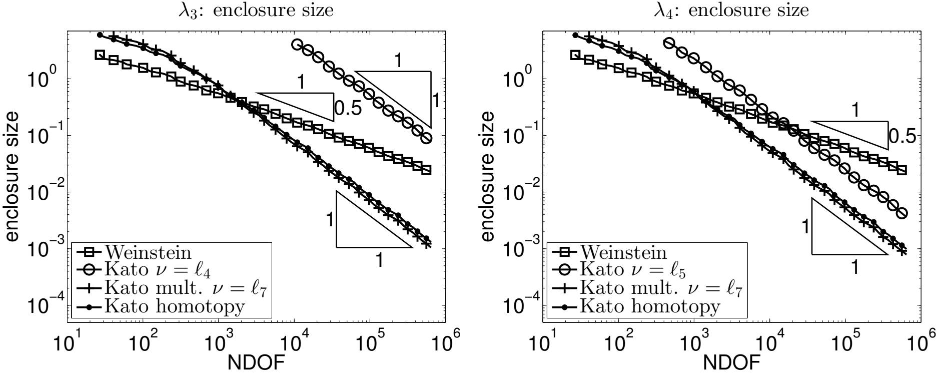

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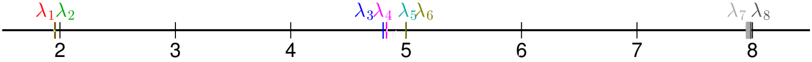

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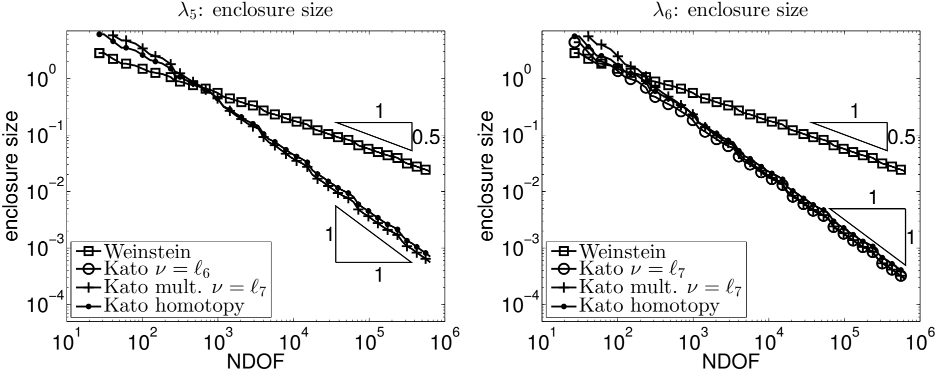

Conclusions

- Good for general symmetric elliptic second-order operators.
- Mixed boundary conditions (e.g. Steklov problem).
- Standard conforming finite element technology.
- Natural for adaptive mesh refinement.
- A priori information on spectrum needed.
- Combination of both Weinstein's and Kato's bound is useful.
- Homotopy method enables a guaranteed choice of ν.

Thank you for your attention

Tomáš Vejchodský (vejchod@math.cas.cz)

Institute of Mathematics
Czech Academy of Sciences

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