Flux reconstructions and lower bounds on eigenvalues

Tomáš Vejchodský (vejchod@math.cas.cz)

Institute of Mathematics
Czech Academy of Sciences

Supported by Neuron Fund for Support of Science, project no. 24/2016

Humboldt University of Berlin, November 16, 2017

Outline

- Introduction
- Lehmann–Goerisch method
- Flux reconstruction
- Numerical examples

Eigenvalue problem

Find $u_i \in V$, $u_i \neq 0$, and $\lambda_i \in \mathbb{R}$ such that

$$a(u_i, v) = \lambda_i b(u_i, v) \quad \forall v \in V.$$

Setting

- V ... Hilbert space
- a ... V-elliptic, symmetric bilinear form on V
- b ... symmetric positive (semi)definite bilinear form on V
- Let $|\cdot|_b$ be compact with respect to $\|\cdot\|_a$. (Any sequence bounded in $\|\cdot\|_a$ contains a Cauchy subsequence in $|\cdot|_b$.)

Facts

- $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ countable sequence of eigenvalues
- $b(u_i, u_j) = \delta_{ij} \quad \forall i, j = 1, 2, \ldots$
- $|v|_b^2 = \sum_{i=1}^{\infty} |b(v, u_i)|^2$ for all $v \in V$
- $\|v\|_a^2 = \sum_{i=1}^{\infty} \lambda_i |b(v, u_i)|^2$ for all $v \in V$ (if b positive definite)

Example: Laplace eigenvalue problem

$$\begin{aligned} -\Delta u_i &= \lambda_i u_i && \text{in } \Omega \\ u_i &= 0 && \text{on } \partial\Omega \end{aligned}$$

Example: Laplace eigenvalue problem

$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i \quad \text{in } \Omega \\
u_i &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

Weak formulation

$\lambda_i > 0, u_i \in V : \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V$

Notation:

$V = H^1_0(\Omega), \quad a(u,v) = (\nabla u, \nabla v), \quad b(u,v) = (u,v)$

Example: Laplace eigenvalue problem

$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i \quad && \text{in } \Omega \\
u_i &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Weak formulation

$\lambda_i > 0, u_i \in V: \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V$

Finite element method

$\Lambda_{h,i} > 0, u_{h,i} \in V_h: \quad (\nabla u_{h,i}, \nabla v_h) = \Lambda_{h,i}(u_{h,i}, v_h) \quad \forall v_h \in V_h$

Notation:

$V = H_0^1(\Omega), \quad a(u,v) = (\nabla u, \nabla v), \quad b(u,v) = (u,v)$

$V_h = \{v_h \in V : v_h|_K \in P_p(K) \quad \forall K \in \mathcal{T}_h\}$

Rayleigh–Ritz method

Theorem

Let

- $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N \in V$ ... linearly independent
- $A_{0,ij} = a(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = b(\tilde{u}_i, \tilde{u}_j)$
- $\Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_N$ be eigenvalues of $A_0 \mathbf{x}_n = \Lambda_n A_1 \mathbf{x}_n$

Then

$$\lambda_n \leq \Lambda_n \quad \forall n = 1, 2, \ldots, N.$$

Lower bounds – history

Standard (conforming) approach:
Temple (1928), Weinstein (1937), Kato (1949), Lehmann (1949), Goerisch (1985), ...

Nonconforming FEM:
Carstensen (2013), Gedicke (2014), Gallistl (2013), Xuefeng LIU (2015), ...

Many results: M.G. Armentano, G. Barrenechea, H. Behnke, R.G. Duran, L. Grubišić, Jun Hu, J.R. Kuttler, Y.A. Kuznetsov, Fubiao Lin, Qun Lin, Xuefeng Liu, M. Plum, S.I. Repin, V.G. Sigillito, M. Vohralík, Hehu Xie, Yidu Yang, Zhimin Zhang, ... *many others*

Lehmann method

Theorem

Let $\Lambda_N < \rho \leq \lambda_{N+1}$

- $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N \in V$ be linearly independent
- $A_{0,ij} = a(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = b(\tilde{u}_i, \tilde{u}_j)$
- $w_i \in V : \quad a(w_i, v) = b(\tilde{u}_i, v) \quad \forall v \in V$
  $A_{2,ij} = a(w_i, w_j)$
- $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N : \quad (A_0 - \rho A_1)\mathbf{x} = \mu(A_0 - 2\rho A_1 + \rho^2 A_2)\mathbf{x}$

Then

$$\rho - \frac{\rho}{1 - \mu_{N+1-n}} \leq \lambda_n \quad n = 1, 2, \ldots, N$$

Lehmann method

Theorem

Let $\Lambda_N < \rho \le \lambda_{N+1}$

- $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N \in V$ be lin. indep.
- $A_{0,ij} = a(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = b(\tilde{u}_i, \tilde{u}_j)$
- $w_i \in V : \quad a(w_i, v) = b(\tilde{u}_i, v) \quad \forall v \in V$
  $A_{2,ij} = a(w_i, w_j)$
- $\mu_1 \le \mu_2 \le \cdots \le \mu_N : \quad (A_0 - \rho A_1)\mathbf{x} = \mu(A_0 - 2\rho A_1 + \rho^2 A_2)\mathbf{x}$

Then

$$\rho - \frac{\rho}{1 - \mu_{N+1-n}} \le \lambda_n \quad n = 1, 2, \ldots, N$$

Observation

$$a(w_i, v) = b(\tilde{u}_i, v) \approx \frac{1}{\Lambda_{h,i}} a(\tilde{u}_i, v) \quad \forall v \in V$$

$$\Rightarrow w_i \approx \frac{1}{\Lambda_{h,i}} \tilde{u}_i$$

Lehmann–Goerisch method

Theorem

Let $\Lambda_N < \rho \leq \lambda_{N+1}$

- $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N \in V$ be linearly independent
- $A_{0,ij} = a(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = b(\tilde{u}_i, \tilde{u}_j)$
- X ... vector space
  $\mathcal{B}$ ... positive semidefinite symmetric bilinear form on X
  $T : V \to X$ ... linear operator: $\mathcal{B}(Tu, Tv) = a(u, v) \quad \forall u, v \in V$
  $\hat{\mathbf{w}}_i \in X : \quad \mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \quad \forall v \in V$
  $\hat{A}_{2,ij} = \mathcal{B}(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j)$
- $\hat{\mu}_1 \leq \hat{\mu}_2 \leq \cdots \leq \hat{\mu}_N : \quad (A_0 - \rho A_1)\hat{\mathbf{x}} = \hat{\mu}(A_0 - 2\rho A_1 + \rho^2 \hat{A}_2)\hat{\mathbf{x}}$

Then

$$\rho - \frac{\rho}{1 - \hat{\mu}_{N+1-n}} \leq \lambda_n \quad n = 1, 2, \ldots, N$$

How to find good $\hat{\mathbf{w}}_i$?

Need

$\Rightarrow$ $\hat{A}_2 \approx A_2$

$\Rightarrow$ $\mathcal{B}(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j) \approx a(w_i, w_j) = \mathcal{B}(Tw_i, Tw_j)$

$\Rightarrow$ $\hat{\mathbf{w}}_i \approx Tw_i \approx \frac{1}{\Lambda_{h,i}} T\tilde{u}_i$ (using Observation $w_i \approx \frac{1}{\Lambda_{h,i}}\tilde{u}_i$)

Natural idea

make $|\hat{\mathbf{w}}_i - \frac{1}{\Lambda_{h,i}} T\tilde{u}_i|^2_{\mathcal{B}}$ small

Example: Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Setting

- $V = H_0^1(\Omega)$, $a(u,v) = (\nabla u, \nabla v) + \gamma(u,v)$, $b(u,v) = (u,v)$
- $X = \left[L^2(\Omega)\right]^3$
- $\mathcal{B}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = (\hat{u}_1, \hat{v}_1) + (\hat{u}_2, \hat{v}_2) + \gamma(\hat{u}_3, \hat{v}_3)$
- $Tu = \begin{pmatrix} \nabla u \\ u \end{pmatrix}$

Facts

(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

Example: Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Setting

- $V = H_0^1(\Omega),\ a(u,v) = (\nabla u, \nabla v) + \gamma(u,v),\ b(u,v) = (u,v)$
- $X = \left[L^2(\Omega)\right]^3$
- $\mathcal{B}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = (\hat{u}_1, \hat{v}_1) + (\hat{u}_2, \hat{v}_2) + \gamma(\hat{u}_3, \hat{v}_3)$
- $Tu = \begin{pmatrix} \nabla u \\ u \end{pmatrix}$

Facts

(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

(b) $\mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \Leftarrow \hat{\mathbf{w}}_i = \begin{pmatrix} \boldsymbol{\sigma}_i \\ \hat{w}_{i,3} \end{pmatrix} \quad \boldsymbol{\sigma}_i \in \mathbf{H}(\mathrm{div}, \Omega)$

$$(\boldsymbol{\sigma}_i, \nabla v) + \gamma(\hat{w}_{i,3}, v) = (\tilde{u}_i, v) \quad \forall v \in V$$

$$-(\mathrm{div}\, \boldsymbol{\sigma}_i, v) + \gamma(\hat{w}_{i,3}, v) = (\tilde{u}_i, v) \quad \forall v \in V$$

$$\hat{w}_{i,3} = \frac{1}{\gamma}(\tilde{u}_i + \mathrm{div}\, \boldsymbol{\sigma}_i)$$

Example: Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Setting

- $V = H_0^1(\Omega),\ a(u,v) = (\nabla u, \nabla v) + \gamma(u,v),\ b(u,v) = (u,v)$
- $X = \left[L^2(\Omega)\right]^3$
- $\mathcal{B}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = (\hat{u}_1, \hat{v}_1) + (\hat{u}_2, \hat{v}_2) + \gamma(\hat{u}_3, \hat{v}_3)$
- $Tu = \begin{pmatrix} \nabla u \\ u \end{pmatrix}$

Facts

(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

(b) $\mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \Leftarrow \hat{\mathbf{w}}_i = \begin{pmatrix} \boldsymbol{\sigma}_i \\ \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i) \end{pmatrix} \quad \boldsymbol{\sigma}_i \in \mathbf{H}(\operatorname{div}, \Omega)$

Example: Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Setting

- $V = H_0^1(\Omega),\ a(u,v) = (\nabla u, \nabla v) + \gamma(u,v),\ b(u,v) = (u,v)$
- $X = \left[L^2(\Omega)\right]^3$
- $\mathcal{B}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = (\hat{u}_1, \hat{v}_1) + (\hat{u}_2, \hat{v}_2) + \gamma(\hat{u}_3, \hat{v}_3)$
- $Tu = \begin{pmatrix} \nabla u \\ u \end{pmatrix}$

Facts

(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

(b) $\mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \Leftarrow \hat{\mathbf{w}}_i = \begin{pmatrix} \boldsymbol{\sigma}_i \\ \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i) \end{pmatrix} \quad \boldsymbol{\sigma}_i \in \mathbf{H}(\operatorname{div}, \Omega)$

(c) $\hat{A}_{2,ij} = \mathcal{B}(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j) \ \Leftrightarrow \ \hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i, \tilde{u}_j + \operatorname{div} \boldsymbol{\sigma}_j)$

Example: Laplace eigenvalue problem

Theorem (Lehmann–Goerisch)

Let $\Lambda_N < \rho \leq \lambda_{N+1}$, $\gamma > 0$

- $\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N \in V$ be linearly independent
- $A_{0,ij} = (\nabla \tilde{u}_i, \nabla \tilde{u}_j) + \gamma(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = (\tilde{u}_i, \tilde{u}_j)$
- $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \dots, \boldsymbol{\sigma}_N \in \mathbf{H}(\operatorname{div}, \Omega)$ be arbitrary
  $\hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i, \tilde{u}_j + \operatorname{div} \boldsymbol{\sigma}_j)$
- $\hat{\mu}_1 \leq \hat{\mu}_2 \leq \dots \leq \hat{\mu}_N : \quad (A_0 - \rho A_1)\hat{\mathbf{x}} = \hat{\mu}(A_0 - 2\rho A_1 + \rho^2 \hat{A}_2)\hat{\mathbf{x}}$

Then

$$\ell_n = \rho - \frac{\rho}{1 - \hat{\mu}_{N+1-n}} \leq \lambda_n \quad n = 1, 2, \dots, N$$

Choice of $\boldsymbol{\sigma}_i$

minimize $|\hat{\mathbf{w}}_i - \frac{1}{\Lambda_i+\gamma} T\tilde{u}_i|^2_{\mathcal{B}} = \left\| \boldsymbol{\sigma}_i - \frac{\nabla \tilde{u}_i}{\Lambda_i+\gamma} \right\|^2_0 + \frac{1}{\gamma} \left\| \operatorname{div} \boldsymbol{\sigma}_i + \frac{\Lambda_i \tilde{u}_i}{\Lambda_i+\gamma} \right\|^2_0$

(i) Constraint minimization: [Behnke, Mertins, Plum, Wieners 2000]

minimize $\left\| \boldsymbol{\sigma}_i - \frac{\nabla \tilde{u}_i}{\Lambda_i+\gamma} \right\|^2_0$ over $\boldsymbol{\sigma}_i \in \mathbf{W}_h \subset \mathbf{H}(\operatorname{div}, \Omega)$

under the constraint: $\operatorname{div} \boldsymbol{\sigma}_i + \frac{\Lambda_i \tilde{u}_i}{\Lambda_i+\gamma} = 0$

$\Leftrightarrow$

Find $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h$, $q_{h,i} \in Q_h$, $i = 1, 2, \ldots, N$

$$(\boldsymbol{\sigma}_{h,i}, \mathbf{w}_h) + (q_{h,i}, \operatorname{div} \mathbf{w}_h) = \left(\frac{\nabla \tilde{u}_i}{\Lambda_i + \gamma}, \mathbf{w}_h \right) \qquad \forall \mathbf{w}_h \in \mathbf{W}_h$$

$$(\operatorname{div} \boldsymbol{\sigma}_{h,i}, \varphi_h) = \left(-\frac{\Lambda_i \tilde{u}_i}{\Lambda_i + \gamma}, \varphi_h \right) \quad \forall \varphi_h \in Q_h$$

$\mathbf{W}_h = \{\boldsymbol{\sigma}_h \in \mathbf{H}(\operatorname{div}, \Omega) : \boldsymbol{\sigma}_h|_K \in \mathbf{RT}_p(K) \quad \forall K \in \mathcal{T}_h\}$

$Q_h = \{q_h \in L^2(\Omega) : q_h|_K \in P_p(K) \quad \forall K \in \mathcal{T}_h\}$

Choice of $\boldsymbol{\sigma}_i$

minimize $|\hat{\mathbf{w}}_i - \frac{1}{\Lambda_i+\gamma} T\tilde{u}_i|^2_{\mathcal{B}} = \left\|\boldsymbol{\sigma}_i - \frac{\nabla\tilde{u}_i}{\Lambda_i+\gamma}\right\|^2_0 + \frac{1}{\gamma}\left\|\operatorname{div}\boldsymbol{\sigma}_i + \frac{\Lambda_i\tilde{u}_i}{\Lambda_i+\gamma}\right\|^2_0$

(ii) Unconstraint minimization:

Find $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h,\ i = 1, 2, \dots, N$

$$(\boldsymbol{\sigma}_{h,i}, \mathbf{w}_h) + \frac{1}{\gamma}(\operatorname{div}\boldsymbol{\sigma}_{h,i}, \operatorname{div}\mathbf{w}_h) = \left(\frac{\nabla\tilde{u}_i}{\Lambda_i+\gamma}, \mathbf{w}_h\right) - \frac{1}{\gamma}\left(\frac{\Lambda_i\tilde{u}_i}{\Lambda_i+\gamma}, \operatorname{div}\mathbf{w}_h\right)$$

$$\forall \mathbf{w}_h \in \mathbf{W}_h$$

$$\mathbf{W}_h = \{\boldsymbol{\sigma}_h \in \mathbf{H}(\operatorname{div}, \Omega) : \boldsymbol{\sigma}_h|_K \in \mathbf{RT}_p(K) \quad \forall K \in \mathcal{T}_h\}$$

Choice of $\boldsymbol{\sigma}_i$

minimize $|\hat{\mathbf{w}}_i - \frac{1}{\Lambda_i+\gamma} T\tilde{u}_i|^2_{\mathcal{B}} = \left\|\boldsymbol{\sigma}_i - \frac{\nabla\tilde{u}_i}{\Lambda_i+\gamma}\right\|_0^2 + \frac{1}{\gamma}\left\|\operatorname{div}\boldsymbol{\sigma}_i + \frac{\Lambda_i\tilde{u}_i}{\Lambda_i+\gamma}\right\|_0^2$

(iii) Local constraint minimization:

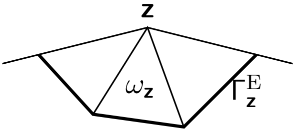
[Braess, Schöberl 2000], [Ern, Vohralík 2013]

$$\boldsymbol{\sigma}_{h,i} = \sum_{\mathbf{z}\in\mathcal{N}_h} \boldsymbol{\sigma}_{\mathbf{z},i},$$

where $\boldsymbol{\sigma}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$ minimizes $\left\|\boldsymbol{\sigma}_{\mathbf{z},i} - \psi_{\mathbf{z}}\frac{\nabla\tilde{u}_i}{\Lambda_i+\gamma}\right\|_{0,\omega_{\mathbf{z}}}^2$

under the constraint: $\operatorname{div}\boldsymbol{\sigma}_{\mathbf{z},i} + \frac{\psi_{\mathbf{z}}\Lambda_i\tilde{u}_i}{\Lambda_i+\gamma} + \frac{\nabla\psi_{\mathbf{z}}\cdot\nabla\tilde{u}_i}{\Lambda_i+\gamma} = 0$ in $\omega_{\mathbf{z}}$

Partition of unity: $\sum_{\mathbf{z}\in\mathcal{N}_h} \psi_{\mathbf{z}} \equiv 1$ in Ω



$\mathbf{W}_{\mathbf{z}} = \{\boldsymbol{\sigma}_{\mathbf{z}} \in \mathbf{H}(\operatorname{div},\omega_{\mathbf{z}}) : \boldsymbol{\sigma}_{\mathbf{z}}|_K \in \mathbf{RT}_p(K)\ \forall K \in \mathcal{T}_{\mathbf{z}}$ and $\boldsymbol{\sigma}_{\mathbf{z}}\cdot\mathbf{n}_{\mathbf{z}} = 0$ on $\Gamma_{\mathbf{z}}^{\mathrm{E}}\}$

$Q_{\mathbf{z}} = \{q_{\mathbf{z}} \in L^2(\omega_{\mathbf{z}}) : q_{\mathbf{z}}|_K \in P_p(K) \quad \forall K \in \mathcal{T}_{\mathbf{z}}\}$

Choice of $\boldsymbol{\sigma}_i$

$$\text{minimize } |\hat{\mathbf{w}}_i - \tfrac{1}{\Lambda_i+\gamma} T\tilde{u}_i|^2_{\mathcal{B}} = \left\|\boldsymbol{\sigma}_i - \tfrac{\nabla \tilde{u}_i}{\Lambda_i+\gamma}\right\|_0^2 + \tfrac{1}{\gamma}\left\|\operatorname{div}\boldsymbol{\sigma}_i + \tfrac{\Lambda_i \tilde{u}_i}{\Lambda_i+\gamma}\right\|_0^2$$

(iii) Local constraint minimization:

[Braess, Schöberl 2000], [Ern, Vohralík 2013]

$$\boldsymbol{\sigma}_{h,i} = \sum_{\mathbf{z}\in\mathcal{N}_h} \boldsymbol{\sigma}_{\mathbf{z},i},$$

Find $\boldsymbol{\sigma}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$, $q_{\mathbf{z},i} \in Q_{\mathbf{z}}$, $i = 1, 2, \dots, N$

$$(\boldsymbol{\sigma}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} + (q_{\mathbf{z},i}, \operatorname{div}\mathbf{w}_h)_{\omega_{\mathbf{z}}} = \left(\psi_{\mathbf{z}} \frac{\nabla \tilde{u}_i}{\Lambda_i + \gamma}, \mathbf{w}_h\right)_{\omega_{\mathbf{z}}} \qquad \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}}$$

$$(\operatorname{div}\boldsymbol{\sigma}_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} = \left(-\frac{\psi_{\mathbf{z}}\Lambda_i \tilde{u}_i}{\Lambda_i + \gamma}, \varphi_h\right)_{\omega_{\mathbf{z}}} + \left(\frac{\nabla\psi_{\mathbf{z}} \cdot \nabla \tilde{u}_i}{\Lambda_i + \gamma}, \varphi_h\right)_{\omega_{\mathbf{z}}} \qquad \forall \varphi_h \in Q_{\mathbf{z}}$$

$$\mathbf{W}_{\mathbf{z}} = \{\boldsymbol{\sigma}_{\mathbf{z}} \in \mathbf{H}(\operatorname{div}, \omega_{\mathbf{z}}) : \boldsymbol{\sigma}_{\mathbf{z}}|_K \in \mathbf{RT}_p(K)\ \forall K \in \mathcal{T}_{\mathbf{z}} \text{ and } \boldsymbol{\sigma}_{\mathbf{z}} \cdot \mathbf{n}_{\mathbf{z}} = 0 \text{ on } \Gamma^{\mathrm{E}}_{\mathbf{z}}\}$$

$$Q_{\mathbf{z}} = \{q_{\mathbf{z}} \in L^2(\omega_{\mathbf{z}}) : q_{\mathbf{z}}|_K \in P_p(K) \quad \forall K \in \mathcal{T}_{\mathbf{z}}\}$$

Choice of $\boldsymbol{\sigma}_i$

minimize $|\hat{\mathbf{w}}_i - \frac{1}{\Lambda_i+\gamma} T\tilde{u}_i|_{\mathcal{B}}^2 = \left\|\boldsymbol{\sigma}_i - \frac{\nabla\tilde{u}_i}{\Lambda_i+\gamma}\right\|_0^2 + \frac{1}{\gamma}\left\|\operatorname{div}\boldsymbol{\sigma}_i + \frac{\Lambda_i\tilde{u}_i}{\Lambda_i+\gamma}\right\|_0^2$

(iv) Local unconstraint minimization:

Find $\boldsymbol{\sigma}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$, $i = 1, 2, \dots, N$

minimizing $\left\|\boldsymbol{\sigma}_{\mathbf{z},i} - \psi_{\mathbf{z}}\frac{\nabla\tilde{u}_i}{\Lambda_i+\gamma}\right\|_{0,\omega_{\mathbf{z}}}^2 + \frac{1}{\gamma}\left\|\operatorname{div}\boldsymbol{\sigma}_{\mathbf{z},i} + \frac{\psi_{\mathbf{z}}\Lambda_i\tilde{u}_i}{\Lambda_i+\gamma}\right\|_{0,\omega_{\mathbf{z}}}^2$

$$(\boldsymbol{\sigma}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} + \frac{1}{\gamma}(\operatorname{div}\boldsymbol{\sigma}_{\mathbf{z},i}, \operatorname{div}\mathbf{w}_h)_{\omega_{\mathbf{z}}} = \left(\psi_{\mathbf{z}}\frac{\nabla\tilde{u}_i}{\Lambda_i+\gamma}, \mathbf{w}_h\right)_{\omega_{\mathbf{z}}} - \frac{1}{\gamma}\left(\frac{\psi_{\mathbf{z}}\Lambda_i\tilde{u}_i}{\Lambda_i+\gamma}, \operatorname{div}\mathbf{w}_h\right)_{\omega_{\mathbf{z}}} \quad \forall\mathbf{w}_h \in \mathbf{W}_{\mathbf{z}}$$

$\mathbf{W}_{\mathbf{z}} = \{\boldsymbol{\sigma}_{\mathbf{z}} \in \mathbf{H}(\operatorname{div}, \omega_{\mathbf{z}}) : \boldsymbol{\sigma}_{\mathbf{z}}|_K \in \mathbf{RT}_p(K)\ \forall K \in \mathcal{T}_{\mathbf{z}}$ and $\boldsymbol{\sigma}_{\mathbf{z}} \cdot \mathbf{n}_{\mathbf{z}} = 0$ on $\Gamma_{\mathbf{z}}^{\mathrm{E}}\}$

Remark 1. Weinstein and Kato bounds

Set $\eta_i = \|\nabla u_{h,i} - (\Lambda_{h,i} + \gamma)\boldsymbol{\sigma}_{h,i}\|_0 \quad i = 1, 2, \dots, N$

$\boldsymbol{\sigma}_{h,i}$ computed by (i) or (iii)

Weinstein bound: $\ell_i^{\mathrm{W}} = \frac{1}{4}\left(-\eta_i + \sqrt{\eta_i^2 + 4\Lambda_{h,i}}\right)^2$

Kato bound: $\ell_i^{\mathrm{K}} = \Lambda_{h,i}\left(1 + \nu\Lambda_{h,i}\sum_{j=i}^{N}\frac{\eta_j^2}{\Lambda_{h,j}^2(\nu - \Lambda_{h,j})}\right)^{-1}$

where $\Lambda_{h,N} < \nu$

Theorem 1.
If $\sqrt{\lambda_{i-1}\lambda_i} \le \Lambda_{h,i} \le \sqrt{\lambda_i\lambda_{i+1}}$ then $\ell_i^{\mathrm{W}} \le \lambda_i$.

Theorem 2.
If $\nu \le \lambda_{N+1}$ then $\ell_i^{\mathrm{K}} \le \lambda_i$ for all $i = 1, 2, \dots, N$.

[Vejchodský, Šebestová 2017]

Remark 2. Adaptive mesh refinement

Residual

$$\varrho_i \in V : \quad (\nabla \varrho_i, \nabla v) = (\nabla u_{h,i}, \nabla v) - \Lambda_{h,i}(u_{h,i}, v) \quad \forall v \in V$$

Theorem

$$\|\nabla \varrho_i\|_0 \leq \eta_i, \quad \text{where } \eta_i = \|\nabla u_{h,i} - (\Lambda_{h,i} + \gamma)\boldsymbol{\sigma}_{h,i}\|_0 .$$

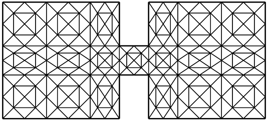
Local error indicators for mesh refinement

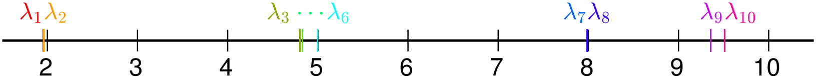
$$\eta_{i,K} = \|\nabla u_{h,i} - (\Lambda_{h,i} + \gamma)\boldsymbol{\sigma}_{h,i}\|_{0,K} \quad \forall K \in \mathcal{T}_h$$

$\boldsymbol{\sigma}_{h,i}$ computed by (i) or (iii)

Example: Dumbbell shaped domain

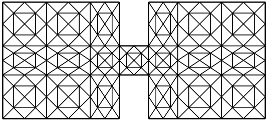
$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$



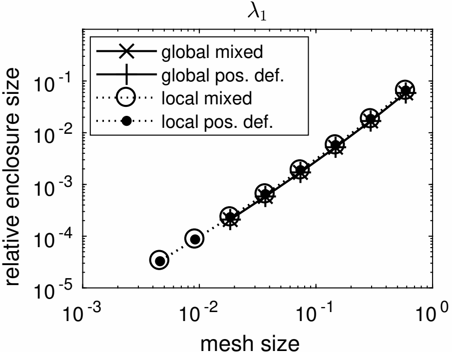


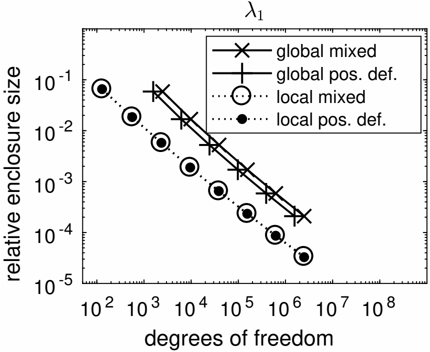
Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$



Uniformly refined meshes:

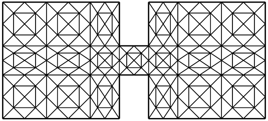




- $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

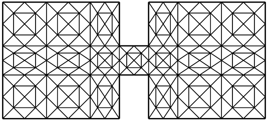
Example: Dumbbell shaped domain

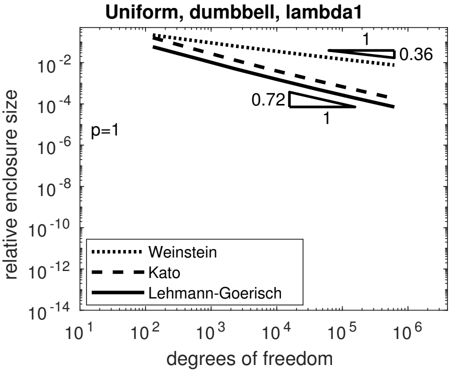
$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i \quad &\text{in } \Omega \\
u_i &= 0 \quad &\text{on } \partial\Omega
\end{aligned}
$$



Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

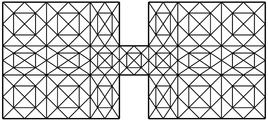


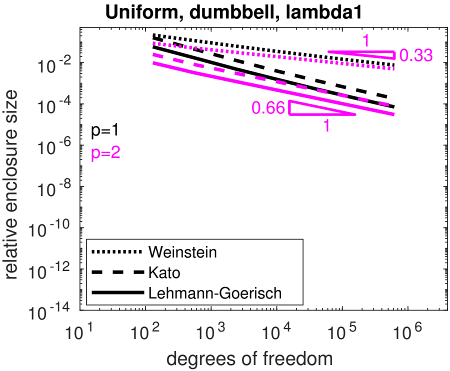


- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

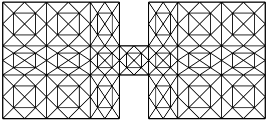


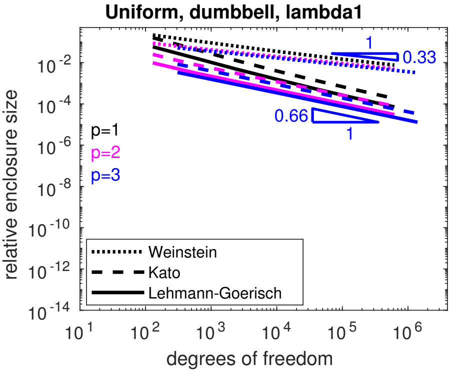


- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

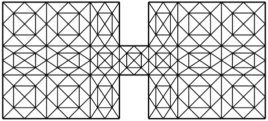


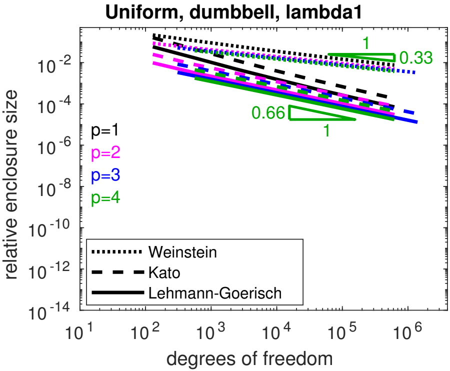


- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

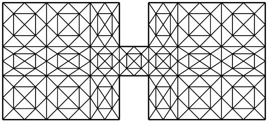


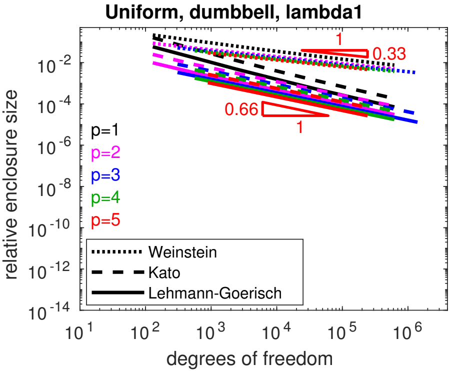


- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

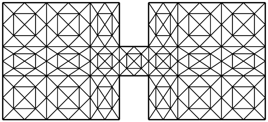


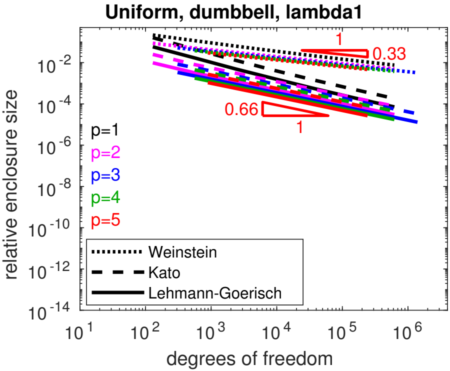


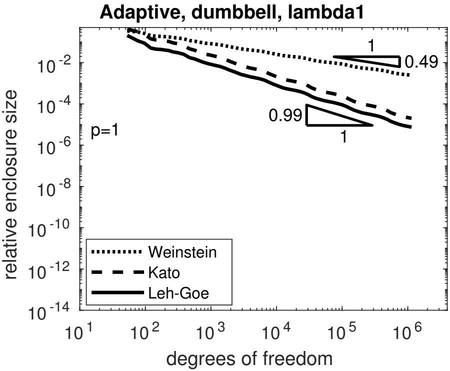
- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$



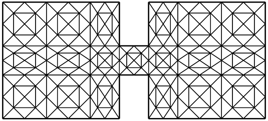


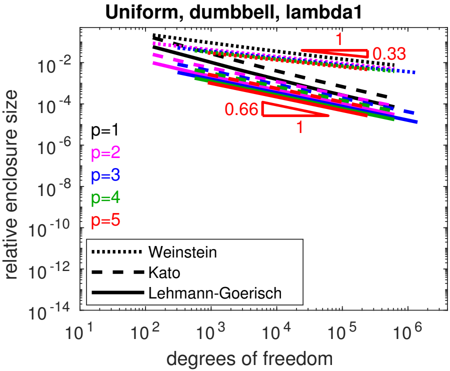


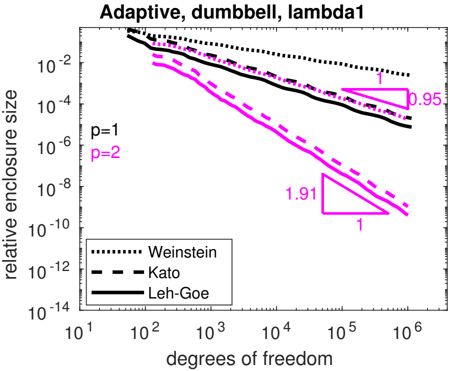
- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$



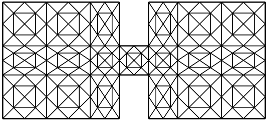


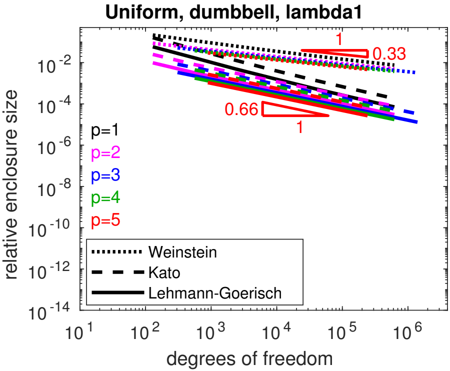


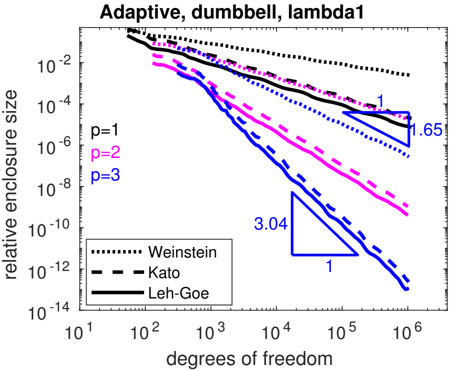
- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i && \text{in } \Omega \\
u_i &= 0 && \text{on } \partial\Omega
\end{aligned}
$$



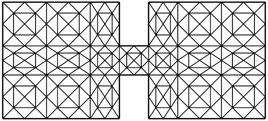


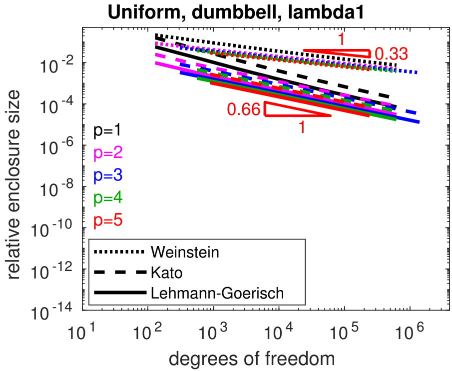


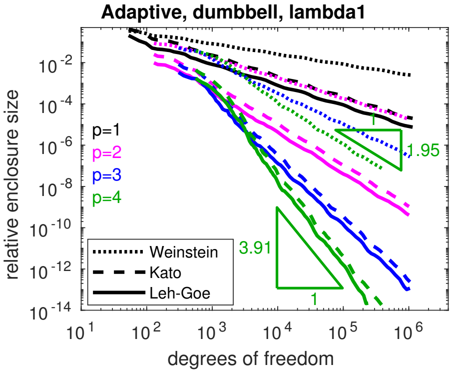
- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$



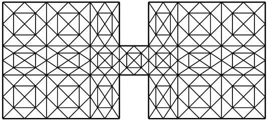


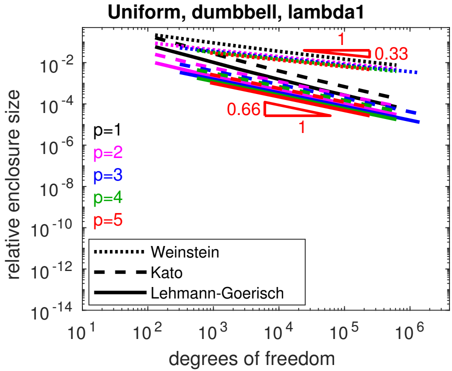


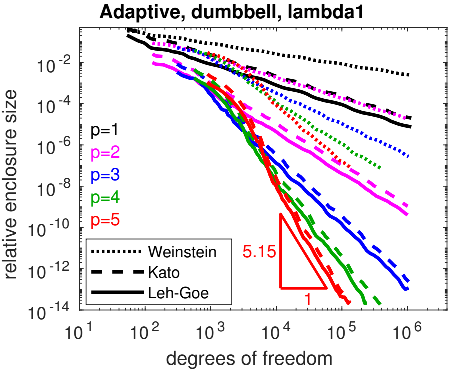
- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$



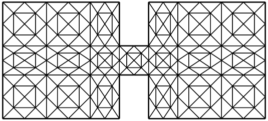


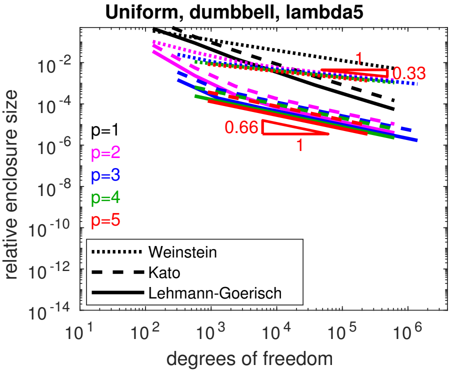


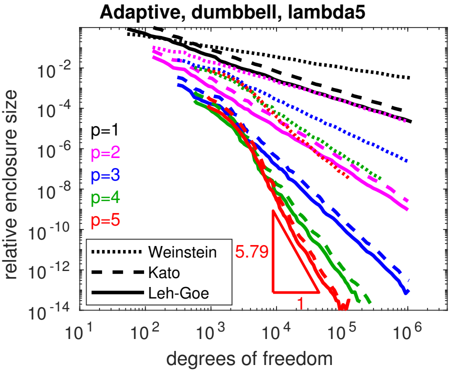
- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$



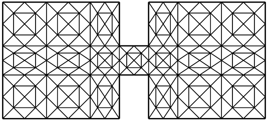




- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, a priori lower bound ρ by Weinstein's method

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$



Tight pairs of eigenvalues:

$$4.996837097248\mathbf{9} \le \lambda_5 \le 4.996837097249\mathbf{0}$$
$$4.996850904101\mathbf{5} \le \lambda_6 \le 4.996850904101\mathbf{6}$$

$$7.98696729210\mathbf{28} \le \lambda_7 \le 7.98696729210\mathbf{38}$$
$$7.98703430682\mathbf{16} \le \lambda_8 \le 7.98703430682\mathbf{27}$$

Conclusions

- Flux reconstructions for source problems

  $\Rightarrow$ good for eigenvalue problems
- Savings in memory requirements
- Parallelization

Flexibility:

- General symmetric elliptic operators
- Higher-order approximations
- Adaptivity