Mathematical theory of fluids in motion

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Abstract

The goal of the course is to present the recent development of the mathematical fluid dynamics in the framework of classical fluid mechanics phenomenological models. In particular, we discuss the Navier-Stokes (viscous) and the Euler (inviscid) systems modelling the motion of a compressible fluid. The theory is developed from fundamental physical principles, the necessary mathematical tools introduced at the moment when needed. In particular, we discuss various concepts of solutions and their relevance in applications. Particular interest is devoted to well-posedness of the initial-value problems and their approximations including possibly certain numerical schemes.

All pictures used in the text thanks to wikipedia

Contents

1 Continuum fluid mechanics

Our goal is to present a *phenomenological theory* of fluids in motion - fluid *dynamics* - based on the principles of classical continuum mechanics. Accordingly, a fluid is described my means of *observable* macroscopic quantities as *mass density*, *bulk velocity*, *internal energy* depending on the *time* t and the reference spatial coordinate $x \in R^N$. Although the physically relevant values are $N = 1, 2, 3$, we mostly focus on the most complex case $N = 3$. If not stated otherwise, the *Eulerian reference system* will be used attached to the physical domain occupied by the fluid in contrast with the *Lagrangean description* related to the hypothetical fluid particles and their trajectories - streamlines - in the physical space.

Leonhard Euler
[1707–1783]

1.1 Mass density, velocity, mass conservation

The distribution of a fluid at a given time t is given by the *mass density* $\varrho = \varrho(t, x)$ - a non-negative scalar function - such that the integral

$$\int_B \varrho(t,x) \ \mathrm{d}x = M(B)$$

gives the total mass of the fluid contained in a given set B at the time t.

The *motion* of (hypothetical) fluid particles is determined by the *velocity field* $\mathbf{u}(t, x) \in R^N$. The trajectory of a bulk of fluid occupying at the initial instant $t = 0$ the set B is given by

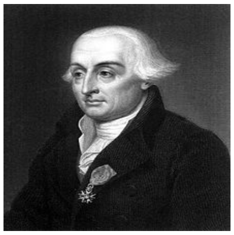
$$\mathbf{X}(t, B), \ t \geq 0, \ \text{where } \frac{\partial}{\partial t}\mathbf{X}(t, x) = \mathbf{u}(t, \mathbf{X}(t, x)), \ \mathbf{X}(0, x) = x, \ x \in B.$$

The individual trajectories $t \mapsto \mathbf{X}(t, x)$ are called *streamlines*. The velocity field must enjoy certain regularity in the $x-$ variable for the streamlines to be well defined on a time interval I, specifically

$$\nabla_x \mathbf{u} \in L^1(I; L^\infty_{\mathrm{loc}}(R^N, R^N)). \tag{1.1}$$

Remark 1.1. *As we shall see below, the question whether or solutions of a fluid model enjoy the desired regularity (1.1) is mostly an open problem, in particular in higher space dimensions, see the review paper by Fefferman [12]. This partially explains why the Eulerian rather than Lagrangean description is used in the mathematical theory.*

1.1.1 Eulerian vs. Lagrangean description



Joseph-Louis Lagrange
[1736–1813]

Given a velocity field $\mathbf{u}$ generating a family of streamlines $\mathbf{X} = \mathbf{X}(t, x)$ in the physical space Ω, a quantity Q can be expressed in terms of the Eulerian variables as

$$Q = Q_E(t, x),\ t \in I,\ x \in \mathbf{X}(t, B),$$

or, in the Lagrangean variables

$$Q = Q_L(t, Y),\ t \in I,\ Y \in B,$$

where

$$Q_L(t, Y) = Q_E(t, \mathbf{X}(t, Y)) \text{ or } Q_L(t, \mathbf{X}^{-1}(t, x)) = Q_E(t, x).$$

In particular, the time derivative in the Lagrangean setting corresponds to the *material* derivative in the Euler coordinates:

$$\partial_t Q_L(t, Y) = \partial_t Q_E(t, \mathbf{X}(t, x)) + \nabla_x Q_E(t, \mathbf{X}(t, x)) \cdot \mathbf{u}(t, \mathbf{X}(t, x)).$$

At first glance it may seem that the Lagrangean description is simpler; whence more suitable for a mathematical treatment. On the other hand, transforming *spatial gradients* requires invertibility of the mapping $\mathbf{X}$ and therefore certain regularity of the velocity field that is often out of reach of the available analytical methods.

1.1.2 Mass transport

Consider a piece B of the physical space $\Omega \subset R^N$ containing a fluid of the density ϱ moving with the (Eulerian) velocity $\mathbf{u} = \mathbf{u}(t, x)$.

The physical principle of MASS CONSERVATION asserts: The change of the total mass of the fluid contained in B during a time interval $t_1 < t_2$,

$$\int_B \varrho(t_2, x)\ \mathrm{d}x\ \mathrm{d}t - \int_B \varrho(t_1, x)\ \mathrm{d}x\ \mathrm{d}t,$$

= (equals)
the total out/in flux of the mass through the boundary ∂_B

$$-\int_{t_1}^{t_2} \int_{\partial B} \varrho(t, x)\mathbf{u}(t, x) \cdot \mathbf{n}(x)\ \mathrm{dS}_x\ \mathrm{d}t,$$

where $\mathbf{n}$ denotes the outer norm vector to ∂B.

The above relation must hold for any time interval $[t_1, t_2]$ and any volume element B. Assuming that all quantities are sufficiently smooth (differentiable), we may use the Gauss-Green theorem and perform the limit $t_2 \to t_1$ to deduce a differential form of the principle of mass conservation - *equation of continuity*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0. \tag{1.2}$$

It is remarkable that the same principle can be derived from an apparently weaker statement

$$\int_I \int_\Omega [\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi] \ \mathrm{d}x \ \ \mathrm{d}t = 0 \text{ for any } \varphi \in C^1_c(I \times \Omega), \tag{1.3}$$

which is usually termed the *weak formulation* of (1.2). Indeed it is enough to take $\varphi_\varepsilon \in C^\infty_c(I \times \Omega)$ a suitable approximation of the characteristic function $1_{[t_1,t_2]\times B}$ and let $\varepsilon \to 0$. If ∂B is smooth, one can consider a family of Lipschitz functions

$$\varphi_\varepsilon(t,x) = \min \left\{ \frac{1}{\varepsilon} \mathrm{dist}[x, \partial B]; 1 \right\} \times \min \left\{ \frac{1}{\varepsilon} \min\{t - t_1; t_2 - t_1\}; 1 \right\}.$$

1.2 Momentum equation

In order to determine the velocity field $\mathbf{u}$ we need a relation between the changes of the momentum $\varrho\mathbf{u}$ and the material forces acting on a volume element of the fluid. We consider two kinds of such forces: (i) *stress forces* acting on any surface element shared by two adjacent parts of the fluid, (ii) *bulk* or volumic forces. The stress is characterized by the *Cauchy stress tensor* $\mathbb{T}$ producing the stress $\mathbb{T} \cdot \mathbf{n}$ acting on a unit surface element characterized by a normal vector $\mathbf{n}$.

Augustin-Louis Cauchy
[1789–1857]

Mathematical formulation of NEWTON'S SECOND LAW reads

$$\int_B \varrho \mathbf{u}(t_2, x) \ \mathrm{d}x - \int_B \varrho \mathbf{u}(t_1, x) \ \mathrm{d}x$$

$$= - \int_{t_1}^{t_2} \int_{\partial B} \varrho(t,x)\mathbf{u}(t,x) \ (\mathbf{u}(t,x) \cdot \mathbf{n}(x)) \ \mathrm{dS}_x \, \mathrm{d}t$$

$$+ \int_{t_1}^{t_2} \int_{\partial B} \mathbb{T}(t,x) \cdot \mathbf{n}(x) \ \mathrm{dS}_x \, \mathrm{d}t + \int_{t_1}^{t_2} \int_B \varrho(t,x) \mathbf{f}(t,x) \, \mathrm{d}x \ \ \mathrm{d}t,$$

for any B and any $t_1 < t_2$.

Remark 1.2. *Note that the above relation is* vectorial *relating N components of the momentum $\varrho\mathbf{u}$ to the stress forces.*

Similarly to Section 1.1.2, we may deduce the differential form of Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T} + \varrho \mathbf{f}, \tag{1.4}$$

or its *weak formulation*

$$\int_I \int_\Omega [(\varrho \mathbf{u}) \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi}] \ \mathrm{d}x \, \mathrm{d}t = \int_I \int_\Omega \mathbb{T} : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t - \int_I \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \ \mathrm{d}t \tag{1.5}$$

for any $\boldsymbol{\varphi} \in C^1_c(I \times \Omega; R^N)$.

Remark 1.3. *As the density ϱ satisfies the equation of continuity (1.2) we may always write*

$$\partial_t(\varrho Q) + \mathrm{div}_x(\varrho Q \mathbf{u}) = \varrho \left(\partial_t Q + \mathbf{u} \cdot \nabla_x Q \right),$$

where the expression on the right hand side is nothing other than the material derivative of a quantity Q multiplied on the mass density.

1.3 Cauchy stress in fluids, examples of fluid equations

The system of equations (1.2), (1.4) is not closed, a description of the Cauchy stress $\mathbb{T}$ is needed in terms of ϱ, $\mathbf{u}$ or other quantities as the case may be. The following statement is often used as a *mathematical definition* of fluid.

A FLUID is characterized by STOKES' LAW

$$\mathbb{T} = \mathbb{S} - p\mathbb{I},$$

where $\mathbb{S}$ is *viscous* stress tensor and p a scalar quantity called *pressure*.

1.3.1 Inviscid fluids, compressible Euler system

We start with an example of inviscid (or perfect) fluids for which $\mathbb{S} = 0$. In the simplest case, the pressure is just a function of the density and we obtain

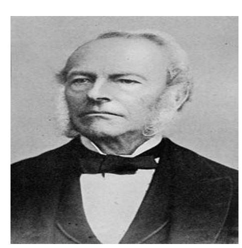
COMPRESSIBLE EULER SYSTEM

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.6}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0. \tag{1.7}$$

If $N = 1$, system (1.6, (1.7) can be rewritten in the Lagrangian (mass) coordinates by introducing new independent variables

$$[t, x] \mapsto [t, y(t, x) = \int_{-\infty}^{x} \varrho(t, z) \ \mathrm{d}z].$$



George Gabriel Stokes
[1819–1903]

The Lagrangian velocity $v = v(t, y)$ satisfies

$$v\left(t, \int_{-\infty}^{x} \varrho(t,z)\ \mathrm{d}z\right) = u(t,x);$$

whence

$$\partial_t v - \varrho u \partial_y v = \partial_t u, \ \ \varrho \partial_y v = \partial_x u,$$

and (1.6) reads

$$\partial_t U - \partial_y v = 0, \tag{1.8}$$

while (1.7) gives rise to

$$\partial_t v + \partial_y p\left(\frac{1}{U}\right) = 0 \tag{1.9}$$

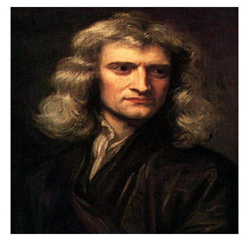
where $U = \frac{1}{\varrho}$ is the *specific volume*. Problem (1.8), (1.9) is called *p-system*.

The Euler system formally conserves energy. Taking the scalar product of (1.7) with $\mathbf{u}$ we obtain, by means of straightforward manipulation

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) + \mathrm{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + p(\varrho)\right)\mathbf{u}\right] - p(\varrho)\mathrm{div}_x\mathbf{u} = 0. \tag{1.10}$$

Moreover, multiplying (1.6) on $b'(\varrho)$ we deduce the *renormalized equation of continuity*

$$\partial_t b(\varrho) + \mathrm{div}_x\left(b(\varrho)\mathbf{u}\right) + \left[b'(\varrho)\varrho - b(\varrho)\right]\ \mathrm{div}_x\mathbf{u} = 0, \tag{1.11}$$



Isaac Newton [1642–1727]

which, together with (1.10), gives rise to the total energy balance

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right) + \mathrm{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + p(\varrho)\right)\mathbf{u}\right] = 0, \ \text{with } P(\varrho) = \varrho\int_1^{\varrho}\frac{p(z)}{z^2}\ \mathrm{d}z. \tag{1.12}$$

1.3.2 Viscous fluid, compressible Navier-Stokes system

We consider the simplest possible example of a viscous fluid, where the viscous stress tensor is a *linear* function of the velocity gradient. For isotropic fluid, the associated $\mathbb{S}$ is given by *Newton's rheological law*

$$\mathbb{S}(\nabla_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{N}\mathrm{div}_x\mathbf{u}\right) + \eta\mathrm{div}_x\mathbf{u}\mathbb{I}, \tag{1.13}$$

where μ and η are scalar quantities taken to be constant here for the sake of simplicity. Accordingly, we obtain

Compressible Navier-Stokes system

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{1.14}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}), \tag{1.15}$$

$$\mathbb{S}(\nabla_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{N}\mathrm{div}_x\mathbf{u}\right) + \eta\mathrm{div}_x\mathbf{u}\mathbb{I}. \tag{1.16}$$

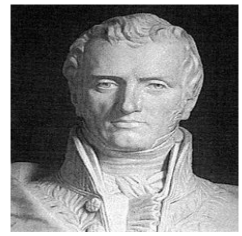
Similarly to (1.12), we write the total energy balance

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) + p(\varrho) \right) \mathbf{u} - \mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u} \right] \\ = -\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}. \tag{1.17}$$

As an *physically admissible* system should not produce energy, the source term $\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}$ must be non-negative for any admissible process, in particular

$$\mu \geq 0, \ \eta \geq 0.$$

We speak about *viscous fluids* if $\mu > 0$.



Claude-Louis Navier [1785–1836]

1.4 First law of thermodynamics - complete fluid systems

As we have seen in the preceding part, the description by means of purely "mechanical" quantities like ϱ and $\mathbf{u}$ is not complete from the physics point of view; the resulting compressible Navier-Stokes system dissipates (mechanical) energy. For the First law of thermodynamics to hold, one has to introduce a new quantity called (specific) *internal energy* e. Alternatively, we may consider the *absolute temperature* ϑ and suppose that both $p = p(\varrho, \vartheta)$ and $e = e(\varrho, \vartheta)$ are given functions of the state variables ϱ, ϑ. The functional dependence of p and e on ϱ and ϑ is called *equation of state* and characterizes the material properties of a given fluid.

Jean Baptiste Joseph Fourier [1768–1830]

We rewrite the energy balance (1.17) as

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + p(\varrho, \vartheta) \right) \mathbf{u} - \mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u} \right] \\ = -\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u} \tag{1.18}$$

and append the system by a similar relation for the internal energy

$$\partial_t \left(\varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\varrho e(\varrho, \vartheta) \mathbf{u} \right] + \mathrm{div}_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}, \tag{1.19}$$

where $\mathbf{q}$ is a new quantity that represents the diffuse flux of the internal energy. In accordance with the First law of thermodynamics, the total energy of the system must be conserved; whence

$$\partial_t \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \right] \\ + \mathrm{div}_x \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \mathbf{u} + p(\varrho, \vartheta) \mathbf{u} - \mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u} + \mathbf{q} \right] = 0. \tag{1.20}$$

The internal energy flux very often coincides with the heat flux, the latter being given by *Fourier's law*

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{1.21}$$

Thus we have derived

NAVIER-STOKES-FOURIER SYSTEM

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.22}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.23}$$

$$\partial_t \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \right] + \mathrm{div}_x \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \mathbf{u} + p(\varrho, \vartheta) \mathbf{u} - \mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u} + \mathbf{q} \right] = 0, \tag{1.24}$$

with

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{1.25}$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{1.26}$$

The "inviscid" version is known as

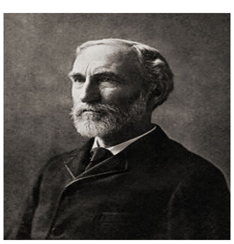
COMPLETE EULER SYSTEM

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.27}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0, \tag{1.28}$$

$$\partial_t \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \right] + \mathrm{div}_x \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \mathbf{u} + p(\varrho, \vartheta) \mathbf{u} \right] = 0. \tag{1.29}$$

1.5 Second law of thermodynamics



Willard Gibbs
[1839–1903]

The Second law of thermodynamics encodes the time *irreversibility* of the fluid evolution characteristic for viscous and heat conducting fluid. We recall the internal energy balance

$$\varrho \partial_t e(\varrho, \vartheta) + \varrho \mathbf{u} \cdot \nabla_x e(\varrho, \vartheta) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u},$$

where

$$\varrho \partial_t e(\varrho, \vartheta) + \varrho \mathbf{u} \cdot \nabla_x e(\varrho, \vartheta) = \varrho De(\varrho, \vartheta) \cdot [\partial_t \varrho, \partial_t \vartheta] + \varrho \mathbf{u} \cdot De(\varrho, \vartheta) \cdot [\nabla_x \varrho, \nabla_x \vartheta].$$

Suppose that e and p are interrelated through *Gibbs' equation*

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + D\left(\frac{1}{\varrho} \right) p(\varrho, \vartheta), \tag{1.30}$$

where s is a new quantity called (specific) *entropy.* Accordingly,

$$\begin{aligned}\varrho\partial_t e(\varrho,\vartheta)+\varrho\mathbf{u}\cdot\nabla_x e(\varrho,\vartheta) &= \varrho De(\varrho,\vartheta)\cdot[\partial_t\varrho,\partial_t\vartheta]+\varrho\mathbf{u}\cdot De(\varrho,\vartheta)\cdot[\nabla_x\varrho,\nabla_x\vartheta]\\ &= \vartheta\varrho Ds(\varrho,\vartheta)\cdot[\partial_t\varrho,\partial_t\vartheta]+\vartheta\varrho\mathbf{u}\cdot Ds(\varrho,\vartheta)\cdot[\nabla_x\varrho,\nabla_x\vartheta]\\ &+p(\varrho,\vartheta)\frac{1}{\varrho}\left(\partial_t\varrho+\mathbf{u}\cdot\nabla_x\varrho\right)\\ &= \vartheta\left[\partial_t(\varrho s(\varrho,\vartheta))+\mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u})\right]-p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u};\end{aligned}$$

whence (1.19) can be rewritten as

$$\partial_t\left(\varrho s(\varrho,\vartheta)\right)+\mathrm{div}_x\left[\varrho s(\varrho,\vartheta)\mathbf{u}\right]+\frac{1}{\vartheta}\mathrm{div}_x\mathbf{q}=\frac{1}{\vartheta}\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u},$$

where, furthermore,

$$\frac{1}{\vartheta}\mathrm{div}_x\mathbf{q}=\mathrm{div}_x\left(\frac{1}{\vartheta}\mathbf{q}\right)+\frac{1}{\vartheta^2}\mathbf{q}\cdot\nabla_x\vartheta,$$

and, consequently, we end up with the *entropy balance equation*

$$\partial_t\left(\varrho s(\varrho,\vartheta)\right)+\mathrm{div}_x\left[\varrho s(\varrho,\vartheta)\mathbf{u}\right]+\mathrm{div}_x\left(\frac{1}{\vartheta}\mathbf{q}\right)=\frac{1}{\vartheta}\left[\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right].\tag{1.31}$$

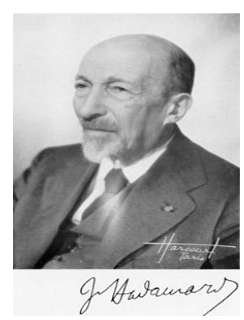
If $\mathbf{q}=-\kappa\nabla_x\vartheta$ is given by Fourier's law, the *entropy production rate* represented by the right-hand side of (1.31) is non-negative as long as $\kappa\geq 0$. In other words, the entropy is produced in the course of the process in accordance with the Second law of thermodynamics.

2 Various concepts of solutions to the equations and systems of mathematical fluid dynamics

A typical problem for an evolutionary equation is the initial-value or Cauchy problem. Knowing the state of the system at an initial time, say $t=0$, solve the evolutionary equation for this initial data. If the physical system (fluid) is confined to a bounded domain, the boundary behavior must be prescribed. To avoid the difficulties created by the influence boundaries on the fluid motion, we focus on problems posed on the whole physical space R^N or on the periodic (flat) torus

$$\mathcal{T}^N=\left([0,1]|_{\{0,1\}}\right)^N.$$

In the latter case, we simply imposed the periodic boundary conditions, that may be seen as a useful though not very realistic approximation. Evolutionary problems are expected to be *well posed* in the sense of Hadamard:



Jacques Hadamard
[1865–1963]

- solutions exist for any physically admissible choice of the initial data;
- solutions are uniquely determined by the initial data;
- solutions depend in a continuous way on the initial data.

Our goal is to address these issues for the evolutionary equations arising in fluid dynamics.

2.1 Possible blow-up of smooth solutions

Solutions of non-linear problems similar to the compressible Euler system (1.6), (1.7) may develop singularities in a finite time no matter how smooth and/or small the data are. Note that "small" here means sufficiently close to the equilibrium state $\varrho = \text{const}$, $\mathbf{u} = 0$.

The classical example of blow up is provided by the 1-D Burgers equation

$$\partial_t V + V \partial_x V = 0, \ V(0) = V_0. \tag{2.1}$$

Indeed it is easy to observe that smooth solutions must satisfy

$$V(t, x + V_0(x)t) = V_0(x), \ t \geq 0. \tag{2.2}$$

Obviously, if $V_0(x_1) > V_0(x_2)$ for some $x_1 < x_2$, we get

$$x = x_1 + tV_0(x_1) = x_2 + tV_0(x_2) \text{ for } t = \frac{x_2 - x_1}{V_0(x_1) - V_0(x_2)} > 0; \text{ whence } V(t,x) = \left\{ \begin{array}{l} V_0(x_1) \\ V_0(x_2) \end{array} \right.$$

Now we show how this construction can be adapted to the compressible Euler system. We restrict ourselves to the case of one space dimension, where (1.6), (1.7) can be rewritten in the Lagrangean coordinates as the $p-$system (1.8), (1.9), namely

$$\begin{aligned} \partial_t s - \partial_x v &= 0 & (2.3) \\ \partial_t v - \partial_x P(s) &= 0, \ P' > 0. & (2.4) \end{aligned}$$

Bernhard Riemann
[1826–1866]

This system can be written in terms of the so-called Riemann invariants, namely

$$\begin{aligned} \sqrt{P'(s)}\partial_t s - \sqrt{P'(s)}\partial_x v &= 0 \\ \partial_t v - \sqrt{P'(s)}\sqrt{P'(s)}\partial_x P(s) &= 0. \end{aligned}$$

Introducing $Z(s) = \int_0^s \sqrt{P'(z)} \ {\rm s}z$ we get

$$\begin{aligned} \partial_t Z - A(Z)\partial_x v &= 0 \\ \partial_t v - A(Z)\partial_x Z &= 0, \end{aligned}$$

or

$$\begin{aligned} \partial_t R_1 - A\left(\frac{R_1 + R_2}{2}\right)\partial_x R_1 &= 0 \\ \partial_t R_2 + A\left(\frac{R_1 + R_2}{2}\right)\partial_x R_2 &= 0, \end{aligned}$$

where we have introduced the Riemann invariants

$$R_1 = v + z, \ R_2 = Z - v.$$

Setting either R_1 or R_2 constant, we end up with a non-linear scalar equation of type (2.1).

We conclude that solutions of the compressible Euler system (1.6), (1.7) may, and indeed do, develop singularities for a generic class of smooth initial data.

Remark 2.1. *Let us point out that similar construction* fails *for the compressible Navier–Stokes system (1.14–1.16). As shown by Kazhikhov [17], this problem admits global in time smooth solutions for smooth initial data if $N = 1$. Similar results for $N = 2, 3$ are not known, see however Kazhikhov and Vaigant [29].*

2.2 Classical - strong solutions

Ideally, a system of partial differential equations should admit strong or classical solutions possessing all the necessary derivatives and enjoying certain continuity to attain the initial data. Sometimes we speak about strong solutions if all derivatives involved in equations exist in the generalized sense as integrable functions. Once the *existence* of a classical solution is established, the questions of uniqueness and/or continuous dependence on the data are usually an easy task given the high regularity of solutions. Unfortunately, classical solutions of most problems in mathematical fluid dynamics are known to exist only on short time intervals - *local eistence* - or globally in time but for the initial data that are sufficiently close to an equilibrium solution - *global existence for "small" data.*

2.2.1 Local existence for the compressible Euler and Navier-Stokes system

Consider the Euler/Navier-Stokes system describing the motion of a compressible fluid introduced in Section 1.3.2:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.5}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{2.6}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{2.7}$$

with $\mu = \eta = 0$ for the inviscid (Euler) system, and $\mu > 0$ for the viscous (Navier-Stokes) fluid. For the sake of simplicity, we consider the periodic boundary conditions, meaning the physical space $\Omega = \mathcal{T}^N$ is the flat torus, and prescribe the initial state

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \tag{2.8}$$

where ϱ_0 and $\mathbf{u}_0$ are as smooth as needed, and $\varrho_0 > 0$ to the degenerate vacuum regime. Problem (2.5–2.8) is known to be locally well posed in the Sobolev scale $W^{m,2}(\Omega)$ for m large enough, see e.g. Majda [20], Kleinerman and Majda [18].

Theorem 2.2. *Let $p \in C^\infty(0,\infty)$, $p'(\varrho) > 0$ for $\varrho > 0$, $\mu \geq 0$, $\eta \geq 0$, and*

$$\varrho_0 > 0, \varrho_0 \in W^{m,2}(\Omega),\ \mathbf{u}_0 \in W^{m,2}(\Omega; R^N) \text{ for } m > \left[\frac{N}{2}\right] + 1.$$

Then there exists a positive time $T > 0$ such that problem (2.5–2.8) admits a strong solution $[\varrho, \mathbf{u}]$ in $(0,T) \times \Omega$, unique in the class

$$\varrho \in C([0,T]; W^{m,2}(\Omega)),\ \partial_t \varrho \in C([0,T]; W^{m-1,2}(\Omega)),$$
$$\mathbf{u} \in C([0,T]; W^{m,2}(\Omega; R^N)),\ \partial_t \mathbf{u} \in C([0,T]; W^{m-k,2}(\Omega; R^N)),$$

$k = 1$ if $\mu = \eta = 0$, $k = 2$ if $\mu > 0$ or $\eta > 0$.

Remark 2.3. *Obviously, the solution $[\varrho, \mathbf{u}]$ is as smooth as we wish thanks to the Sobolev embedding relation*

$$W^{k,2}(\Omega) \hookrightarrow C(\Omega),\ k > \left[\frac{N}{2}\right].$$

The Sobolev spaces, based on the concept of generalized derivatives and Lebesgue integration, form a natural framework for the so called energy method used in the proof of local existence. The *a priori* bounds are usually established in the Sobolev framework rather than the framework of spaces of classical continuous and Hölders continuous functions. The desired regularity is then obtained by means of the *embedding theorems.* We refer to the monographs by Adams [1], Maz'ya [23], Pick et al. [26] or Ziemer [30] for the basic properties of the Sobolev spaces.

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2.2.2 Rewriting the Euler system as symmetric hyperbolic

Let us show very briefly the leading idea behind the proof of Theorem **??**, namely how to rewrite the Euler problem as a *symmetric hyperbolic system.* To this end, we make a simplifying but still physically relevant assumption that the pressure p is given by the isentropic constitutive relation $p(\varrho) = a\varrho^\gamma$, $\gamma > 1$. With the new choice of independent variables

$$r = \sqrt{\frac{2a\gamma}{\gamma - 1}} \varrho^{\frac{\gamma-1}{2}},$$

we may rewrite the Euler system in the form

$$\partial_t r + \mathbf{u} \cdot \nabla_x r + \frac{\gamma - 1}{2} r \mathrm{div}_x \mathbf{u} = 0 \tag{2.9}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + r \nabla_x r \quad = \quad 0. \tag{2.10}$$

To obtain suitable *a priori* estimates, we differentiate the equations $([N/2]+1)-$times with respect to the $x-$variable and then multiply (2.9) on $\partial_x^\alpha \varrho$, and (2.10) on $\partial_x^{alpha} \mathbf{u}$, $\alpha = [N/2]+1$. We obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\partial_x^\alpha \varrho|^2 + |\partial_x^\alpha \mathbf{u}|^2 \ \ \mathrm{d}x \leq c + \left(\int_\Omega |\partial_x^\alpha \varrho|^2 + |\partial_x^\alpha \mathbf{u}|^2 \ \ \mathrm{d}x \right)^M , \ M > 0$$

from we deduce local–in–time boundedness of the Sobolev norm

$$\|\partial_x^\alpha \varrho\|_{L^2(\Omega)} + \|\partial_x^\alpha \mathbf{u}\|^2_{L^2(\Omega)} .$$

This operation requires the embedding relation

$$W^{\alpha,2} \hookrightarrow C \text{ for } \alpha > \frac{N}{2}.$$

2.2.3 Global existence for small initial data

The Navier-Stokes system admits global-in-time solutions provided the initial data are close enough to an equilibrium state. Here, we present a possible result in this direction that was essentially shown by Matsumura and Nishida [21], [22] for $N = 3$.

Theorem 2.4. *Let $p \in C^\infty(0,\infty)$, $p'(\varrho) > 0$ for $\varrho > 0$, $\mu > 0$, $\eta \geq 0$. Let a positive constant $\overline{\varrho} > 0$ be given.*

Then there exists $\varepsilon > 0$ such that for any initial data

$$\varrho_0 \in W^{3,2}(\Omega), \ \mathbf{u}_0 \in W^{3,2}(\Omega; R^3), \ \int_\Omega (\varrho_0 - \overline{\varrho}) \ \mathrm{d}x = 0,$$

$$\|\varrho_0 - \overline{\varrho}\|_{W^{3,2}(\Omega)} + \|\mathbf{u}_0\|_{W^{3,2}(\Omega;R^3)} < \varepsilon,$$

the Navier–Stokes problem (2.5–2.8) admits a unique strong solution $[\varrho, \mathbf{u}]$ defined on the time interval $(0,\infty)$,

$$\varrho \in C([0,T]; W^{3,2}(\Omega)), \ \partial_t \varrho \in C([0,T]; W^{2,2}(\Omega)),$$
$$\mathbf{u} \in C([0,T]; W^{3,2}(\Omega; R^3)), \ \partial_t \mathbf{u} \in C([0,T]; W^{1,2}(\Omega; R^3))$$

such that

$$\varrho(t,\cdot) \to \overline{\varrho} \text{ in } W^{3,2}(\Omega), \ \mathbf{u}(t,\cdot) \to 0 \text{ in } W^{3,2}(\Omega; R^3) \text{ as } t \to \infty.$$

Remark 2.5. *Solutions in Theorem 2.4 are more regular provided we increase accordingly the regularity of the initial data.*

A similar result for the inviscid Euler system is not available. As we shall see below, solutions of the compressible Euler system develop singularities in a finite lap of time for a fairly generic class of smooth and even small initial data.

2.3 Weak solutions

Weak solutions satisfy the equations in the sense of distributions. A single equation in the weak formulation is replaced by an (infinite) family of integral identities satisfied by sufficiently smooth test functions. The weak formulation of the Euler/Navier Stokes system (1.14), (1.15) reads

$$\int_0^T \int_\Omega [\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi] \ \mathrm{d}x \ \mathrm{d}t = 0,$$

for any test function $\varphi \in C_c^1((0,T) \times \Omega)$,

$$\int_0^T \int_\Omega [\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho) \mathrm{div}_x \varphi] \ \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any test function $\varphi \in C_c^1((0,T) \times \Omega; R^N)$.

Note that for the Euler system $\mathbb{S} \equiv 0$ and therefore less regularity for the weak solution is required. It follows from the weak formulation that

$$t \mapsto \int_\Omega \varrho(t, \cdot) \varphi \ \mathrm{d}x \text{ and } t \mapsto \int_\Omega (\varrho \mathbf{u})(t, \cdot) \cdot \varphi \ \mathrm{d}x$$

can be seen as *continuous* scalar function of time for any test function φ. Accordingly, it is more convenient to write the weak formulation in the form

$$\left[\int_\Omega \varrho \varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega [\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi] \ \mathrm{d}x \ \mathrm{d}t,$$

for any $0 \le \tau_1 \le \tau_2 \le T$ and for any test function $\varphi \in C_c^1([0,T] \times \Omega)$,

$$\left[\int_\Omega \varrho \mathbf{u} \cdot \varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_0^T \int_\Omega [\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho) \mathrm{div}_x \varphi] \ \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t \tag{2.11}$$

for any $0 \le \tau_1 \le \tau_2 \le T$ and for any test function $\varphi \in C_c^1([0,T] \times \Omega; R^N)$.

Remark 2.6. *Note that the weak formulation (2.11) already includes satisfaction of the initial conditions. Observe that it is more natural to formulate the initial state in terms of the momentum* $\varrho \mathbf{u}$ *rather that the velocity* $\mathbf{u}$ *as the former is always* weakly *continuous.*

2.3.1 Weak solutions to the compressible Euler system

The class of weak solutions is obviously larger than that of classical solutions as basically no smoothness is required. The price to pay is a dramatic lost of well posedness as the following result shows (see Chiodaroli [5], EF [13]).

Theorem 2.7. *Let $p \in C^3(0,\infty)$ and let $T > 0$ be given. Let the initial data belong to the class*

$$\varrho_0 \in C^3(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in C^3(\Omega; R^N),\ \Omega = \mathcal{T}^N,\ N = 2,3.$$

Then the compressible Euler system (1.6), (1.7 admits infinitely many *weak solutions emanating from the initial state $[\varrho_0, \mathbf{u}_0$ and belonging to the class*

$$\varrho \in L^\infty((0,T)\times\Omega),\ \varrho > 0,\ \mathbf{u} \in L^\infty((0,T)\times\Omega; R^N).$$

Theorem 2.7 can be shown by the method of *convex integration* developed recently in the context of fluid mechanics by DeLellis and Székelyhidi [9], [8], [10]. It shows that on one hand the class of weak solutions, at leats for the inviscid fluid flows, is large enough to provide positive existence results, on the other hand it is too large to ensure uniqueness.

Remark 2.8. *We emphasize that results similar to Theorem 2.7 for* viscous *fluids, meaning for the Navier-Stokes system, are not known.*

2.4 Dissipative (weak) solutions

The solutions, the existence of which is claimed in Theorem 2.7, are produced in a rather non-constructive way by adding oscillatory components to a quantity called sub-solution. As a result, they typically produce energy, meaning violate the energy balance (1.12). Thus we may try to eliminate them by adding a weak counterpart of the energy balance as an integral part of the definition of weak solution.

We say the $[\varrho, \mathbf{u}]$ is a *dissipative* weak solution to the Euler/Navier Stokes system if, in addition to the weak formulation (2.11), the total *energy inequality*

$$\begin{aligned} \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right] (\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \\ \leq \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0) \right]\ \mathrm{d}x \end{aligned} \tag{2.12}$$

holds for a.a. $\tau \in (0,T)$.

Remark 2.9. *Recall that for the Euler system $\mathbb{S}(\nabla_x \mathbf{u}) \equiv 0$ and (2.12) simply says that the total energy of the system is non-increasing in time.*

Remember that the Navier-Stokes system is already dissipative so adding inequality in (2.12) does not violate any physical principle. In addition, we have extrapolated this argument also to the "conservative" Euler system.

2.5 Relative energy

The dissipative solutions introduced in the previous section satisfy an extended version of the energy inequality (2.12) known as *relative energy inequality.* We introduce the *relative energy functional*

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) \right] \ \mathrm{d}x. \tag{2.13}$$

If

$$\varrho \mapsto p(\varrho) \text{ is strictly increasing in } (0, \infty),$$

then the pressure potential P is strictly convex and $\mathcal{E}$ represents a kind of (non-symmetric) distance function between $[\varrho, \mathbf{u}]$ and $[r, \mathbf{U}]$.

Seeing that

$$\begin{aligned} &\mathcal{E}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right) \\ &= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x - \int_\Omega \varrho\mathbf{u} \cdot \mathbf{U} \ \mathrm{d}x + \int_\Omega \varrho|\mathbf{U}|^2 \ \mathrm{d}x - \int_\Omega P'(r)\varrho \ \mathrm{d}x + \int_\Omega p(r) \ \mathrm{d}x \end{aligned}$$

we easily observe that wa may evaluate the time difference

$$\left[\mathcal{E}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right) \right]_{t=0}^{t=\tau}$$

as soon as $[\varrho, \mathbf{u}]$ is a *dissipative* (weak) solution to the Euler/Navier-Stokes system, and $[r, \mathbf{U}]$ is a pair of sufficiently smooth "test functions", $r > 0$. Indeed we successively compute

- $$\left[\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \leq 0$$

  in accordance with the energy inequality (2.12), keeping in mind that $\mathbb{S} \equiv 0$ for the Euler system;

- taking $\mathbf{U}$ as a test function in the weak formulation of the momentum equation:

  $$\begin{aligned} \left[\int_\Omega \varrho\mathbf{u} \cdot \mathbf{U} \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} &= \int_0^T \int_\Omega \left[\varrho\mathbf{u} \cdot \partial_t \mathbf{U} + \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{U} + p(\varrho)\mathrm{div}_x \mathbf{U} \right] \ \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t; \end{aligned}$$

- taking $\frac{1}{2}|\mathbf{U}|^2$ and $P'(r)$ as test functions in the weak formulation of the equation of continuity:

  $$\left[\int_\Omega \varrho \frac{1}{2}|\mathbf{U}|^2 \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho\mathbf{U} \cdot \partial_t \mathbf{U} + \varrho\mathbf{u} \cdot \mathbf{U} \cdot \nabla_x \mathbf{U} \right] \ \mathrm{d}x \ \mathrm{d}t,$$

  and

  $$\left[\int_\Omega \varrho P'(r) \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho P''(r)\partial_t r + P''(r)\varrho\mathbf{u} \cdot \nabla_x r \right] \ \mathrm{d}x \ \mathrm{d}t.$$

Summing up the previous observations we obtain

RELATIVE ENERGY INEQUALITY

$$\begin{aligned}
\left[\mathcal{E}\left(\varrho,\mathbf{u}\,\Big|\,r,\mathbf{U}\right)\right]_{t=0}^{t=\tau} &+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x \\
&\leq \int_0^\tau \int_\Omega \varrho\left[\partial_t \mathbf{U} + \mathbf{u}\cdot\nabla_x\mathbf{U}\right]\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{U} - p(\varrho)\mathrm{div}_x\mathbf{U}\right]\ \mathrm{d}x\,\mathrm{d}t \\
&- \int_{\tau_1}^{\tau_2} \int_\Omega \left[(\varrho - r)P''(r)\partial_t r + P''(r)\varrho\mathbf{u}\cdot\nabla_x r\right]\ \mathrm{d}x\ \mathrm{d}t,
\end{aligned} \tag{2.14}$$

where we have used the identity

$$p'(r) = rP''(r).$$

3 Weak vs. strong solutions

Our goal in this section is to show the *weak-strong uniqueness* property for the dissipative solutions of the Euler/Navier-Stokes system. In other words, the weak and strong solutions emanating from the same initial data coincided as long as the latter exists.

3.1 Weak-strong uniqueness

Suppose that the Euler/Navier-Stokes system admits a strong solution $[\varrho = r, \mathbf{u} = \mathbf{U}]$ on a time interval $(0,T)$, $r > 0$. Let $[\varrho, \mathbf{u}]$ be a dissipative weak solution of the same problem with the same initial data. Taking $[r, \mathbf{U}]$ as test functions in the relative energy inequality (2.14), and using the fact that

$$\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U} = -\frac{1}{r}\nabla_x p(r) + \frac{1}{r}\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U}),$$

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we obtain

$$
\begin{aligned}
\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(\tau) &+ \int_0^\tau \int_\Omega (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \\
&\leq \int_0^\tau \int_\Omega \varrho |\nabla_x \mathbf{U}| |\mathbf{u} - \mathbf{U}|^2 \ \mathrm{d}x \\
&- \int_0^\tau \int_\Omega \varrho P''(r) \nabla_x r \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[(\varrho - r) P''(r) \partial_t r + P''(r) \varrho \mathbf{u} \cdot \nabla_x r + p(\varrho) \mathrm{div}_x \mathbf{U} \right] \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left(1 - \frac{\varrho}{r}\right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \right] \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

Furthermore, as r satisfies

$$
\partial_t r + \mathbf{U} \cdot \nabla_x \mathbf{r} = -r \mathrm{div}_x \mathbf{U},
$$

we get, performing several by-parts integrations,

$$
\begin{aligned}
&- \int_0^\tau \int_\Omega \varrho P''(r) \nabla_x r \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[(\varrho - r) P''(r) \partial_t r + P''(r) \varrho \mathbf{u} \cdot \nabla_x r + p(\varrho) \mathrm{div}_x \mathbf{U} \right] \ \mathrm{d}x \ \mathrm{d}t \\
&= - \int_0^\tau \int_\Omega \left[(\varrho - r) P''(r) \partial_t r + P''(r) \varrho \mathbf{U} \cdot \nabla_x r + p(\varrho) \mathrm{div}_x \mathbf{U} \right] \ \mathrm{d}x \ \mathrm{d}t \\
&= - \int_0^\tau \int_\Omega \mathrm{div}_x \mathbf{U} \Big(p(\varrho) - p'(r)(\varrho - r) - p(r) \Big) \ \mathrm{d}x \ \mathrm{d}t.
\end{aligned}
$$

Thus we conclude that

$$
\begin{aligned}
\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(\tau) &+ \int_0^\tau \int_\Omega (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \\
&\leq \int_0^\tau \int_\Omega \varrho |\nabla_x \mathbf{U}| |\mathbf{u} - \mathbf{U}|^2 \ \mathrm{d}x \\
&- \int_0^\tau \int_\Omega \mathrm{div}_x \mathbf{U} \Big(p(\varrho) - p'(r)(\varrho - r) - p(r) \Big) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left(1 - \frac{\varrho}{r}\right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \right] \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
\tag{3.1}
$$

3.1.1 Weak-strong uniqueness for the Euler system

Relation (3.1) implies $\varrho = r$, $\mathbf{u} = \mathbf{U}$ by means of the standard Gronwall lemma. More specifically, in the case of the Euler system, the strong solution must be at least globally Lipschitz and we need

$$
p(\varrho) - p'(r)(\varrho - r) - p(r) \text{ to be dominated by } P(\varrho) - P'(r)(\varrho - r) - P(r).
$$

Note that this is true provided, for instance,

$$p \in C[0,\infty) \cap C^2(0,\infty),\ p'(\varrho) > 0 \text{ for } \varrho > 0,\ \liminf_{\varrho \to \infty} p'(\varrho) > 0,\ \liminf_{\varrho \to \infty} \frac{P(\varrho)}{p(\varrho)} > 0. \tag{3.2}$$

We have shown the following result.

Theorem 3.1. *Let the pressure p satisfy (3.2). Let the compressible Euler system (1.6), (1.7) admits a solution* $[\tilde{\varrho}, \tilde{\mathbf{u}}]$, $\tilde{\varrho} > 0$ *that is Lipschitz in* $[0,T] \times \Omega$, $\Omega = \mathcal{T}^N$, $N = 1,2,3$. *Let* $[\varrho, \mathbf{u}]$, $\varrho \geq 0$ *be a dissipative weak solution of the same problem,*

$$\varrho(0,\cdot) = \tilde{\varrho}(0,\cdot),\ \ \varrho\mathbf{u}(0,\cdot) = \tilde{\varrho}\tilde{\mathbf{u}}(0,\cdot).$$

Then

$$\varrho = \tilde{\varrho},\ \ \mathbf{u} = \tilde{\mathbf{u}} \text{ a.a. in } (0,T) \times \Omega.$$

3.1.2 Weak-strong uniqueness for the Navier-Stokes system

If viscosity is present, we have to handle the last integral in (3.1), namely

$$\int_0^\tau \int_\Omega \left[\left(1 - \frac{\varrho}{r} \right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \right] \ \mathrm{d}x \, \mathrm{d}t.$$

Clearly, this integral can be dominated by the relative entropy functional on the set, where

$$0 < \underline{\varrho} \leq \varrho \leq \overline{\varrho},$$

where $\underline{\varrho}$, $\overline{\varrho}$ are positive constants.

Now, we distinguish two cases: (i) ϱ is large $\varrho \geq \overline{\varrho}$, (ii) ϱ is small $0 \leq \varrho \leq \underline{\varrho}$. If ϱ is large, specifically $\overline{\varrho} > 2 \max\{r\}$, we get

$$| \left(1 - \frac{\varrho}{r} \right) (\mathbf{u} - \mathbf{U})| \leq c_1 |\sqrt{\varrho}| |\sqrt{\varrho} (\mathbf{u} - \mathbf{U})| \leq c_2 \left(\varrho + \varrho |\mathbf{u} - \mathbf{U}|^2 \right),$$

where the right-hand side is dominated by the relative energy functional.

If ϱ is small, specifically $\varrho \leq \underline{\varrho} < \frac{1}{2} \min\{r\}$, then

$$\begin{aligned}
&\left| \int_{\varrho \leq \underline{\varrho}} \left[\left(1 - \frac{\varrho}{r} \right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \right] \ \mathrm{d}x \ \mathrm{d}t \right| \\
&\leq \delta \int_0^\tau \int_\Omega |\mathbf{u} - \mathbf{U}|^2 \ \mathrm{d}x + c(\delta) \int_{\varrho \leq \underline{\varrho}} |\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})|^2 \left(1 - \frac{\varrho}{r} \right)^2 \ \mathrm{d}x \ \mathrm{d}t \\
&\leq \delta \int_0^\tau \int_\Omega \varrho |\mathbf{u} - \mathbf{U}|^2 + (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \\
&+ c(\delta) \int_{\varrho \leq \underline{\varrho}} |\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})|^2 \left(1 - \frac{\varrho}{r} \right)^2 \ \mathrm{d}x \ \mathrm{d}t
\end{aligned}$$

for any $\delta > 0$, where the last integral can be absorbed by the left-hand side of (3.1) provided $\delta > 0$ is small enough.

We have shown an analogue of Theorem 3.1 for the Navier-Stokes system.

Theorem 3.2. *Let the pressure p satisfy (3.2). Let the compressible Navier-Stokes system (1.14–1.16) admits a solution $[\tilde{\varrho}, \tilde{\mathbf{u}}]$, $\tilde{\varrho} > 0$ that is Lipschitz in $[0, T] \times \Omega$, $\Omega = \mathcal{T}^N$,*

$$\mathbf{u} \in L^1(0, T; W^{2,\infty}(\Omega; R^N)), \ \ N = 1, 2, 3.$$

Let $[\varrho, \mathbf{u}]$, $\varrho \geq 0$ be a dissipative weak solution of the same problem,

$$\varrho(0, \cdot) = \tilde{\varrho}(0, \cdot), \ \ \varrho\mathbf{u}(0, \cdot) = \tilde{\varrho}\tilde{\mathbf{u}}(0, \cdot).$$

Then

$$\varrho = \tilde{\varrho}, \ \ \mathbf{u} = \tilde{\mathbf{u}} \ \textit{a.a. in} \ (0, T) \times \Omega.$$

3.2 Wild dissipative solutions of the Euler system

In the light of the weak-strong uniqueness result established in Theorem 3.2, it may seem that imposing the energy inequality eliminates the “wild” solutions, the existence of which is claimed in Theorem 2.7. However, this is true only to certain extent as shown in the following result, see [13].

Theorem 3.3. *Let $\Omega = \mathcal{T}^N$, $N = 2, 3$, $T > 0$, and let $p \in C^3(0, \infty)$. Let*

$$\varrho_0 \in C^3(\Omega), \ \ \varrho_0 > 0 \ \textit{in} \ \Omega,$$

be a given density distribution.

Then there exists u_0,

$$u_0 \in L^\infty(\Omega; R^N)$$

such that the corresponding initial-value problem for the Euler system (1.6), (1.7) admits infinitely many dissipative (weak) solutions in $(0, T) \times \Omega$.

Remark 3.4. *Note carefully that there is no contradiction with the weak-strong uniqueness result as the initial velocity* $\mathbf{u}_0$ is not *regular. We also repeat that a similar result in the context of viscous fluids, specifically for the compressible Navier-Stokes, is not available.*

3.2.1 Admissible weak solutions to the compressible Euler system

In the context of hyperbolic conservation law, it is customary to strengthen the global energy inequality to a local one introduced in (1.12). We may say that $[\varrho, \mathbf{u}]$ is an *admissible weak solution* to the Euler system if

$$
\begin{aligned}
&\left[\int_\Omega \varrho\varphi \ \mathrm{d}x\right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2}\int_\Omega [\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi] \ \mathrm{d}x \ \mathrm{d}t,\\
&\text{for any } 0 \le \tau_1 \le \tau_2 \le T \text{ and for any test function } \varphi \in C_c^1([0,T]\times\Omega),\\
&\left[\int_\Omega \varrho\mathbf{u}\cdot\varphi \ \mathrm{d}x\right]_{t=\tau_1}^{t=\tau_2} = \int_0^T\int_\Omega [\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho)\mathrm{div}_x\varphi] \ \mathrm{d}x\,\mathrm{d}t\\
&\text{for any } 0 \le \tau_1 \le \tau_2 \le T \text{ and for any test function } \varphi \in C_c^1([0,T]\times\Omega; R^N)\\
&\left[\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right]\varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} \ge \int_0^\tau\int_\Omega \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\partial_t\varphi\right] \ \mathrm{d}x\,\mathrm{d}t\\
&\qquad + \int_0^\tau\int_\Omega \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + p(\varrho)\right)\mathbf{u}\cdot\nabla_x\varphi\right] \ \mathrm{d}x\,\mathrm{d}t\\
&\text{for a.a. } 0 \le \tau \le T \text{ and for any test function } \varphi \in C_c^1([0,T]\times\Omega), \ \varphi \ge 0.
\end{aligned} \tag{3.3}
$$

Note that the notion of admissible weak solution is stronger than the dissipative weak solution as a *local* version of the energy inequality is required. Still the following result holds, see Chiodaroli [5].

Theorem 3.5. *Let $\Omega = \mathcal{T}^N$, $N = 2, 3$, and let $p \in C^3(0, \infty)$. Let*

$$\varrho_0 \in C^3(\Omega), \ \ \varrho_0 > 0 \ \textit{in} \ \Omega,$$

be a given density distribution.

Then there exist $T > 0$ and u_0,

$$u_0 \in L^\infty(\Omega; R^N),$$

such that the corresponding initial-value problem for the Euler system (1.6), (1.7) admits infinitely many admissible weak solutions in $(0, T) \times \Omega$.

The previous result is local in time. However, we also have the following, see [13].

Theorem 3.6. *Let $\Omega = \mathcal{T}^N$, $N = 2, 3$, $p \in C^3(0, \infty)$, $\overline{\varrho} > 0$, and $T > 0$ be given.*

Then there exists $\varepsilon > 0$ such that for any

$$\varrho_0 \in C^2(\Omega),\ |\varrho_0 - \overline{\varrho}| < \varepsilon \text{ in } \Omega$$

there is u_0,

$$u_0 \in L^\infty(\Omega; R^N),$$

such that the corresponding initial-value problem for the Euler system (1.6), (1.7) admits infinitely many admissible weak solutions in $(0, T) \times \Omega$.

In the light of the above results it may still seem that one could save the game by taking smooth initial data and working in the class of admissible solutions for the Euler system. Unfortunately, even in this case the presence of "wild solutions" cannot be avoided, see Chiodaroli, DeLellis, Kreml [6].

Theorem 3.7. *Let $\Omega = R^2$, $N = 2, 3$, $p \in C^3(0, \infty)$.*

Then there exist such initial data

$$\varrho_0 \in W^{1,\infty}(\Omega),\ \mathbf{u}_0 \in W^{1,\infty}(\Omega; R^2),\ \varrho_0 > 0 \text{ uniformly in } R^2$$

such that the corresponding initial-value problem for the Euler system (1.6), (1.7) admits infinitely many admissible weak solutions in $(0, \infty) \times \Omega$.

Remark 3.8. *Note that the initial data belong to the class, where the weak-strong uniqueness holds. Thus the corresponding solution is smooth (Lipschitz) on some interval $[0, T_{\rm crit})$, looses regularity at $T_{\rm crit}$ and bifurcates into branches of admissible " wild" solutions after the critical time.*

4 Stability of the solution set

Our goal is to identify stability properties of the solution set, in particular, the available *a priori* bounds and weak sequential stability of families of solutions.

4.1 A priori bounds

A priori bounds reflect the piece of information transferred on the solutions from the data. Optimally, the solutions may stay at least as regular as the data for any positive time but this is may not be true in general.

As a matter of fact, only little regularity is known to survive for the compressible Euler/Navier-Stokes system characterized by the energy estimates.

4.1.1 Energy estimates

Dissipative solutions of the Euler/Navier-Stokes system satisfy the global energy inequality (2.12),

$$\begin{aligned} &\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \\ &\leq \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \ \mathrm{d}x, \end{aligned} \tag{4.1}$$

which, on condition that the initial energy

$$\int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \ \mathrm{d}x$$

is bounded, give rise to *a priori* bounds

$$\begin{aligned} \sup_{t \in (0,T)} \|\sqrt{\varrho} \mathbf{u}(t, \cdot)\|_{L^2(\Omega; R^N)} &\leq c(\text{data}) \\ \sup_{t \in (0,T)} \|P(\varrho)(t, \cdot)\|_{L^1(\Omega; R)} &\leq c(\text{data}) \end{aligned} \tag{4.2}$$

that are the same for the Euler and Navier-Stokes system. For the sake of simplicity, suppose that the pressure p is given by the *isentropic* state equation

$$p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1, \tag{4.3}$$

in which case the pressure potential $P(\varrho)$ can be taken in the form

$$P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma,$$

and the second estimate in (4.2) reads

$$\sup_{t \in (0,T)} \|\varrho(t, \cdot)\|_{L^\gamma(\Omega; R)} \leq c(\text{data}). \tag{4.4}$$

We remark that (4.2), (4.4) are basically all available *a priori* estimates for the compressible Euler system.

4.1.2 Estimates based on dissipation

For the Navier-Stokes system, we get an important extra bound

$$\left\| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \mathbb{I} \right\|_{L^2((0,T) \times \Omega; R^{N \times N})} \leq c(\text{data}). \tag{4.5}$$

As $\mathbf{u}$ satisfies the periodic boundary conditions, it is easy to deduce from (4.5) that

$$\|\nabla_x \mathbf{u}\|_{L^2((0,T)\times\Omega;R^{N\times N})} \leq c(\text{data}). \tag{4.6}$$

Now, we show that (4.6), together with (4.4) and the mass conservation principle

$$\int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = \int_\Omega \varrho_0\ \mathrm{d}x = M_0 > 0 \tag{4.7}$$

that can be easily deduced from (2.11), imply that

$$\int_0^T \|\mathbf{u}(t,\cdot)\|^2_{W^{1,2}(\Omega;R^N)}\ \mathrm{d}t \leq c(\text{data}). \tag{4.8}$$

The desired estimate will follow from a version of Poincaré's inequality that may be of independent interest.

Lemma 4.1. *Let $0 \leq r \leq k$ be a bounded measurable function such that*

$$\int_\Omega r\ \mathrm{d}x \geq M > 0.$$

Then

$$\|v\|_{L^2(\Omega)} \leq c(k,M)\left[\int_\Omega r|v|^2\ \mathrm{d}x + \int_\Omega |\nabla_x v|^2\ \mathrm{d}x\right]$$

for any $v \in W^{1,2}(\Omega)$.

Proof: Assuming a contrary we construct a sequence $\{v_n\}_{n=1}^\infty \subset W^{1,2}(\Omega)$ such that

$$\|v_n\|_{L^2(\Omega)} = 1 \text{ and } \|v_n\|_{L^2(\Omega)} \geq n\left[\int_\Omega r_n|v_n|^2\ \mathrm{d}x + \int_\Omega |\nabla_x v_n|^2\ \mathrm{d}x\right],$$

where r_n satisfies the hypotheses of the lemma. Using compactness of the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, $q < 6$ if $N \leq 3$, we may suppose

$$v_n \to v \text{ in } L^2(\Omega),\ \nabla_x v_n \to 0 \text{ in } L^2(\Omega;R^N);\ \text{whence } v = \overline{v} \neq 0 - \text{a constant}. \tag{4.9}$$

On the other hand, by the same token

$$|\mathbf{v}_n|^2 \to |v|^2 = |\overline{v}|^2 \text{ in } L^q(\Omega),\ q < 3,\ r_n \to r \text{ weakly-(*) in } L^\infty(\Omega),\ \int_\Omega r\ \mathrm{d}x > M > 0,$$

and

$$\int_\Omega r_n|v_n|^2\ \mathrm{d}x \to 0 = \int_\Omega r|\overline{v}|^2\ \mathrm{d}x > M|\overline{v}|^2$$

in contrast with (4.9)

□

Thus, in order to apply Lemma 4.1, it is enough to check that (4.4), (4.7) implies the existence of $k \geq 0$ such that

$$r_k = \min\{\varrho, k\} \text{ satisfies } \int_\Omega r_k \ \mathrm{d}x \geq M_0.$$

To see this, we write

$$M_0 = \int_\Omega \varrho \ \mathrm{d}x = \int_\Omega r_k \ \mathrm{d}x + \int_{\varrho \geq k} \varrho \ \mathrm{d}x = \int_\Omega r_k \ \mathrm{d}x + \int_{\varrho \geq k} \varrho^{1-\gamma} \varrho^\gamma \ \mathrm{d}x$$

$$\leq \int_\Omega r_k \ \mathrm{d}x + k^{1-\gamma} c(\text{data}).$$

4.2 Stability of strong solutions for the Euler system

The *a priori* bounds (4.2) based on finiteness of the total energy available for the Euler system are rather poor but almost optimal. Indeed we do not expect to get any bounds on the derivatives that would prevent formation of shocks characteristic for the inviscid fluids, see e.g. Smoller [27]. Thus we concentrate on families of (weak) solutions to the Euler system satisfying solely (4.2), For technical reasons, it will be more convenient to work with the density ϱ and the momentum $\mathbf{m} = \varrho \mathbf{u}$ as independent variables. The Euler system then reads

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \tag{4.10}$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \tag{4.11}$$

with the convention that $m = 0$ whenever $\varrho = 0$ for the convective term in (4.11) to be well defined.

We consider the dissipative solutions of (4.10), (4.11), namely

$$\left[\int_\Omega \varrho \varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t, \tag{4.12}$$

$$\left[\int_\Omega \mathbf{m} \cdot \varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega \left[\mathbf{m} \cdot \partial_t \varphi + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t + \int_{\tau_1}^{\tau_2} \int_\Omega p(\varrho) \mathrm{div}_x \varphi \ \mathrm{d}x \, \mathrm{d}t, \tag{4.13}$$

$$\left[\int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \ \mathrm{d}x \right]_{t=0}^{t=\tau} \leq 0. \tag{4.14}$$

4.2.1 Weak sequential stability

Suppose we have a family $\{\varrho_\varepsilon, \mathbf{m}_\varepsilon\}_{\varepsilon > 0}$ of solutions to (4.12–4.14) defined in a space-time cylinder $(0, T) \times \Omega$. The energy estimates just yield all nonlinearities uniformly bounded in the (non-reflexive) space $L^1((0, T) \times \Omega)$, or, better, in $L^\infty(0, T; L^1(\Omega))$. There are two disturbing phenomena that may occur to bounded sequences in L^1.

- **Oscillations.** A sequence $\{h_\varepsilon\}_{\varepsilon>0}$ behaves like

$$h_\varepsilon(t,x) \approx h\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) \text{ for some } h \text{ periodic in } t \text{ and } x.$$

- **Concentrations.** A sequence $\{h_\varepsilon\}_{\varepsilon>0}$ behaves like

$$h_\varepsilon(t,x) \approx \frac{1}{\varepsilon^{N+1}} h\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) \text{ for some } h \in C_c^\infty(\{|(t,x)|<1\}).$$

Both phenomena may, of course, occur simultaneously. In the subsequent analysis, oscillations in the family $\{\varrho_\varepsilon, \mathbf{m}_\varepsilon\}_{\varepsilon>0}$ will be controlled by the equations (4.12), (4.13), while concentrations will be handled by means of the energy inequality (4.14). It is important to observe that all non-linearities in (4.12), (4.13) are dominated by the energy for large values of arguments, specifically

$$\left|\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right| \leq c\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}\right),\ p(\varrho)\leq cP(\varrho) \text{ for all } \varrho >> 1. \tag{4.15}$$

As the space $L^1((0,T)\times\Omega)$ is embedded in the space of signed *measures* $\mathcal{M}([0,T]\times\Omega)$ that happens to be the dual to $C([0,T]\times\Omega)$, we may assume that

$$\begin{aligned}
\varrho_\varepsilon &\to \{\varrho\} \text{ weakly-(*) in } \mathcal{M}([0,T]\times\Omega),\\
\mathbf{m}_\varepsilon &\to \{\mathbf{m}\} \text{ weakly-(*) in } \mathcal{M}([0,T]\times\Omega;R^N),\\
p(\varrho_\varepsilon) &\to \{p\} \text{ weakly-(*) in } \mathcal{M}([0,T]\times\Omega),\\
P(\varrho_\varepsilon) &\to \{P\} \text{ weakly-(*) in } \mathcal{M}([0,T]\times\Omega),\\
\frac{\mathbf{m}_\varepsilon\otimes\mathbf{m}_\varepsilon}{\varrho_\varepsilon} &\to \{\mathbb{M}\} \text{ weakly-(*) in } \mathcal{M}([0,T]\times\Omega;R^{N\times N})
\end{aligned}$$

passing to suitable subsequences as the case may be.

Consequently, we may let $\varepsilon\to 0$ in (4.12), (4.13) obtaining

$$\begin{gathered}
-\int_\Omega \varrho_0\varphi \ \mathrm{d}x = \int_0^T\int_\Omega \left[\{\varrho\}\partial_t\varphi + \{\mathbf{m}\}\cdot\nabla_x\varphi\right] \ \mathrm{d}x\,\mathrm{d}t,\\
\text{for all } \varphi\in C_c^1([0,T)\times\Omega),
\end{gathered} \tag{4.16}$$

$$\begin{gathered}
-\int_\Omega \mathbf{m}_0\cdot\varphi \ \mathrm{d}x = \int_0^T\int_\Omega \left[\{\mathbf{m}\}\cdot\partial_t\varphi + \{\mathbb{M}\}:\nabla_x\varphi + \{p\}\mathrm{div}_x\varphi\right] \ \mathrm{d}x\,\mathrm{d}t,\\
\text{for all } \varphi\in C_c^1([0,T)\times\Omega;R^N),
\end{gathered} \tag{4.17}$$

where ϱ_0, $\mathbf{m}_0$ stand for the limit of the initial data.

As a matter of fact, the limit for ϱ_ε, $\mathbf{m}_\varepsilon$ can be "improved". It follows from (4.4) that

$$\varrho_\varepsilon \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)),$$

where, obviously, ϱ can be identified with $\{\varrho\}$. Similarly,

$$\mathbf{m}_\varepsilon = \sqrt{\varrho_\varepsilon}\frac{\mathbf{m}_\varepsilon}{\sqrt{\varrho_\varepsilon}},$$

where, by virtue of (4.2),

$$\left\{\frac{\mathbf{m}_\varepsilon}{\sqrt{\varrho_\varepsilon}}\right\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^2(\Omega));$$

whence, applying Hölder's inequality and (4.4), we may infer that

$$\mathbf{m}_\varepsilon \to \mathbf{m}(\equiv \{\mathbf{m}\}) \text{ weakly-(*) in } L^{\frac{2\gamma}{\gamma-1}}(\Omega;R^N)).$$

Finally, we may consider

$$\varphi_\delta = \psi_\delta(t)\varphi,\ \varphi \in \mathbb{C}^1([0,T]\times\Omega)$$

$$\psi_\delta \in C^\infty(R),\ \psi_\delta' \leq 0,\ \psi_\delta(t) = \left\{ \begin{array}{l} 1 \text{ for } t \leq \tau, \\ 0 \text{ for} t \geq \tau+\delta \end{array}\right.$$

as a test function in (4.16), (4.17) and perform the limit $\delta \to 0$ to conclude that

$$\begin{gathered}\int_\Omega \varrho\varphi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \varrho_0\varphi\ \mathrm{d}x = \int_0^\tau\int_\Omega [\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi]\ \mathrm{d}x\,\mathrm{d}t, \\ \text{for a.a. } \tau\in(0,T), \text{ and for all } \varphi \in C^1([0,T]\times\Omega),\end{gathered} \tag{4.18}$$

$$\begin{gathered}\int_\Omega \mathbf{m}\cdot\varphi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \mathbf{m}_0\cdot\varphi\ \mathrm{d}x = \int_0^\tau\int_\Omega [\mathbf{m}\cdot\partial_t\varphi + \{\mathbb{M}\}:\nabla_x\varphi + \{p\}\mathrm{div}_x\varphi]\ \mathrm{d}x\,\mathrm{d}t, \\ \text{for a.a. } \tau\in(0,T), \text{ and for all } \varphi \in C^1([0,T]\times\Omega;R^N),\end{gathered} \tag{4.19}$$

where τ correspond to the Lebesgue points of the mappings

$$t\mapsto \varrho(t,\cdot)\in L^\gamma(\Omega),\ t\mapsto \mathbf{m}(t,\cdot) \text{ in } L^{2\gamma/(\gamma-1)}(\Omega;R^N).$$

Finally, we may apply the same arguments to the energy inequality (4.14) deducing

$$\int_\Omega \left[\frac{N}{2}\mathrm{trace}\{\mathbb{M}\} + \{P\}\right](\tau,\cdot)\ \mathrm{d}x \leq \int_\Omega E_0\ \mathrm{d}x \text{ for a.a. } \tau\in(0,T), \tag{4.20}$$

where E_0 denotes the limit of the initial energy, and

$$\int_\Omega \left[\frac{N}{2}\mathrm{trace}\{\mathbb{M}\} + \{P\}\right](\tau,\cdot)\ \mathrm{d}x = \lim_{\delta\to 0}\frac{1}{2\delta}\int_{\tau-\delta}^{\tau+\delta}\int_\Omega\left[\frac{N}{2}\mathrm{trace}\{\mathbb{M}\} + \{P\}\right]\ \mathrm{d}x\,\mathrm{d}t \tag{4.21}$$

for a.a. $\tau\in(0,T)$, where the limit is understood in the space of measures $\mathcal{M}(\Omega)$.

4.2.2 Oscillations - Young measures

There is an efficient tool to describe oscillations in weakly convergent sequences - the associated Young measure. We report the following fundamental result, see Pedregal [25, Chapter 6, Theorem 6.2] (also Ball [3]).

Theorem 4.1. *Let* $\{\mathbf{v}_n\}_{n=1}^{\infty}$, $\mathbf{v}_n : Q \subset R^N \to R^M$ *be a sequence of functions bounded in* $L^1(Q; R^M)$, *where* Q *is a domain in* R^N.

Then there exist a subsequence (not relabeled) and a parameterized family $\{\nu_y\}_{y\in Q}$ *of probability measures on* R^M *depending measurably on* $y \in Q$ *with the following property:*

For any Caratheodory function $\Phi = \Phi(y, z)$, $y \in Q$, $z \in R^M$ *such that*

$$\Phi(\cdot, \mathbf{v}_n) \to \overline{\Phi} \text{ weakly in } L^1(Q),$$

we have

$$\overline{\Phi}(y) = \int_{R^M} \psi(y, z)\ \mathrm{d}\nu_y(z) \text{ for a.a. } y \in Q.$$

Our goal is to rewrite the limit system in terms of the Young measure associated to the family $[\varrho_\varepsilon, \mathbf{m}_\varepsilon] \in R^{1+N}$. Accordingly, the Young measure is a parameterized family of probability measures supported on the set $[0, \infty) \times R^N$ (the densities are supposed non-negative)

$$\nu_{t,x} : (t, x) \in (0, T) \times \Omega \to \mathcal{P}([0, \infty) \times R^N),\ \nu \in L^\infty_{\text{weak}-(*)}((0, T) \times \Omega; \mathcal{P}(R^{1+N})),$$

with the barycenter

$$\langle \nu_{t,x}; \varrho \rangle = \varrho(t, x),\ \langle \nu_{t,x}; \mathbf{m} \rangle = \mathbf{m}(t, x) \text{ for a.a. } (t, x) \in (0, T) \times \Omega.$$

For a function $H = H(\varrho, \mathbf{m})$ we define

$$\overline{H(\varrho, \mathbf{m})}(t, x) \equiv \langle \nu_{t,x}; H(\varrho, \mathbf{m}) \rangle \text{ for a.a. } (t, x) \in (0, T) \times \Omega. \tag{4.22}$$

provided the latter exists.

Remark 4.2. *Here, we should distinguish between*

$$\langle \nu_{t,x}; F(\varrho, \mathbf{m} \rangle$$

meaning the application of the Young measure to the Borel function $F : R^1 \times R^N \to R$, *and the composition*

$$F(\varrho, \mathbf{m}) : (0, T) \times \Omega \to R.$$

From Theorem 4.1 we know that $\overline{H(\varrho, \mathbf{m})}$ is well defined L^1-function (meaning for a.a. (t,x)) provided

$$H(\varrho_\varepsilon, \mathbf{m}_\varepsilon) \to \overline{H(\varrho, \mathbf{m})} \text{ weakly in } L^1((0,T)\times\Omega). \tag{4.23}$$

We show that $\overline{H(\varrho, \mathbf{m})}$ can be defined through formula (4.22), meaning that the function $H(\varrho, \mathbf{m})$, $[\varrho, \mathbf{m}] \in R^{1+N}$ is $\nu_{t,x}$ integrable for a.a. (t,x), and the function

$$(t,x) \mapsto \langle \nu_{t,x}; H(\varrho, \mathbf{m})\rangle \text{ is integrable in } (0,T)\times\Omega,$$

whenever

$$\int_0^T \int_\Omega |H(\varrho_\varepsilon, \mathbf{m}_\varepsilon)| \, \mathrm{d}x \, \mathrm{d}t \le c(\text{data}) \text{ uniformly for } \varepsilon \to 0. \tag{4.24}$$

Note that (4.24) in general *does not* imply (4.23).

Since the Lebesgue integral is absolutely convergent, it is enough to consider $|H|$, or, equivalently, to assume that $H \ge 0$. We take a family of cut-off functions

$$T_k(Z) = \min\{Z, k\}.$$

As $T_k(H)$ are bounded, the functions $T_k(H)$ are $\nu_{t,x}$ integrable for a.a. (t,x),

$$(t,x) \mapsto \langle \nu_{t,x}; T_k(H(\varrho, \mathbf{m}))\rangle \in L^1((0,T)\times\Omega),$$
$$\|(t,x) \mapsto \langle \nu_{t,x}; T_k(H(\varrho, \mathbf{m}))\rangle\|_{L^1((0,T)\times\Omega)} \le c(\text{data})$$

uniformly for $k \to \infty$. However,

$$T_k(H(\varrho, \mathbf{m})) \nearrow H(\varrho, \mathbf{m}) \text{ for any } [\varrho, \mathbf{m}] \in R^{N+1} \text{ as } k \to \infty;$$

whence, by monotone convergence theorem, H is $\nu_{t,x}$ integrable (the integral may equal ∞). On the other hand,

$$\overline{T_k(H(\varrho, \mathbf{m}))} \nearrow \overline{H(\varrho, \mathbf{m})} \text{ in } L^1((0,T)\times\Omega);$$

whence

$$\langle \nu_{t,x}; T_k(H(s, \mathbf{v}))\rangle \to \overline{H(\varrho, \mathbf{m})}(t,x) = \langle \nu_{t,x}; H(\varrho, \mathbf{m})\rangle \text{ for a.a. } (t,x).$$

Thus $\langle \nu_{t,x}; H(\varrho, \mathbf{m})\rangle$ is finite for a.a. $(t,x) \in (0,T)\times\Omega$.

Remark 4.3. *The function* $\overline{H(\varrho, \mathbf{m})}$ *- the limit of* $\overline{T_k(H(\varrho, \mathbf{m}))}$ *for* $k \to \infty$ *is known as the* biting limit *of* $\{H(\varrho_\varepsilon, \mathbf{u}_\varepsilon)\}_{\varepsilon > 0}$*, cf. Ball and Murat [4].*

We infer that

$$\text{weak-(*) limit in } \mathcal{M} \text{ of } H(\varrho_\varepsilon, \mathbf{m}_\varepsilon) \equiv \{H\} = \overline{H(\varrho, \mathbf{m})} + D_H \tag{4.25}$$

for any H such that

$$\int_0^T \int_\Omega |H(\varrho_\varepsilon, \mathbf{m}_\varepsilon)| \, \mathrm{d}x \, \mathrm{d}t \le c(\text{data}),$$

where $D_H \in \mathcal{M}([0,T]\times\Omega)$ is called *concentration defect measure.* From the previous discussion, we deduce a useful estimate for the concentration defect measure D_H, namely,

$$
\begin{aligned}
\|D_H\|_{\mathcal{M}(Q)} &= \sup_{\|\varphi\|_{C(Q)}\leq 1} \int_Q D_H\varphi \ \mathrm{d}x\,\mathrm{d}t \\
&\leq \lim_{k\to\infty}\left(\lim_{\varepsilon\to 0}\int_{|H(\varrho_\varepsilon,\mathbf{m}_\varepsilon)|\geq k,\ (t,x)\in Q} (|H(\varrho_\varepsilon,\mathbf{m}_\varepsilon)|-k)\ \mathrm{d}x\,\mathrm{d}t\right)
\end{aligned}
\tag{4.26}
$$

Consequently, equations (4.18), (4.19) can be written in the form

$$
\begin{gathered}
\int_\Omega \varrho\varphi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \varrho_0\varphi\ \mathrm{d}x = \int_0^\tau\int_\Omega [\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi]\ \ \mathrm{d}x\,\mathrm{d}t, \\
\text{for a.a. } \tau\in(0,T), \text{ and for all } \varphi\in C^1([0,T]\times\Omega),
\end{gathered}
\tag{4.27}
$$

$$
\begin{aligned}
&\int_\Omega \mathbf{m}\cdot\varphi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \mathbf{m}_0\cdot\varphi\ \mathrm{d}x \\
&= \int_0^\tau\int_\Omega \left[\mathbf{m}\cdot\partial_t\varphi + \overline{\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right)} : \nabla_x\varphi + \overline{p(\varrho)}\mathrm{div}_x\varphi + \mathbb{D}_m : \nabla_x\varphi\right]\ \mathrm{d}x\,\mathrm{d}t, \\
&\text{for a.a. } \tau\in(0,T),\ \text{and for all } \varphi\in C^1([0,T]\times\Omega;R^N),
\end{aligned}
\tag{4.28}
$$

together with the energy inequality

$$
\int_\Omega\left[\frac{1}{2}\overline{\left(\frac{|\mathbf{m}|^2}{\varrho}\right)} + \overline{P(\varrho)} + D\right](\tau,\cdot)\ \mathrm{d}x \leq \int_\Omega E_0\ \mathrm{d}x \text{ for a.a. } \tau\in(0,T),
\tag{4.29}
$$

where $D\in L^\infty(0,T)$ is the energy concentration defect satisfying, in view of (4.15), (4.26),

$$
\int_0^\tau\int_\Omega |\mathbb{D}_m|\ \mathrm{d}x\ \ \mathrm{d}t \leq c\int_0^\tau D(t)\ \ \mathrm{d}t \text{ for a.a. } \tau\in(0,T).
\tag{4.30}
$$

4.2.3 Measure-valued solutions

At this stage, we may forget the way how the Young and dissipative measures were obtained and define *measure-valued solution* of the Euler system (4.10), (4.11) as the pair

- the parameterized measure $\nu_{t,x}\in L^\infty_{\text{weak}-(*)}\Big((0,T)\times\Omega;\mathcal{P}([0,\infty)\times R^N)\Big)$;
- the dissipation defect $D\in L^\infty(0,T)$

satisfying

$$
\begin{gathered}
\int_\Omega \langle\nu_{\tau,x};\varrho\rangle\,\varphi\ \mathrm{d}x - \int_\Omega\langle\nu_{0,x};\varrho\rangle\,\varphi\ \mathrm{d}x = \int_0^\tau\int_\Omega [\langle\nu_{t,x};\varrho\rangle\,\partial_t\varphi + \langle\nu_{t,x};\mathbf{m}\rangle\cdot\nabla_x\varphi]\ \ \mathrm{d}x\,\mathrm{d}t, \\
\text{for a.a. } \tau\in(0,T), \text{ and for all } \varphi\in C^1([0,T]\times\Omega);
\end{gathered}
\tag{4.31}
$$

$$\int_\Omega \langle \nu_{\tau,x}; \mathbf{m}\rangle \cdot \varphi \ {\rm d}x - \int_\Omega \langle \nu_{0,x}; \mathbf{m}\rangle \cdot \varphi \ {\rm d}x$$
$$= \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; \mathbf{m}\rangle \cdot \partial_t \varphi + \left\langle \nu_{t,x}; \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} \right\rangle : \nabla_x \varphi + \langle \nu_{t,x}; p(\varrho)\rangle \, {\rm div}_x \varphi + \mathbb{D}_m : \nabla_x \varphi \right] \ {\rm d}x\,{\rm d}t, \tag{4.32}$$

for a.a. $\tau \in (0,T)$, and for all $\varphi \in C^1([0,T]\times\Omega; R^N)$,

$$\int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2}\left(\frac{|\mathbf{m}|^2}{\varrho}\right) + P(\varrho) \right\rangle \ {\rm d}x + D(\tau) \le \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2}\left(\frac{|\mathbf{m}|^2}{\varrho}\right) + P(\varrho) \right\rangle \ {\rm d}x \text{ for a.a. } \tau \in (0,T), \tag{4.33}$$

where

$$\int_0^\tau \int_\Omega |\mathbb{D}_m| \ {\rm d}x \ {\rm d}t \le c \int_0^\tau D(t) \ {\rm d}t \text{ for a.a. } \tau \in (0,T). \tag{4.34}$$

Remark 4.4. *Note that the measure ν_0 plays the role of "initial conditions". If we assume that the data were controlled in the limit $\varepsilon \to 0$, the initial measure coincides with*

$$\mathbf{u}_{0,x} = \delta_{\varrho_0(x);\mathbf{m}_0(x)} \textit{ for a.a. } x \in \Omega.$$

Remark 4.5. *The function*

$$H : \ [\varrho, \mathbf{m}] \mapsto \frac{|\mathbf{m}|^2}{\varrho}$$

is defined as

$$H[\varrho, 0] = 0 \textit{ for all } \varrho \in R, \ H[\varrho, \mathbf{m}] = \infty \textit{ for } \varrho \le 0, \ \mathbf{m} \ne 0.$$

Then H is convex, lower-semi continuous and can be approximated from below by a sequence of continuous functions.

4.2.4 Weak-strong uniqueness in the class of measure-valued solutions

The proof of the weak-strong uniqueness property in the context of *measure-valued* solutions follows the same lines as in Section 3. We introduce an analogue of the relative energy:

$$\mathcal{E}\left(\nu = \nu_{t,x}(\varrho,\mathbf{m}) \ \Big| \ r, \mathbf{U}\right) = \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}\left(|\mathbf{m} - \varrho\mathbf{U}|^2\right) + P(\varrho) - P'(r)(\varrho - r) - P(r) \right\rangle \ {\rm d}x \tag{4.35}$$

for C^1-smooth functions r and $\mathbf{U}$.

Next, we successively compute

$$\int_\Omega \langle \nu_{\tau,x}; \mathbf{m}\rangle \cdot \mathbf{U} \ {\rm d}x - \int_\Omega \langle \nu_0; \mathbf{m}\rangle \cdot \mathbf{U}_0 \ {\rm d}x$$
$$= \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; \mathbf{m}\rangle \cdot \partial_t \mathbf{U} + \left\langle \nu_{t,x}; \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} \right\rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(\varrho)\rangle \, {\rm div}_x \mathbf{U} \right] \ {\rm d}x\,{\rm d}t$$
$$+ \int_0^\tau \int_\Omega \mathbb{D}_m : \nabla_x \mathbf{U} \ {\rm d}x\,{\rm d}t;$$

$$\int_\Omega \frac{1}{2} \langle \nu_{\tau,x}; \varrho \rangle |\mathbf{U}|^2 \ \mathrm{d}x - \frac{1}{2} \int_\Omega \langle \nu_{0,x}; \varrho \rangle |\mathbf{U}|^2 \ \mathrm{d}x$$
$$= \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; \varrho \rangle \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U} \right] \ \mathrm{d}x \, \mathrm{d}t;$$

and

$$\int_\Omega \langle \nu_{\tau,x}; \varrho \rangle P'(r) \ \mathrm{d}x - \int_\Omega \langle \nu_{0,x}; \varrho \rangle P'(r) \ \mathrm{d}x$$
$$= \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; \varrho \rangle \partial_t P'(r) + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \nabla_x P'(r) \right] \ \mathrm{d}x \, \mathrm{d}t.$$

Summing up the previous relations, we obtain a measure-valued analogue of the relative energy inequality (2.14):

$$\begin{aligned}
&\left[\int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho} \left(|\mathbf{m} - \varrho \mathbf{U}|^2 \right) + P(\varrho) - P'(r)(\varrho - r) - P(r) \right\rangle \ \mathrm{d}x \right]_{t=0}^{t=\tau} + D(\tau) \\
&\leq - \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; \mathbf{m} \rangle \cdot \partial_t \mathbf{U} + \left\langle \nu_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(\varrho) \rangle \operatorname{div}_x \mathbf{U} \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; \varrho \rangle \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U} \right] \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; \varrho \rangle \partial_t P'(r) + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \nabla_x P'(r) - r \partial_t P'(r) \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega |\mathbb{D}_m : \nabla_x \mathbf{U}| \ \mathrm{d}x \, \mathrm{d}t,
\end{aligned} \tag{4.36}$$

where, furthermore, the expression on the right-hand side can be rewritten as

$$\begin{aligned}
&\int_0^\tau \int_\Omega \left[p(r) - \langle \nu_{t,x}; p(\varrho) \rangle \right] \operatorname{div}_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \langle \nu_{t,x}; \varrho \mathbf{U} - \mathbf{m} \rangle \cdot \left[\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \right] \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; \varrho \rangle \partial_t P'(r) + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \nabla_x P'(r) - r \partial_t P'(r) + p(r) \operatorname{div}_x \mathbf{U} \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega |\mathbb{D}_m : \nabla_x \mathbf{U}| \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{(\mathbf{m} - \varrho \mathbf{U}) \otimes (\varrho \mathbf{U} - \mathbf{m})}{\varrho} \right\rangle : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

Now we assume in addition that r, $\mathbf{U}$ is a smooth solution of the Euler system,

$$\partial_t r + \mathbf{U} \cdot \nabla_x r = - r \operatorname{div}_x \mathbf{U}, \ \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} = - \frac{p'(r)}{r} \nabla_x r.$$

Consequently,

$$
\begin{aligned}
&\left[\int_\Omega \left\langle \nu_{t,x}; \frac{1}{2\varrho}\left(|\mathbf{m}-\varrho\mathbf{U}|^2\right) + P(\varrho) - P'(r)(\varrho - r) - P(r) \right\rangle \ \mathrm{d}x\right]_{t=0}^{t=\tau} + D(\tau)\\
&\leq \int_0^\tau \int_\Omega \left[p(r) + \langle \nu_{t,x}; \varrho\rangle\, p'(r) - \langle \nu_{t,x}; p(\varrho)\rangle\right] \mathrm{div}_x \mathbf{U} \ \mathrm{d}x\,\mathrm{d}t\\
&+ \int_0^\tau \int_\Omega \left[r\partial_t P'(r) - p(r)\mathrm{div}_x\mathbf{U}\right] \ \mathrm{d}x\,\mathrm{d}t\\
&+ \int_0^\tau \int_\Omega |\mathbb{D}_m : \nabla_x \mathbf{U}| \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{(\mathbf{m}-\varrho\mathbf{U})\otimes(\varrho\mathbf{U}-\mathbf{m})}{\varrho}\right\rangle : \nabla_x \mathbf{U} \ \mathrm{d}x\,\mathrm{d}t\\
&= -\int_0^\tau \int_\Omega \Big\langle \nu_{t,x}; p(\varrho) - p'(r)(\varrho - r) - p(r)\Big\rangle \mathrm{div}_x \mathbf{U} \ \mathrm{d}x\,\mathrm{d}t\\
&+ \int_0^\tau \int_\Omega |\mathbb{D}_m : \nabla_x \mathbf{U}| \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \left\langle \nu_{t,x}; \frac{(\mathbf{m}-\varrho\mathbf{U})\otimes(\varrho\mathbf{U}-\mathbf{m})}{\varrho}\right\rangle : \nabla_x \mathbf{U} \ \mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{4.37}
$$

Thus, in view of (4.34), the integral on the right-hand side of (4.37) can be "absorbed" by the left-hand side by means of a Gronwall argument as soon as

$$
\int_\Omega \left\langle \nu_0; \frac{1}{2\varrho}\left(|\mathbf{m}-\varrho\mathbf{U}|^2\right) + P(\varrho) - P'(r)(\varrho - r) - P(r) \right\rangle \ \mathrm{d}x,
$$

meaning

$$
\nu_{0,x} = \delta_{r(x), r(x)\mathbf{U}(x)} \text{ for a.a. } x \in \Omega.
$$

We have obtained the following results, cf. Gwiazda, Swierczewska-Gwiazda, Wiedemann [16].

Theorem 4.6. *Let ϱ, $\mathbf{U}$ be a C^1 solution of the compressible Euler system (1.6), (1.7) in $[0,T]\times\Omega$. Let $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$, be a measure-valued solution of the same system (in terms of ϱ and the momentum $\mathbf{m}$), with a dissipation defect D and such that*

$$
\nu_{0,x} = \delta_{r(0,x), r\mathbf{U}(0,x)} \ \textit{for a.a. } x \in \Omega.
$$

Then $D = 0$ and

$$
\nu_{t,x} = \delta_{r(t,x), r\mathbf{U}(t,x)} \ \textit{for a.a. } (t,x) \in (0,T)\times\Omega.
$$

Remark 4.7. *Regularity of the strong solution can be relaxed to globally Lipschitz in $(0,T)\times\Omega$ by a simple density argument.*

Remark 4.8. *A similar result holds for the compressible Navier-Stokes system, see [14].*

We point out that not all measure-valued solutions to the compressible Euler system can be identified as asymptotic limits of families of weak solutions, see [7] for a counterexample.

5 Weak sequential stability of solutions to the Navier-Stokes system

Our goal will be to show that, in contrast with the Euler system, a family of (weak, dissipative) solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ to the compressible Navier-Stokes system is weakly compact provided the initial data are. In particular, any accumulation point of such a family is again a weak (dissipative) solution of the same problem.

5.1 Integrability

We recall the *a priori* bounds that follow directly from the energy inequality, see Section 4.1:

$$\begin{aligned} \sup_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot)\|_{L^\gamma(\Omega)} \leq c(\text{data}), \\ \sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon(t,\cdot)\|_{L^2(\Omega;R^N)} \leq c(\text{data}) \\ \sup_{t\in(0,T)} \|\varrho_\varepsilon\mathbf{u}_\varepsilon(t,\cdot)\|_{L^{2\gamma/(\gamma+1)}(\Omega;R^N)} \leq c(\text{data}) \\ \int_0^T \|\mathbf{u}_\varepsilon(t,\cdot)\|^2_{W^{1,2}(\Omega;R^N)} \ \mathrm{d}t \leq c(\text{data}). \end{aligned} \tag{5.1}$$

We get immediately

$$\sup_{t\in(0,T)} \|p(\varrho_\varepsilon)(t,\cdot)\|_{L^1(\Omega)} \leq c(\text{data}), \tag{5.2}$$

and, by virtue of Hölder's inequality,

$$\|\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\|_{L^p(\Omega;R^{N\times N})} \leq \|\sqrt{\varrho_\varepsilon}\|_{L^{2\gamma}(\Omega)}\|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega;R^N)}\|\mathbf{u}_\varepsilon\|_{L^q(\Omega;R^N)},$$
$$\text{where } \frac{1}{p} = \frac{1}{2\gamma}+\frac{1}{2}+\frac{1}{q},\ q = \left\{ \begin{array}{l} \geq 1 \text{ arbitrary finite if } N=2, \\ 6 \text{ if } N=3 \end{array} \right. ,$$

where q corresponds to the Sobolev embedding relation,

$$W^{1,2}(\Omega) \hookrightarrow L^q(\Omega).$$

Consequently, we may take $p>1$ and conclude

$$\|\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\|_{L^2(0,T;L^p(\Omega;R^N))} \leq c(\text{data}) \text{ if } \gamma = \left\{ \begin{array}{l} >1 \text{ if } N=2, \\ \gamma > \frac{3}{2} \text{ if } N=3, \end{array} \right. \tag{5.3}$$

in which case

$$1 \leq p < \frac{2\gamma}{\gamma+1} \text{ if } N=2,\ 1 \leq p \leq \frac{6\gamma}{4\gamma+3}.$$

5.2 Pressure estimates

It remains to "improve" the space integrability of the pressure. This may impose even more restrictions on the adiabatic exponent γ, however, fortunately, it is not the case. In order to simplify the presentation, we start with the case

$$\gamma > 3.$$

The idea is to express the pressure by means of the momentum equation (1.15). Formally, we obtain

$$\nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S} - \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) - \partial_t(\varrho \mathbf{u})$$

whence

$$\Delta_x p(\varrho) = \mathrm{div}_x \mathrm{div}_x \left(\mathbb{S} - \varrho \mathbf{u} \otimes \mathbf{u} \right) - \partial_t \mathrm{div}_x(\varrho \mathbf{u}),$$

and, consequently,

$$p(\varrho) = \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x \left(\mathbb{S} - \varrho \mathbf{u} \otimes \mathbf{u} \right) - \partial_t \Delta_x^{-1} \mathrm{div}_x [\varrho \mathbf{u}] + \frac{1}{|\Omega|} \int_\Omega p(\varrho) \ \mathrm{d}x \ \mathrm{d}x.$$

Note that the first term on the right-hand side is under control in view of (5.1), (5.3), however, we do not have any information on

$$\partial_t \Delta_x^{-1} \mathrm{div}_x [\varrho \mathbf{u}].$$

To overcome this problem, we multiply the identity by ϱ and perform by parts integration obtaining

$$\begin{aligned}
&\int_0^T \int_\Omega p(\varrho)\varrho \ \mathrm{d}x \ \mathrm{d}t \\
&= \int_0^T \int_\Omega \varrho \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x \left(\mathbb{S} - \varrho \mathbf{u} \otimes \mathbf{u} \right) \ \mathrm{d}x \, \mathrm{d}t + \frac{1}{|\Omega|} \int_0^T \int_\Omega \varrho \ \mathrm{d}x \int_\Omega p(\varrho) \ \mathrm{d}x \, \mathrm{d}t \\
&- \left[\int_\Omega \varrho \Delta_x^{-1} \mathrm{div}_x [\varrho \mathbf{u}] \ \mathrm{d}x \right]_{t=0}^{t=T} + \int_0^T \int_\Omega \Delta_x^{-1} \mathrm{div}_x [\varrho \mathbf{u}] \partial_t \varrho \ \mathrm{d}x \ \mathrm{d}t \qquad (5.4) \\
&= \int_0^T \int_\Omega \varrho \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x \left(\mathbb{S} - \varrho \mathbf{u} \otimes \mathbf{u} \right) \ \mathrm{d}x \, \mathrm{d}t + \frac{1}{|\Omega|} \int_0^T \int_\Omega \varrho \ \mathrm{d}x \int_\Omega p(\varrho) \ \mathrm{d}x \, \mathrm{d}t \\
&- \left[\int_\Omega \varrho \Delta_x^{-1} \mathrm{div}_x [\varrho \mathbf{u}] \ \mathrm{d}x \right]_{t=0}^{t=T} + \int_0^T \int_\Omega [\varrho \mathbf{u}] \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x [\varrho \mathbf{u}] \ \mathrm{d}x \ \mathrm{d}t,
\end{aligned}$$

where all integrals now contain only spatial derivatives. Seeing that the pseudodifferential operators $\mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x$ preserve the L^p-norm, we may estimate the pressure integral

$$\int_0^T \int_\Omega p(\varrho)\varrho \ \mathrm{d}x \approx \int_0^T \int_\Omega \varrho^{\gamma+1} \ \mathrm{d}x \, \mathrm{d}t \approx \int_0^T \int_\Omega p(\varrho)^{\frac{\gamma+1}{\gamma}} \ \mathrm{d}x \, \mathrm{d}t$$

using the energy bounds (5.1–5.3) deduced in the preceding section. Indeed, if $\gamma > 3$, $N = 3$, relations

(5.1) imply

$$\sup_{t\in(0,T)} \|\varrho_\varepsilon \mathbf{u}_\varepsilon(t,\cdot)\|_{L^{3/2}(\Omega;R^3)} \leq c(\text{data}),$$

$$\int_0^T \|\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\|_{L^{3/2}(\Omega;R^{3\times 3})} \leq c(\text{data})$$

$$\int_0^T \|\varrho_\varepsilon \mathbf{u}\|^2_{L^2(\Omega;R^3)} \leq c(\text{data}),$$

therefore, using the Sobolev embedding

$$W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega)$$

we easily observe that all integrals on the right-hand side of (5.4) remain bounded in terms of the initial data. Consequently, we may conclude

$$\int_0^T \int_\Omega p(\varrho_\varepsilon)^{\frac{\gamma+1}{\gamma}} \ \mathrm{d}x\,\mathrm{d}t \leq c(\text{data}) \text{ provided } \gamma > 3. \tag{5.5}$$

Of course, the same holds also for $N = 2$.

To carry out this programme in the context of weak solutions, it is enough to consider the quantity

$$\varphi = \nabla_x \Delta_x^{-1} \left(\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_\varepsilon \ \mathrm{d}x \right) \tag{5.6}$$

in the weak formulation (2.11) for the momentum equation.

5.3 Refined pressure estimates, renormalized equation of continuity

For $\gamma < 3$, we may need to replace (5.6) by

$$\varphi = \nabla_x \Delta_x^{-1} \left(\varrho_\varepsilon^\beta - \frac{1}{|\Omega|} \int_\Omega \varrho_\varepsilon^\beta \ \mathrm{d}x \right), \ \beta > 0,$$

which gives rise to a term

$$\partial_t \varrho^\beta$$

in the formal calculations. We therefore need to express the *time derivative* of a composition $b(\varrho)$ of ϱ with a non-linear function b. Formally, the relevant relation may be derived by multiplying the equation of continuity on $b'(\varrho)$:

$$\partial_t b(\varrho) + \mathrm{div}_x \left(b(\varrho)\mathbf{u} \right) + \left[b'(\varrho)\varrho - b(\varrho) \right] \mathrm{div}_x \mathbf{u} = 0 \tag{5.7}$$

or, in the weak formulation,

$$\begin{aligned} \left[\int_\Omega b(\varrho)\varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} &= \int_{\tau_1}^{\tau_2} \int_\Omega \left[b(\varrho)\partial_t \varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi \right] \ \mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\tau_1}^{\tau_2} \int_\Omega \left[b(\varrho) - b'(\varrho)\varrho \right] \mathrm{div}_x \mathbf{u} \varphi \ \mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{5.8}$$

for any $\varphi \in C_c^\infty([0,T]\times\Omega)$.

Equation (5.7) or (5.8) is called *renormalized equation of continuity.* Renormalization plays an important role in the theory of weak solutions to the transport equations, see the pioneering work by DiPerna and Lions [11], or the more recent development by Ambrosio [2].

The weak formulation (5.8) can be derived for *weak* solutions of the equation of continuity enjoying certain integrability properties. In particular, if

$$\mathbf{u} \in L^q(0,T;W^{1,p}(\Omega)),\ \varrho \in L^{q'}(0,T;W^{1,p'}(\Omega)),\ \frac{1}{p}+\frac{1}{p'}=1,\ \frac{1}{q}+\frac{1}{q'}=1.$$

In general, satisfaction of (5.8) has to be included as an integral part of the definition of weak solutions to the compressible Navier-Stokes system.

The basic idea employed in [11] was to apply *mollifiers* to the continuity equation. Such an operation can be performed even at the level of weak solutions if we realize that mollifying via spatial convolution is just taking a special class of test functions. More specifically, consider a family of *regularizing kernels* $\{\kappa_\delta\}_{\delta>0}$,

$$\kappa_\delta(z) = \frac{1}{\delta^N}\kappa\left(\frac{|z|}{\delta}\right), \kappa \in C_c^\infty(R),\ \mathrm{supp}\ \kappa \subset (-1,1),$$

$$\kappa \geq 0,\ \kappa(-y)=k(y),\ \kappa(y) \text{ decreasing for } y\geq 0,\ \int_R \kappa(y)\ \mathrm{d}y = 1.$$

We report the following technical result usually attributed to Friedrichs, see [15, Lemma 10.12].

Lemma 5.1. *Let*

$$h_\delta = \kappa_\delta * \mathrm{div}_x(r\mathbf{U}) - \mathrm{div}_x\left([\kappa_\delta * r]\mathbf{U}\right),$$

where $\{\kappa_\delta\}_{\delta>0}$ *is a family of regularizing kernels, and where the symbol* $*$ *denotes convolution in the* $x-$*variable. Let*

$$r \in L^p(\Omega),\ \mathbf{U} \in W^{1,p'}(\Omega),\ \frac{1}{p}+\frac{1}{p'}=1,\ p>1 \text{ finite.}$$

Then

$$\|h_\delta\|_{L^1(\Omega)} \leq c\|r\|_{L^p(\Omega)}\|\mathbf{U}\|_{W^{1,p'}(\Omega)},\ \text{and } h_\delta \to 0 \text{ in } L^1(\Omega) \text{ as } \delta \to 0.$$

As a direct application of Friedrich's lemma, we may show that

- ϱ, $\mathbf{u}$ satisfying the equation of continuity in the weak sense satisfy also its renormalized version as long as

$$\varrho \in L^2((0,T)\times\Omega),\ \mathbf{U} \in L^2(0,T;W^{1,2}(\Omega;R^N)).$$

- the density component of ϱ of a renormalized solution is *strongly continuous in time*,

$$t \mapsto \varrho(t,\cdot) \text{ is a continuous mapping from } [0,T] \text{ to } L^1(\Omega).$$

To see the first property, just regularize the equation of continuity by κ_δ (use $\kappa_\delta(y-\cdot)$ as a test function):

$$\partial_t[\varrho]_\delta + \mathrm{div}_x\left([\varrho]_\delta\mathbf{u}\right) = \mathrm{div}_x\left([\varrho]_\delta\mathbf{u}\right) - \left[\mathrm{div}_x\left(\varrho\mathbf{u}\right)\right]_\delta,$$

where $[h]_\delta = \kappa_\delta * h$. As now everything is smooth, we get easily

$$\partial_t b\left([\varrho]_\delta\right) + \mathrm{div}_x\left(b\left([\varrho]_\delta \mathbf{u}\right)\right) + \left(b'\left([\varrho]_\delta\right)[\varrho]_\delta - b\left([\varrho]_\delta\right)\right)\mathrm{div}_x \mathbf{u}$$
$$= b'\left([\varrho]_\delta\right)\left(\mathrm{div}_x\left([\varrho]_\delta \mathbf{u}\right) - \left[\mathrm{div}_x\left(\varrho \mathbf{u}\right)\right]_\delta\right).$$

Now, at least for b' bounded we may let $\delta \to 0$ to obtain the desired result.

In order to see strong continuity, consider $b(\varrho)$ for b, b' bounded, $b' > 0$. As $b(\varrho)$ satisfies the renormalized equation, we conclude that

$$b \in C_{\mathrm{weak}}([0,T]; L^p(\Omega)) \text{ for any } 1 \le p < \infty$$

provided b is defined through limits of integral averages

$$\int_\Omega b(\varrho)(\tau,\cdot)\varphi \ \mathrm{d}x = \lim_{\delta \to 0} \frac{1}{2\delta} \int_{-\delta}^{\delta} \int_\Omega b(\varrho)(\tau,\cdot)\varphi \ \mathrm{d}x\,\mathrm{d}t,\ \ \tau \in [0,T],$$

with the corresponding one-sided limits at the extremal points $\tau = 0, T$. Regularizing (5.8) by means of a family of convolution kernels, we get, for $b = b(\varrho)$,

$$\partial_t [b]_\delta + \mathrm{div}_x\left([b]_\delta \mathbf{u}\right) = \left[(b(\varrho) - b'(\varrho)\varrho)\mathrm{div}_x \mathbf{u}\right]_\delta,$$

where the right-hand side is a function belonging to $L^2(0,T;C^k(\Omega))$ for any k finite. Thus $[b]_\delta$ is Hölder continuous in time ranging in the space of smooth functions.

Now, we may perform "second renormalization" deducing

$$\partial_t [b]_\delta^2 + \mathrm{div}_x\left([b]_\delta^2 \mathbf{u}\right) = 2[b]_\delta\left[(b(\varrho) - b'(\varrho)\varrho)\mathrm{div}_x \mathbf{u}\right]_\delta - [b]_\delta^2 \mathrm{div}_x \mathbf{u}.$$

Taking the limit for $\delta \to 0$ we get that

- $$[b]_\delta(t,\cdot) \to b(t,\cdot) \text{ in } L^1(\Omega),\ [b]_\delta^2(t,\cdot) \to b^2(t,\cdot) \text{ in } L^1(\Omega) \text{ for } \textit{any } t \in [0,T];$$

- $$[b]_\delta \to b \text{ in } C_{\mathrm{weak}}(0,T;L^1(\Omega)),\ [b]_\delta^2 \to b^2 \text{ in } C_{\mathrm{weak}}(0,T;L^1(\Omega)).$$

Remark 5.1. *As a matter of fact,*

$$[b]_\delta^2 \to \overline{b^2} \ \textit{in } C_{\mathrm{weak}}(0,T;L^1(\Omega)),$$

however, since we also have strong point-wise convergence, $\overline{b^2} = b^2$.

Thus for $t \to \tau$, we get

$$b(t,\cdot) \to b(\tau,\cdot) \text{ weakly in } L^2(\Omega),\ \int_\Omega b^2(t,\cdot)\ \mathrm{d}x \to \int_\Omega b^2(\tau,\cdot)\ \mathrm{d}x;$$

whence b is strongly continuous. This implies strong continuity of ϱ.

5.4 Weak compactness and continuity of the convective terms

Here again, for the sake of simplicity, we assume that $\gamma > 3$, and, to handle the most difficult case, $N = 3$. In view of the uniform bounds established in the preceding section, all non-linearities appearing in the weak formulation (2.11) are equi-integrable - weakly precompact in $L^1((0,T)\times\Omega)$, consequently, passing to suitable subsequences as the case may be, we deduce

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T]; L^\gamma(\Omega)),\ \mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)), \tag{5.9}$$

where

$$\left[\int_\Omega \varrho\varphi \ {\rm d}x\right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2}\int_\Omega \left[\varrho\partial_t\varphi + \overline{\varrho\mathbf{u}}\cdot\nabla_x\varphi\right] \ {\rm d}x \ {\rm d}t,$$

for any $0 \le \tau_1 \le \tau_2 \le T$ and for any test function $\varphi \in C^1_c([0,T]\times\Omega)$,

$$\begin{aligned}\left[\int_\Omega \overline{\varrho\mathbf{u}}\cdot\varphi \ {\rm d}x\right]_{t=\tau_1}^{t=\tau_2} &= \int_0^T\int_\Omega \left[\overline{\varrho\mathbf{u}}\cdot\partial_t\varphi + \overline{\varrho\mathbf{u}\otimes\mathbf{u}} : \nabla_x\varphi + \overline{p(\varrho)}{\rm div}_x\varphi\right] \ {\rm d}x\,{\rm d}t \\ &\quad - \int_0^T\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\varphi \ {\rm d}x \ {\rm d}t\end{aligned} \tag{5.10}$$

for any $0 \le \tau_1 \le \tau_2 \le T$ and for any test function $\varphi \in C^1_c([0,T]\times\Omega; R^N)$,

together with the energy inequality

$$\begin{aligned}\int_\Omega \left[\frac{1}{2}\overline{\varrho|\mathbf{u}|^2} + \overline{P(\varrho)}\right](\tau,\cdot) \ {\rm d}x + \int_0^\tau\int_\Omega \overline{\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}} \ {\rm d}x \ {\rm d}t \\ \le \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0)\right] \ {\rm d}x\end{aligned} \tag{5.11}$$

for a.a. $\tau \in (0,T)$, where bars denote the weak limit of compositions. In addition, using convexity of the functions

$$[\varrho, \mathbf{m}] \mapsto \frac{|\mathbf{m}|^2}{\varrho},\ \varrho \mapsto P(\varrho),$$

we may deduce from (5.11) that

$$\begin{aligned}\int_\Omega \left[\frac{1}{2}\frac{|\overline{\varrho\mathbf{u}}|^2}{\varrho} + P(\varrho)\right](\tau,\cdot) \ {\rm d}x + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} \ {\rm d}x \ {\rm d}t \\ \le \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0)\right] \ {\rm d}x\end{aligned} \tag{5.12}$$

for a.a. $\tau \in (0,T)$. Note that, in contrast with the Euler system studied in Section 4, there are no concentration term. Our goal in the remaining part of this section will be to show that all "bars" can be removed in (5.11–5.12).

We start with a simple observation that $L^\gamma(\Omega)$ is compactly embedded into the dual $W^{-1,2}(\Omega)$ of the Sobolev space $W^{1,2}(\Omega)$ as soon as $\gamma > \frac{6}{5}$ as

$$W^{1,2}(\Omega) \hookrightarrow\hookrightarrow \text{ (compactly) in } L^q(\Omega),\ 1 \le q < 6.$$

Consequently, (5.9) implies

$$\varrho_\varepsilon \to \varrho \text{ (strongly) in } L^2(0,T;W^{-1,2}(\Omega)),$$

and

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \text{ in } C_{\rm weak}(0,T;L^{2\gamma/(\gamma+1)}(\Omega;R^3)), \tag{5.13}$$

in other words $\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}$. Now, keeping in mind that $\gamma > 3$, we get $\frac{2\gamma}{\gamma+1} > \frac{3}{2}$, and repeating the same arguments we deduec from (5.9), (5.13) that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^2(0,T;L^{3/2}(\Omega;R^{3\times 3})).$$

Thus (5.10), (5.12) give rise to

$$\left[\int_\Omega \varrho \varphi \ {\rm d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \ {\rm d}x \ {\rm d}t,$$

for any $0 \leq \tau_1 \leq \tau_2 \leq T$ and for any test function $\varphi \in C^1_c([0,T] \times \Omega)$,

$$\begin{aligned} \left[\int_\Omega \varrho \mathbf{u} \cdot \varphi \ {\rm d}x \right]_{t=\tau_1}^{t=\tau_2} &= \int_0^T \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \overline{p(\varrho)} {\rm div}_x \varphi \right] \ {\rm d}x \, {\rm d}t \\ &- \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \ {\rm d}x \ {\rm d}t \end{aligned} \tag{5.14}$$

for any $0 \leq \tau_1 \leq \tau_2 \leq T$ and for any test function $\varphi \in C^1_c([0,T] \times \Omega; R^N)$,

$$\begin{aligned} \int_\Omega \left[\frac{1}{2} \frac{|\varrho \mathbf{u}|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \ {\rm d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ {\rm d}x \ {\rm d}t \\ \leq \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \ {\rm d}x \text{ for a.a. } \tau \in (0,T). \end{aligned}$$

Thus it remains to show that

$$\overline{p(\varrho)} = p(\varrho).$$

Note that this relation for strictly convex pressure is *equivalent* to the strong (a.a. pointwise) convergence of the sequence of densities. As we have no information on boundedness of derivatives of ϱ_ε, this is a non-trivial task.

5.5 Strong convergence of the density

We first observe that ϱ, $\mathbf{u}$ satisfy the equation of continuity. Next, as $\gamma > 3$, we have $\varrho \in L^\infty(0,T;L^3(\Omega))$ and $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega))$; whence the regularization method delineated in Section 5.3 can be applied to recover the renormalized equation of continuity (5.8) for the limit solution, in particular,

$$\begin{aligned} \left[\int_\Omega \varrho \log(\varrho) \varphi \ {\rm d}x \right]_{t=\tau_1}^{t=\tau_2} &= \int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho \log(\varrho) \partial_t \varphi + \varrho \log(\varrho) \mathbf{u} \cdot \nabla_x \varphi \right] \ {\rm d}x \, {\rm d}t \\ &- \int_{\tau_1}^{\tau_2} \int_\Omega \varrho {\rm div}_x \mathbf{u} \varphi \ {\rm d}x \, {\rm d}t \end{aligned} \tag{5.15}$$

for any $\varphi \in C^\infty_c([0,T] \times \Omega)$.

By the same token, we have the same equation at the $\varepsilon-$level:

$$
\begin{aligned}
\left[\int_\Omega \varrho_\varepsilon \log(\varrho_\varepsilon)\varphi \ \mathrm{d}x\right]_{t=\tau_1}^{t=\tau_2} &= \int_{\tau_1}^{\tau_2}\int_\Omega \left[\varrho_\varepsilon \log(\varrho_\varepsilon)\partial_t\varphi + \varrho_\varepsilon \log(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x\varphi\right] \ \mathrm{d}x\,\mathrm{d}t \\
&- \int_{\tau_1}^{\tau_2}\int_\Omega \varrho_\varepsilon \mathrm{div}_x \mathbf{u}_\varepsilon \varphi \ \mathrm{d}x\,\mathrm{d}t \\
&\text{for any } \varphi \in C_c^\infty([0,T]\times\Omega);
\end{aligned}
$$

whence, letting $\varepsilon \to 0$,

$$
\begin{aligned}
\left[\int_\Omega \overline{\varrho \log(\varrho)}\varphi \ \mathrm{d}x\right]_{t=\tau_1}^{t=\tau_2} &= \int_{\tau_1}^{\tau_2}\int_\Omega \left[\overline{\varrho \log(\varrho)}\partial_t\varphi + \overline{\varrho \log(\varrho)\mathbf{u}} \cdot \nabla_x\varphi\right] \ \mathrm{d}x\,\mathrm{d}t \\
&- \int_{\tau_1}^{\tau_2}\int_\Omega \overline{\varrho \mathrm{div}_x \mathbf{u}} \varphi \ \mathrm{d}x\,\mathrm{d}t \\
&\text{for any } \varphi \in C_c^\infty([0,T]\times\Omega).
\end{aligned} \tag{5.16}
$$

Subtracting (5.15) from (5.16) we conclude

$$
\left[\int_\Omega \left[\overline{\varrho \log(\varrho)} - \varrho\log(\varrho)\right] \ \mathrm{d}x\right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2}\int_\Omega \left[\varrho \mathrm{div}_x\mathbf{u} - \overline{\varrho \mathrm{div}_x \mathbf{u}}\right] \ \mathrm{d}x\,\mathrm{d}t. \tag{5.17}
$$

Note that $\varrho \mapsto \varrho\log(\varrho)$ is strictly convex, therefore

$$
\overline{\varrho \log(\varrho)} - \varrho\log(\varrho) \geq 0,
$$

whereas

$$
\overline{\varrho \log(\varrho)} - \varrho\log(\varrho) = 0
$$

is in fact equivalent to the desired strong convergence of $\{\varrho_\varepsilon\}_{\varepsilon>0}$. Thus supposing the initial data converge strongly, meaning

$$
\overline{\varrho_0 \log(\varrho_0)} - \varrho_0\log(\varrho_0),
$$

we need to know more about the quantity $\varrho\mathrm{div}_x\mathbf{u} - \overline{\varrho\mathrm{div}_x\mathbf{u}}$.

Fortunately, or rather miraculously, we have the relation

$$
\overline{p(\varrho)\varrho} - \overline{p(\varrho)}\varrho = \left(\frac{4}{3}\mu + \eta\right)\left(\overline{\varrho\mathrm{div}_x\mathbf{u}} - \varrho\mathrm{div}_x\mathbf{u}\right) \tag{5.18}
$$

discovered by P.-L.Lions [19], called usually the *effective viscous flux identity*. As the pressure is non-decreasing, we immediately deduce

$$
\overline{p(\varrho)\varrho} - \overline{p(\varrho)}\varrho = \lim_{\varepsilon\to 0}(p(\varrho_\varepsilon) - p(\varrho))(\varrho_\varepsilon - \varrho) \geq 0
$$

yielding the desired sign of the right-hand side of (5.17).

5.5.1 Effective viscous flux

Our goal will be to show the effective viscous flux identity (5.18). It can be deduced from the Navier-Stokes system in a similar way we have obtained the pressure estimates in Section 5.2. Specifically, we take

$$\varphi = \nabla_x \Delta_x^{-1} \left[\varrho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \varrho_\varepsilon \ \mathrm{d}x \right]$$

as a test function in the momentum equation obtaining

$$\begin{aligned}
&\int_0^T \int_\Omega \left[p(\varrho_\varepsilon)\varrho_\varepsilon - \left(\frac{4}{3}\mu + \eta \right) \varrho_\varepsilon \mathrm{div}_x \mathbf{u}_\varepsilon \right] \ \mathrm{d}x \ \mathrm{d}t \\
&\quad = - \int_0^T \int_\Omega \varrho_\varepsilon \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x \left[\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \right] \ \mathrm{d}x \, \mathrm{d}t + \frac{1}{|\Omega|} \int_0^T \int_\Omega \varrho_\varepsilon \ \mathrm{d}x \int_\Omega p(\varrho_\varepsilon) \ \mathrm{d}x \, \mathrm{d}t \\
&\quad - \left[\int_\Omega \varrho_\varepsilon \Delta_x^{-1} \mathrm{div}_x [\varrho_\varepsilon \mathbf{u}_\varepsilon] \ \mathrm{d}x \right]_{t=0}^{t=T} + \int_0^T \int_\Omega [\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x [\varrho_\varepsilon \mathbf{u}_\varepsilon] \ \mathrm{d}x \ \mathrm{d}t.
\end{aligned} \tag{5.19}$$

Similarly, applying the same treatment to the limit equation with

$$\varphi \nabla_x \Delta_x^{-1} \left[\varrho - \frac{1}{|\Omega|} \int_\Omega \varrho \ \mathrm{d}x \right]$$

we get

$$\begin{aligned}
&\int_0^T \int_\Omega \left[\overline{p(\varrho)}\varrho - \left(\frac{4}{3}\mu + \eta \right) \varrho \mathrm{div}_x \mathbf{u} \right] \ \mathrm{d}x \ \mathrm{d}t \\
&\quad = - \int_0^T \int_\Omega \varrho \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x \left[\varrho \mathbf{u} \otimes \mathbf{u} \right] \ \mathrm{d}x \, \mathrm{d}t + \frac{1}{|\Omega|} \int_0^T \int_\Omega \varrho \ \mathrm{d}x \int_\Omega \overline{p(\varrho)} \ \mathrm{d}x \, \mathrm{d}t \\
&\quad - \left[\int_\Omega \varrho \Delta_x^{-1} \mathrm{div}_x [\varrho \mathbf{u}] \ \mathrm{d}x \right]_{t=0}^{t=T} + \int_0^T \int_\Omega [\varrho \mathbf{u}] \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x [\varrho \mathbf{u}] \ \mathrm{d}x \ \mathrm{d}t.
\end{aligned} \tag{5.20}$$

Consequently, the effective viscous flux identity (5.18) will follow provided we show that

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_0^T \int_\Omega [\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x [\varrho_\varepsilon \mathbf{u}_\varepsilon] \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \varrho_\varepsilon \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x \left[\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \right] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_0^T \int_\Omega [\varrho \mathbf{u}] \cdot \nabla_x \Delta_x^{-1} \mathrm{div}_x [\varrho \mathbf{u}] \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_\Omega \varrho \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x \left[\varrho \mathbf{u} \otimes \mathbf{u} \right] \ \mathrm{d}x \, \mathrm{d}t,
\end{aligned}$$

which can be conveniently rewritten as

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \mathbf{u}_\varepsilon \cdot \left[\varrho_\varepsilon \nabla_x \Delta_x^{-1} \mathrm{div}_x [\varrho_\varepsilon \mathbf{u}_\varepsilon] - \varrho_\varepsilon \mathbf{u}_\varepsilon \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x [\varrho_\varepsilon] \right] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \mathbf{u} \cdot \left[\varrho \nabla_x \Delta_x^{-1} \mathrm{div}_x [\varrho \mathbf{u}] - \varrho \mathbf{u} \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x [\varrho] \right] \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

Now, repeating the same argument as when showing compactness of convective terms, we observe that it is enough to show

$$\left[\varrho_\varepsilon \nabla_x \Delta_x^{-1} \mathrm{div}_x[\varrho_\varepsilon \mathbf{u}_\varepsilon] - \varrho_\varepsilon \mathbf{u}_\varepsilon \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x[\varrho_\varepsilon]\right] \to \left[\varrho \nabla_x \Delta_x^{-1} \mathrm{div}_x[\varrho \mathbf{u}] - \varrho \mathbf{u} \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x[\varrho]\right] \tag{5.21}$$

in $L^2(0,T;W^{-1,2}(\Omega;R^3))$. To this end, we need the following general result.

Lemma 5.2. *Let*

$$\mathbf{V}_n \to \mathbf{V} \text{ weakly in } L^p(\Omega;R^N),\ \mathbf{W}_n \to \mathbf{W} \text{ weakly in } L^q(\Omega;R^N),\ \frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1.$$

Then

$$V_n^i \partial_{x_i} \Delta^{-1} \partial_{x_j}[W_n^j] - W_n^i \partial_{x_i} \Delta^{-1} \partial_{x_j}[V_n^j] \to V^i \partial_{x_i} \Delta^{-1} \partial_{x_j}[W^j] - W^i \partial_{x_i} \Delta^{-1} \partial_{x_j}[V^j]$$

in $L^r(\Omega)$.

Lemma 5.2 follows from the celebrated Div-Curl Lemma by Murat and Tartar, see [24], [28].

Lemma 5.3. *Let* $Q \subset \mathbb{R}^N$ *be an open set. Assume*

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly in } L^p(Q;\mathbb{R}^N),$$

$$\mathbf{V}_n \to \mathbf{V} \text{ weakly in } L^q(Q;\mathbb{R}^N),$$

where

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1.$$

In addition, let

$$\left.\begin{array}{r} \mathrm{div}\ \mathbf{U}_n \equiv \nabla \cdot \mathbf{U}_n, \\ \mathrm{curl}\ \mathbf{V}_n \equiv (\nabla \mathbf{V}_n - \nabla^T \mathbf{V}_n) \end{array}\right\} \textit{be precompact in} \left\{ \begin{array}{l} W^{-1,s}(Q), \\ W^{-1,s}(Q;\mathbb{R}^{N\times N}), \end{array}\right.$$

for a certain $s>1$. *Then*

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \text{ weakly in } L^r(Q).$$

Proof of Lemma 5.2:

Simply write

$$\begin{aligned} &V_n^i \partial_{x_i} \Delta^{-1} \partial_{x_j}[W_n^j] - W_n^i \partial_{x_i} \Delta^{-1} \partial_{x_j}[V_n^j] \\ &= \left(V_n^i - \partial_{x_i} \Delta^{-1} \partial_{x_j}[V_n^j]\right) \partial_{x_i} \Delta^{-1} \partial_{x_j}[W_n^j] - \left(W_n^i - \partial_{x_i} \Delta^{-1} \partial_{x_j}[W_n^j]\right) \partial_{x_i} \Delta^{-1} \partial_{x_j}[V_n^j], \end{aligned}$$

and observe that

$$\partial_{x_i} \Delta^{-1} \partial_{x_j}[V_n^j],\ \partial_{x_i} \Delta^{-1} \partial_{x_j}[W_n^j],\ i=1,\dots,N$$

represent N components of a gradient (whence **curl** $=0$), while

$$\left(V_n^i - \partial_{x_i} \Delta^{-1} \partial_{x_j}[V_n^j]\right),\ \left(W_n^i - \partial_{x_i} \Delta^{-1} \partial_{x_j}[W_n^j]\right),\ i=1,\dots,N$$

are solenoidal.

$\square$

Now, seeing that ϱ_ε satisfy (5.9), (5.13), where

$$\frac{1}{\gamma} + \frac{\gamma+1}{2\gamma} = \frac{1}{q} < 1, \ L^q(\Omega) \hookrightarrow\hookrightarrow W^{-2,2}(\Omega),$$

the convergence in (5.21) takes place in $L^p(0,T;W^{-2,2}(\Omega))$ for any $1 \le p < \infty$. Next, as $\gamma > 3$, we get

$$\left[\varrho_\varepsilon \nabla_x \Delta_x^{-1} \mathrm{div}_x[\varrho_\varepsilon \mathbf{u}_\varepsilon] - \varrho_\varepsilon \mathbf{u}_\varepsilon \mathrm{div}_x \Delta_x^{-1} \mathrm{div}_x[\varrho_\varepsilon]\right] \in L^2(0,T;L^r(\Omega;R^3)), \ r = \frac{6\gamma}{12+\gamma} > \frac{6}{5};$$

whence

$$L^r(\Omega) \hookrightarrow\hookrightarrow W^{-\alpha,2}(\Omega) \text{ for a certain } \alpha < 1,$$

and, by interpolation, the convergence in (5.21) holds in the desired Hilbert space $L^2(0,T;W^{-1,2}(\Omega;R^3))$.

Thus we have completed the proof of the effective viscous flux identity (5.18) and we may infer that

$$\varrho_\varepsilon \to \varrho \text{ in } L^1((0,T)\times\Omega), \text{ in particular } \overline{p(\varrho)} = p(\varrho).$$

Thus we have shown the following result.

Theorem 5.2. *Let* $p(\varrho) = a\varrho^\gamma$, $a > 0$, $\gamma > 3$, $N = 3$. *Let*

$$\{\varrho_\varepsilon\}_{\varepsilon>0}, \ \{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$$

be a family of dissipative weak solutions to the compressible Navier-Stokes system in $(0,T)\times\Omega$, *with the initial data*

$$\varrho_{0,\varepsilon} \in L^\gamma(\Omega), \ \varrho_{0,\varepsilon} > 0, \ \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon} \in L^{2\gamma/(\gamma+1)}(\Omega;R^3),$$

$$\varrho_{0,\varepsilon} \to \varrho_0 \ \text{in } L^\gamma(\Omega), \ \int_\Omega \varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 \ \mathrm{d}x \le c.$$

Then

$$\varrho_\varepsilon \to \varrho \ \text{in } L^1((0,T)\times\Omega), \ \mathbf{u}_\varepsilon \to \mathbf{u} \ \text{weakly in } L^2(0,T;W^{1,2}(\Omega)),$$

where $[\varrho, \mathbf{u}]$ *is a dissipative weak solution to the compressible Navier-Stokes system.*

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References

[1] R.A. Adams. *Sobolev spaces*. Academic Press, New York, 1975.

[2] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.*, **158**:227–260, 2004.

[3] J.M. Ball. A version of the fundamental theorem for Young measures. *In Lect. Notes in Physics 344, Springer-Verlag*, pages 207–215, 1989.

[4] J.M. Ball and F. Murat. Remarks on Chacons biting lemma. *Proc. Amer. Math. Soc.*, **107**:655–663, 1989.

[5] E. Chiodaroli. A counterexample to well-posedness of entropy solutions to the compressible Euler system. *J. Hyperbolic Differ. Equ.*, **11**(3):493–519, 2014.

[6] E. Chiodaroli, C. De Lellis, and O. Kreml. Global ill-posedness of the isentropic system of gas dynamics. *Comm. Pure Appl. Math.*, **68**(7):1157–1190, 2015.

[7] E. Chiodaroli, E. Feireisl, O. Kreml, and E. Wiedemann. $\mathcal{A}$-free rigidity and applications to the compressible Euler system. 2015. arxiv preprint No. 1511.03114, to appear in Anal. Mat. Pura Appl.

[8] C. De Lellis and L. Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, **195**(1):225–260, 2010.

[9] C. De Lellis and L. Székelyhidi, Jr. The *h*-principle and the equations of fluid dynamics. *Bull. Amer. Math. Soc. (N.S.)*, **49**(3):347–375, 2012.

[10] C. De Lellis and L. Székelyhidi, Jr. Dissipative continuous Euler flows. *Invent. Math.*, **193**(2):377–407, 2013.

[11] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98**:511–547, 1989.

[12] C. L. Fefferman. Existence and smoothness of the Navier-Stokes equation. In *The millennium prize problems*, pages 57–67. Clay Math. Inst., Cambridge, MA, 2006.

[13] E. Feireisl. Maximal dissipation and well-posedness for the compressible Euler system. *J. Math. Fluid Mech.*, **16**(3):447–461, 2014.

[14] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann. Dissipative measure-valued solutions to the compressible Navier–Stokes system. *Calc. Var. Partial Differential Equations*, 55(6):55:141, 2016.

[15] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids*. Birkhäuser-Verlag, Basel, 2009.

[16] P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann. Weak-strong uniqueness for measure-valued solutions of some compressible fluid models. *Nonlinearity*, **28**(11):3873–3890, 2015.

[17] A. V. Kazhikhov. Correctness "in the large" of mixed boundary value problems for a model system of equations of a viscous gas. *Dinamika Splošn. Sredy*, 21(Tecenie Zidkost. so Svobod. Granicami):18–47, 188, 1975.

[18] S. Klainerman and A. Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Comm. Pure Appl. Math.*, **34**:481–524, 1981.

[19] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models.* Oxford Science Publication, Oxford, 1998.

[20] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1984.

[21] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, **20**:67–104, 1980.

[22] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of compressible and heat conductive fluids. *Comm. Math. Phys.*, **89**:445–464, 1983.

[23] V.G. Maz'ya. *Sobolev spaces.* Springer, Berlin, 1985.

[24] F. Murat. Compacité par compensation. *Ann. Sc. Norm. Sup. Pisa, Cl. Sci. Ser. 5*, IV:489–507, 1978.

[25] P. Pedregal. *Parametrized measures and variational principles.* Birkhäuser, Basel, 1997.

[26] L. Pick, A. Kufner, O. John, and S. Fu ˇ cík. *Function spaces. Vol. 1*, volume 14 of *De Gruyter Series in Nonlinear Analysis and Applications.* Walter de Gruyter & Co., Berlin, extended edition, 2013.

[27] J. Smoller. *Shock waves and reaction-diffusion equations.* Springer-Verlag, New York, 1967.

[28] L. Tartar. Compensated compactness and applications to partial differential equations. *Nonlinear Anal. and Mech., Heriot-Watt Sympos., L.J. Knopps editor, Research Notes in Math 39, Pitman, Boston*, pages 136–211, 1975.

[29] V. A. Vaigant and A. V. Kazhikhov. On the existence of global solutions to two-dimensional Navier-Stokes equations of a compressible viscous fluid (in Russian). *Sibirskij Mat. Z.*, **36**(6):1283–1316, 1995.

[30] W.P. Ziemer. *Weakly differentiable functions.* Springer-Verlag, New York, 1989.