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Abstract

It is well known that certain pp-wave metrics, belonging to a more general class of Ricci-flat type N, $\tau_i = 0$, Kundt spacetimes, are universal and thus they solve vacuum equations of all gravitational theories with Lagrangian constructed from the metric, the Riemann tensor and its derivatives of arbitrary order. In this paper, we show (in an arbitrary number of dimensions) that in fact all Ricci-flat type N, $\tau_i = 0$, Kundt spacetimes are universal and we also generalize this result in a number of ways by relaxing $\tau_i = 0$, $\Lambda = 0$ and type N conditions.

First, we show that a universal spacetime is necessarily a CSI spacetime, i.e. all curvature invariants constructed from the Riemann tensor and its derivatives are constant. Then we focus on type N where we arrive at a simple necessary and sufficient condition: a type N spacetime is universal if and only if it is an Einstein Kundt spacetime. Similar statement does not hold for type III Kundt spacetimes, however, we prove that a subclass of type III, $\tau_i = 0$, Kundt spacetimes is also universal.

1 Introduction

Already in late 80's and early 90's, it was shown [1,2] that certain pp waves are solutions to all gravitational theories with Lagrangian constructed from the metric, the Riemann tensor and its derivatives of arbitrary order, being thus also classical solutions to string theory. A natural question arises whether there exist other spacetimes with this property. This was answered affirmatively in [3] where further examples of these so called *universal* metrics were given.

Definition 1.1. A metric is called *universal* if all conserved symmetric rank-2 tensors constructed from the metric, the Riemann tensor and its covariant derivatives of arbitrary order are multiples of the metric.

A universal metric thus have $T_{ab} = \lambda g_{ab}$ for any conserved symmetric rank-2 tensor T_{ab}. Such metrics are of particular interest since they are vacuum solutions (possibly with a nonvanishing cosmological constant) to all theories with Lagrangian in the form

$$L = L(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \dots, \nabla_{a_1 \dots a_p} R_{bcde}). \tag{1}$$

In the Riemannian signature, these metrics were studied by Bleecker [4] where it was found that all such metrics are necessarily isotropy-irreducible homogeneous metrics. This is also a sufficient condition. In particular, this means that all polynomial invariants constructed from the Riemann tensor and its derivatives of arbitrary order are constants.

In this paper, we are concerned with the Lorentzian case and we study general properties of universal spacetimes in arbitrary dimension and we identify various classes of these spacetimes.

First, we employ a necessary condition for universal spacetimes derived in appendix B. Since $T^a_{\ a} =$ constant for a universal spacetime, one can prove (see the proof of theorem B.2)

Theorem 1.2. *A universal spacetime is necessarily a CSI spacetime.*

CSI spacetimes are spacetimes with all polynomial invariants constructed from the Riemann tensor and its derivatives of arbitrary order being constant, see e.g. [5]. Therefore, the CSI spacetimes play an important role in the question of universality and as potential solutions for arbitrary theories of gravity. Note, however, that the CSI criterion is not sufficient; on the other hand, there is an interesting reverse result (see appendix B): *Given a CSI metric, g_{ab}, then there exists a class of theories having g_{ab} as a vacuum solution.*

As far as universal spacetimes are concerned, the CSI constraint on universal spacetimes from theorem 1.2 allows us to systematically study necessary and sufficient conditions for universal metrics of types N and III in the classification of [6] (see also [7] for a recent review).

In particular, for type N[1] we derive the following necessary and sufficient condition:

Theorem 1.3. *A type N spacetime is universal if and only if it is an Einstein Kundt spacetime.*

Kundt spacetimes are spacetimes admitting non-expanding, shearfree, twistfree geodetic null congruence, see sec. 3 for discussion of Kundt spacetimes.

Universal metrics of [1, 2] belong to a more general class of Ricci-flat, $\tau_i = 0$, type N Kundt metrics. Furthermore, Einstein type N Kundt metrics also allow for $\tau_i \neq 0$ and for nonvanishing cosmological constant Λ (see sec. 3 for corresponding metrics).

For type III spacetimes, Einstein Kundt condition is not a sufficient condition for a spacetime being universal since further constraints follow. In this case, we prove the following sufficient conditions.

Theorem 1.4. *Einstein type III, $\tau_i = 0$, Kundt spacetimes obeying $C_{acde}{C_b}^{cde} = 0$ are universal.*

Necessary conditions are studied only under some genericity assumptions. For a "generic" type III spacetime, we show that Kundt condition is a necessary condition for universality (see sec. A.2).

Throughout this paper, Kundt spacetimes admitting covariantly constant null vector (CCNV) will be denoted as pp waves. While in four dimensions pp waves are of type N, in five dimensions they are of type N or III and in dimension $n > 5$ pp waves of type N, III and II exist [7]. Theorem 1.3 implies that type N pp waves are universal. On the other hand, type III pp waves are not compatible with necessary condition for type III universal spacetimes, $C_{acde}{C_b}^{cde} = 0$, and therefore are not universal, see sec. 3.

This paper is organized as follows: in section 2, we briefly introduce notation and summarize definitions and results necessary in this paper. Section 3 is devoted to the Kundt metrics and explicit examples of universal metrics within this class. In section 4, we prove necessary and sufficient conditions for type N universal spacetimes (theorem 1.3), while more technical proofs of theorems 1.4 and 1.2 are postponed to appendices A and B, respectively. In section 5, we briefly summarize and discuss the main results. Finally, in appendix C, we give a subset of Bianchi and Ricci identities used in this paper.

2 Preliminaries

In this paper, we employ algebraic classification of the Weyl tensor [6] and higher-dimensional generalizations of the Newman-Penrose [8, 9] and Geroch-Held-Penrose formalisms [10]. We follow notation of [7, 10]. Here, we only briefly summarize definitions and results necessary in this paper (for more thorough introduction see recent review paper [7]).

In an n-dimensional Lorentzian manifold, we work in a null frame with two null vectors $\boldsymbol{\ell}$ and $\boldsymbol{n}$ and $n-2$ spacelike vectors $\boldsymbol{m}^{(i)}$ obeying[2]

$$\ell^a \ell_a = n^a n_a = 0, \qquad \ell^a n_a = 1, \qquad m^{(i)a} m^{(j)}_a = \delta_{ij}. \tag{2}$$

Lorentz transformations between null frames are generated by boosts

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}, \tag{3}$$

null rotations and spins. We say that quantity q has a boost weight b if it transforms under a boost (3) according to

$$\hat{q} = \lambda^{\mathrm{b}} q. \tag{4}$$

[1]Note that while in [3] it was already mentioned without proof that Einstein type N Kundt spacetimes are universal, to our knowledge necessary conditions have been never studied.

[2]Hereafter, coordinate indices $a, b, \ldots$ and frame indices $i, j, \ldots$ take values from 0 to $n-1$ and 2 to $n-1$, respectively.

Obviously, various components of a tensor in a null frame may have distinct integer boost weights. For our purposes, it is convenient to introduce boost order of a tensor with respect to a null frame as a maximum boost weight of its frame components taken over nonvanishing components. Note that in fact boost order of a tensor depends only on the null direction $\boldsymbol{\ell}$.

Type N spacetimes are defined as spacetimes admitting a frame in which only boost-weight -2 components of the Weyl tensor are present. Consequently, the Weyl tensor can be expressed as [6, 7]

$$C_{abcd} = 4\Omega'_{ij}\, \ell_{\{a} m^{(i)}_{\ b} \ell_c m^{(j)}_{\ d\,\}}, \tag{5}$$

where

$$C_{\{abcd\}} \equiv \frac{1}{2}(C_{[ab][cd]} + C_{[cd][ab]}) = C_{abcd}, \tag{6}$$

and Ω'_{ij} is symmetric and traceless. For Einstein type N spacetimes, ℓ is necessarily geodetic [8]. Without loss of generality, we can choose ℓ to be affinely parameterized and a frame parallelly transported along ℓ. Then, the covariant derivatives of the frame vectors in terms of spin coefficients read

$$\ell_{a;b} = L_{11}\ell_a\ell_b + L_{1i}\ell_a m^{(i)}_{\ b} + \tau_i m^{(i)}_{\ a}\ell_b + \rho_{ij} m^{(i)}_{\ a} m^{(j)}_{\ b}, \tag{7}$$

$$n_{a;b} = -L_{11}n_a\ell_b - L_{1i}n_a m^{(i)}_{\ b} + \kappa'_i m^{(i)}_{\ a}\ell_b + \rho'_{ij} m^{(i)}_{\ a} m^{(j)}_{\ b}, \tag{8}$$

$$m^{(i)}_{\ a;b} = -\kappa'_i\ell_a\ell_b - \tau_i n_a\ell_b - \rho'_{ij}\ell_a m^{(j)}_{\ b} + \overset{i}{M}_{j1} m^{(j)}_{\ a}\ell_b - \rho_{ij} n_a m^{(j)}_{\ b} + \overset{i}{M}_{kl} m^{(k)}_{\ a} m^{(l)}_{\ b}. \tag{9}$$

It is often convenient to decompose the optical matrix ρ_{ij} into its trace θ (expansion), tracefree symmetric part σ_{ij} and antisymmetric part A_{ij}

$$\rho_{ij} = \sigma_{ij} + \theta\delta_{ij} + A_{ij}, \qquad \sigma_{ij} \equiv \rho_{(ij)} - \tfrac{1}{n-2}\rho_{kk}\delta_{ij}, \qquad \theta \equiv \tfrac{1}{n-2}\rho_{kk}, \qquad A_{ij} \equiv \rho_{[ij]}. \tag{10}$$

Shear and twist of $\boldsymbol{\ell}$ correspond to the traces $\sigma^2 \equiv \sigma^2_{ii} = \sigma_{ij}\sigma_{ji}$ and $\omega^2 \equiv -A^2_{ii} = -A_{ij}A_{ji}$, respectively.

It can be shown that, for algebraically special Einstein spacetimes, possible forms of ρ_{ij} are rather restricted. In particular, for type N Einstein spacetimes, a parallelly propagated frame can be chosen so that [8, 10]

$$\rho_{ij} = \mathrm{diag}\left(\begin{bmatrix} S & A \\ -A & S \end{bmatrix}, 0, \dots, 0\right). \tag{11}$$

Covariant derivatives in directions of frame vectors $\boldsymbol{\ell}$, $\boldsymbol{n}$ and $\boldsymbol{m^{(i)}}$ will be denoted as D, $\triangle$ and δ_i, respectively, so that

$$\nabla_a = n_a D + m^{(i)}_{\ a}\delta_i + \ell_a \triangle\,. \tag{12}$$

3 The Kundt metrics

As follows from the results discussed above, all universal metrics discussed here (and in fact possibly all universal metrics in general) belong to the Kundt class, defined as spacetimes admitting a null geodetic vector field $\boldsymbol{\ell}$ with vanishing shear, expansion and twist, i.e., $\kappa_i = 0 = \rho_{ij}$ and thus the covariant derivative of $\boldsymbol{\ell}$ simplifies to [11–13]

$$\ell_{a;b} = L_{11}\ell_a\ell_b + \tau_i(\ell_a m^{(i)}_{\ b} + m^{(i)}_{\ a}\ell_b). \tag{13}$$

Before introducing coordinates, let us first discuss various relevant subclasses of Kundt spacetimes:

The Ricci-flat type N Kundt class splits into two subclasses characterized by vanishing/nonvanishing τ_i. It can be shown that for $\tau_i = 0$ class, one can rescale $\boldsymbol{\ell}$ to set $L_{11} = 0$. This class thus represents type N Ricci-flat pp waves (i.e., it admits a covariantly constant null vector - CCNV).

The Ricci-flat type III[3] Kundt class splits again in two subclasses characterized by vanishing/nonvanishing τ_i. In the $\tau_i = 0$ class, representing recurrent spacetimes, L_{11} can be set to zero by boost iff $\Psi'_i = 0$. Thus, for $\tau_i = 0 = \Psi'_i$ this class represents type III Ricci-flat pp waves.

The necessary condition for type III universal spacetimes,

$$C_{acde}C_b^{\ cde} = 0, \tag{14}$$

[3]See (40) for the form of type III Weyl tensor.

is an identity for four-dimensional type III spacetimes. Furthermore, if (14) holds for an n-dimensional spacetime it is also valid for $n+1$ dimensional spacetime obtained by warping the original metric. Since also an algebraic type of an algebraically special spacetime is preserved under warp [14], one can easily construct higher-dimensional type III Kundt spacetimes obeying (14) by warping four-dimensional type III Kundt spacetimes. For $n > 4$ type III, (14) can be expressed as [15]

$$C_{acde}C_b{}^{cde} = \tilde{\Psi}\ell_a\ell_b = 0, \tag{15}$$

where

$$\tilde{\Psi} \equiv \frac{1}{2}\Psi'_{ijk}\Psi'_{ijk} - \Psi'_i\Psi'_i. \tag{16}$$

It is clear that, for type III pp waves, $\tilde{\Psi} \neq 0$ (since $\Psi'_i = 0$, see (59), while $\Psi'_{ijk} \neq 0$) and thus these spacetimes are not universal.

Note that, for type III and N Einstein Kundt spacetimes with $\tau_i = 0$, cosmological constant Λ necessarily vanishes due to the Ricci identity (62), while $\tau_i \neq 0$ class allows for nonvanishing Λ.

For the Kundt class, one can introduce coordinates such that $\boldsymbol{\ell} = \partial_r$ and $\ell_a \mathrm{d}x^a = \mathrm{d}u$ and the metric takes the form [5,16] (see also more recent papers studying the Kundt class [11,13,17])

$$\mathrm{d}s^2 = 2\mathrm{d}u\,[\mathrm{d}r + H(u,r,x^\gamma)\mathrm{d}u + W_\alpha(u,r,x^\gamma)\mathrm{d}x^\alpha] + g_{\alpha\beta}(u,x^\gamma)\mathrm{d}x^\alpha\mathrm{d}x^\beta, \tag{17}$$

where coordinate indices $\alpha,\beta,\gamma = 2\ldots n-1$. The remaining frame vectors can be chosen such that $n_a\mathrm{d}x^a = \mathrm{d}r + H\mathrm{d}u + W_\alpha\mathrm{d}x^\alpha$ and $m^{(i)}_a\mathrm{d}x^a = e^{(i)}_\alpha\mathrm{d}x^\alpha$ with $g_{\alpha\beta} = \delta_{ij}e^{(i)}_\alpha e^{(j)}_\beta$. It follows that $\ell_{a;b} = \frac{1}{2}g_{ab,r}$, $L_{11} = H_{,r}$ and $\tau_i = \frac{1}{2}W_{\alpha,r}\,e^\alpha_{(i)} = \frac{1}{2}W_{i,r}$. Obviously, for g_{ab} independent on r, $\boldsymbol{\ell}$ is a CCNV.

Since, by theorem 1.2, universality implies CSI, we can restrict ourselves to the *Kundt CSI metrics*, where [5,11]

$$\begin{aligned} W_\alpha(u,r,x^\gamma) &= rW^{(1)}_\alpha(u,x^\gamma) + W^{(0)}_\alpha(u,x^\gamma), \\ H(u,r,x^\gamma) &= \frac{r^2}{8}\left(a + W^{(1)}_\alpha W^{(1)\alpha}\right) + rH^{(1)}(u,x^\gamma) + H^{(0)}(u,x^\gamma), \end{aligned} \tag{18}$$

$g_{\alpha\beta}(x^\gamma)$ (note that $g_{\alpha\beta,u} = 0$) is a locally homogeneous space and a is a constant. Note that (18) are necessary but not sufficient conditions for Kundt CSI. Let us now further specialize to the Ricci-flat case.

3.1 Ricci-flat type III and N Kundt spacetimes

The class of Ricci-flat type III and N Kundt spacetimes coincides with Ricci-flat subclass of VSI spacetimes (spacetimes with all curvature invariants vanishing) of [18]. Corresponding metrics, studied in detail in [19], admit the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\,[\mathrm{d}r + H(u,r,x^\gamma)\mathrm{d}u + W_\alpha(u,r,x^\gamma)\mathrm{d}x^\alpha] + \delta_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta, \tag{19}$$

with

$$W_2 = -\frac{2\epsilon r}{x^2}, \tag{20}$$

$$W_M(u,r,x^\gamma) = W^{(0)}_M(u,x^\gamma), \tag{21}$$

$$H(u,r,x^\gamma) = \frac{\epsilon r^2}{2(x^2)^2} + rH^{(1)}(u,x^\gamma) + H^{(0)}(u,x^\gamma) \qquad (\epsilon = 0,\ 1), \tag{22}$$

which for type N reduce to

$$W_2 = -\frac{2\epsilon r}{x^2}, \tag{23}$$

$$W_M = x^N B_{NM}(u) + C_M(u)[x^2 + \epsilon(1-x^2)], \tag{24}$$

$$H = \frac{\epsilon r^2}{2(x^2)^2} + H^{(0)}(u,x^\gamma), \tag{25}$$

where $B_{NM} = B_{[NM]}$, $M,N = 3\ldots n-1$, $\epsilon = 0$ or 1 and $\tau_i = 0$ case corresponds to $\epsilon = 0$.

Universal metrics of [1,2] are of type N with $\tau_i = 0$ and they correspond to a subset of metrics (19), (23) - (25) with $\epsilon = 0$, $B_{NM}(u) = 0$, $C_M(u) = 0$. Note, however, that in four dimensions these two classes coincide since for $n = 4$ and $\epsilon = 0$, $B_{NM}(u)$ vanishes and $C_M(u)$ can be transformed away.

Further conditions follow from the Ricci-flat condition (see [19]).

3.2 Explicit examples of universal metrics

3.2.1 Type N Einstein Kundt

In four dimensions, after a coordinate transformation $r = vQ^2/P^2$, all type N Einstein Kundt metrics can be expressed as [20–22]

$$\mathrm{d}s^2 = 2\frac{Q^2}{P^2}\mathrm{d}u\mathrm{d}v + \left(2k\frac{Q^2}{P^2}v^2 + \frac{(Q^2)_{,u}}{P^2}v - \frac{Q}{P}H\right)\mathrm{d}u^2 + \frac{1}{P^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{26}$$

where

$$P = 1 + \frac{\Lambda}{12}(x^2+y^2), \quad k = \frac{\Lambda}{6}\alpha(u)^2 + \frac{1}{2}\left(\beta(u)^2 + \gamma(u)^2\right),$$
$$Q = \left(1 - \frac{\Lambda}{12}(x^2+y^2)\right)\alpha(u) + \beta(u)x + \gamma(u)y, \quad H = 2f_{1,x} - \frac{\Lambda}{3P}(xf_1 + yf_2),$$

where $\alpha(u), \beta(u), \gamma(u)$ are free functions (see [22] for canonical forms) and $f_1 = f_1(u,x,y)$ and $f_2 = f_2(u,x,y)$ obey $f_{1,x} = f_{2,y}$, $f_{1,y} = -f_{2,x}$.

Higher-dimensional type N Einstein Kundt spacetimes can be generated by warping (26). Note that, in general, singularities appear as a consequence of the warped product unless cosmological constants of both (n and $n+1$ dimensional) solutions are both negative or both zero (see [14] for more details).

Other examples in this class are (A)dS-waves of various kinds. For example,

$$\mathrm{d}s^2 = e^{-pw}\left(2\mathrm{d}u\mathrm{d}v + H(u,w,x^M)\mathrm{d}u^2 + \delta_{MN}\mathrm{d}x^M\mathrm{d}x^N\right) + \mathrm{d}w^2, \tag{27}$$

with p being a constant and H obeying $H_{,KL}\,\delta^{KL} + \left(H_{,ww} - \frac{n-1}{2}pH_{,w}\right)\mathrm{e}^{-pw} = 0$ and

$$\mathrm{d}s^2 = \sinh^2(pw)\left[2\mathrm{d}u\mathrm{d}v + \left(v^2p^2 + H(u,x^M)\right)\mathrm{d}u^2\right] + \mathrm{d}w^2 + \frac{1}{p^2}\cosh^2(pw)\mathrm{d}S^2_{\mathbb{H}^d}, \tag{28}$$

where $\mathrm{d}S^2_{\mathbb{H}^d} = (1 - \frac{1}{4}\delta_{KL}x^Kx^L)^{-2}\delta_{MN}\mathrm{d}x^M\mathrm{d}x^N$ is the 'unit' metric on the d-dimensional hyperbolic space, and H needs to satisfy

$$2\left(1 - \frac{1}{4}\delta_{MN}x^M\,x^N\right)H_{,KL}\,\delta^{KL} + (d-2)H_{,M}\,x^M = 0.$$

3.2.2 Type III, $\tau_i = 0$, Ricci-flat Kundt spacetimes

An explicit example of a four-dimensional type III, $\tau_i = 0$, Ricci-flat Kundt metric (expressed in another coordinates) is [23]

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - x(v + \mathrm{e}^x)\mathrm{d}u^2 + \mathrm{e}^x(\mathrm{d}x^2 + \mathrm{e}^{-2u}\mathrm{d}y^2). \tag{29}$$

One can obtain a higher-dimensional type III universal metric as a direct product of (29) with extra flat dimensions since the algebraic type and conditions (14) and $\tau_i = 0$ will be preserved.

4 The sufficiency and necessity proof for type N

Here, we present the sufficiency and necessity proof for type N.

4.1 The sufficiency proof

Let us start with the following proposition:

Proposition 4.1. *For Einstein type N Kundt spacetimes, boost order of $\nabla^{(k)}C$ (a covariant derivative of an arbitrary order of the Weyl tensor) is at most -2.*

Proof. We prove the above proposition using the balanced scalar approach introduced in [18]. Let us say that a scalar η with boost weight b is 1-balanced if $D^{-b-1}\eta = 0$ for $b < -1$ and $\eta = 0$ for $b \geq -1$ and that a tensor is 1-balanced if all its components are 1-balanced scalars.

Now Ricci and Bianchi equations (see appendix C) imply that for spin coefficients and Weyl components of various boost weights b

$$b=-2: \quad D^3\kappa'_i=0, \quad D\Omega'_{ij}=0, \tag{30}$$

$$b=-1: \quad D^2L_{11}=0, \quad D^2\rho'_{ij}=0, \quad D^2\overset{i}{M}_{j1}=0, \tag{31}$$

$$b=0: \qquad D\tau_i=0, \quad D\overset{i}{M}_{jk}=0. \tag{32}$$

It follows that, for a 1-balanced scalar η, scalars $L_{11}\eta$, $\tau_i\eta$, $\kappa'_i\eta$, $\rho'_{ij}\eta$, $\overset{i}{M}_{j1}\,\eta$ and $\overset{i}{M}_{kl}\,\eta$ are also 1-balanced scalars. Furthermore, from formulas for commutators given in appendix C it follows that $D\eta$, $\delta_i\eta$, $\triangle\eta$ are 1-balanced scalars as well.

In the covariant derivative of a 1-balanced tensor, only the above mentioned terms (like $\tau_i\eta$, $D\eta_i$ etc.) appear and thus

Lemma 4.2. *For Einstein type N Kundt spacetimes, a covariant derivative of a 1-balanced tensor is a 1-balanced tensor.*

Since a type N Weyl tensor (5) is 1-balanced, proposition 4.1 immediately follows. □

Let as now argue that proposition 4.1 implies that, for Einstein type N Kundt spacetimes, it is not possible to construct a nonvanishing rank-2 tensor from $\nabla^{(k)}C$ (and the metric).

First, note that any rank-2 tensor has components of boost weight -2 or higher. From proposition 4.1 it follows that

Lemma 4.3. *In Einstein type N Kundt spacetimes, rank-2 tensors constructed from $\nabla^{(k)}C$, which are quadratic or of higher order in $\nabla^{(k)}C$, vanish.*

We show below that rank-2 tensors linear in $\nabla^{(k)}C$ vanish as well.

For any tensor T, the commutator of covariant derivatives can be expressed as

$$[\nabla_a,\nabla_b]T_{c_1\ldots.c_k}=T_{d\ldots c_k}R^d_{c_1ab}+\cdots+T_{c_1\ldots d}R^d_{c_kab}. \tag{33}$$

When T is a Weyl tensor, rank-2 tensors constructed from RHS of (33) are either quadratic in C (and thus vanish due to lemma 4.3) or contractions of the Weyl tensor with the Ricci tensor or the metric which, for Einstein spacetimes, vanish due to tracelessness of the Weyl tensor.

When expressing rank-2 tensors constructed from the Weyl tensor, we can thus assume that first two covariant derivatives of the Weyl tensor commute. This will be expressed using the following notation

$$C_{abcd;ef}\cong C_{abcd;fe}. \tag{34}$$

Using (34), the Bianchi identities and tracelessness of the Weyl tensor, it follows that all rank-2 tensors constructed from $\nabla^{(2)}C$ vanish.

Now, assume that for some even n all covariant derivatives in $\nabla^{(n)}C$ in an expression of a rank-2 tensor commute and all rank-2 tensors from $\nabla^{(n)}C$ vanish. Using (33), we obtain

$$C_{abcd;e_1\ldots e_ifgh_1\ldots h_j}-C_{abcd;e_1\ldots e_igfh_1\ldots h_j}\cong 0 \tag{35}$$

(where $i+j=n$) since when using (35) in an expression of a rank-2 tensor from $\nabla^{(n+2)}C$ only rank-2 tensors from $\nabla^{(n)}C$ would appear on RHS of (35), but these are zero by our assumption. Using Bianchi identities, it again follows that rank-2 tensors from $\nabla^{(n+2)}C$ vanish. Our assumption for n thus also holds for $n+2$ and therefore these hold for all even n. Thus we arrived at

Lemma 4.4. *In Einstein type N Kundt spacetimes, rank-2 tensors constructed from $\nabla^{(k)}C$ which are linear in $\nabla^{(k)}C$ vanish.*

A direct consequence of lemmas 4.3 and 4.4 is

Proposition 4.5. *All type N Einstein Kundt spacetimes are universal.*

4.2 The necessity proof

Now, let us show that, for type N Einstein spacetimes, CSI implies Kundt.

The simplest non-trivial curvature invariant for type N spacetimes is [24]

$$I_N \equiv C^{a_1 b_1 a_2 b_2;c_1 c_2} C_{a_1 d_1 a_2 d_2;c_1 c_2} C^{e_1 d_1 e_2 d_2;f_1 f_2} C_{e_1 b_1 e_2 b_2;f_1 f_2}. \tag{36}$$

In terms of higher-dimensional GHP quantities it can be shown [18] (see also [25]) to be proportional (via a numerical constant) to

$$I_N \propto \left[(\Omega'_{22})^2 + (\Omega'_{23})^2\right]^2 (S^2 + A^2)^4, \tag{37}$$

where S and A are introduced in (11). For type N Einstein spacetimes, I_N is constant iff $S^2 + A^2 = 0$, which implies Kundt. This can be shown either by expressing DI_N and using the Bianchi and Ricci identities or by looking at the explicit $r-$dependence of I_N in the non-Kundt case [25]

$$I_N \propto \frac{\left[(\Omega'^0_{22})^2 + (\Omega'^0_{23})^2\right]^2}{(r^2 + a_0^2)^6}, \tag{38}$$

where r is an affine parameter along null geodetic integral curves of $\boldsymbol{\ell}$.

We thus arrived at

Lemma 4.6. *CSI Einstein type N spacetimes belong to the Kundt class.*

Together with theorem 1.2 this gives

Proposition 4.7. *Universal Einstein type N spacetimes belong to Kundt class.*

Theorem 1.3 now directly follows from Propositions 4.5 and 4.7.

5 Conclusion

Universal spacetimes are vacuum solutions to all theories with Lagrangian consisting of the metric, the Riemann tensor and its derivatives of arbitrary order. In this work, we study necessary and sufficient conditions for such spacetimes of arbitrary dimension and relate universal spacetimes to other known classes of metrics. First, the necessary condition (theorem 1.2) states that universal spacetimes are CSI (spacetimes with constant curvature invariants).

Then, we focus on type N and III spacetimes. For type N spacetimes, we find simple necessary and sufficient conditions for universality. In this case, theorem 1.3 states that a type N spacetime is universal if and only if it is an Einstein Kundt spacetime. This class of spacetimes consists of pp waves admitting CCNV ($\tau_i = 0, \Lambda = 0$) and Kundt waves ($\tau_i \neq 0, \Lambda$ arbitrary). A four-dimensional metric for these spacetimes is given in eq. (26). In higher dimensions, a metric has the form (17), which simplifies to (19), (23)-(25) in the Ricci-flat case.

For type III, situation is more complicated. We show (theorem 1.4) that type III Einstein, $\tau_i = 0$ Kundt spacetimes obeying $C_{acde} C_b{}^{cde} = 0$ are universal. Condition $C_{acde} C_b{}^{cde} = 0$ is clearly a necessary condition for type III spacetime being universal, on the other hand, we prove that the Kundt condition is a necessary condition only under some genericity assumptions. Although type III, $\tau_i \neq 0$ Kundt spacetimes are not in general universal, necessity of $\tau_i = 0$ condition remains open.

For dimensions $n > 4$, type III Einstein, $\tau_i = 0$ Kundt class contains also type III pp waves admitting CCNV. Interestingly, these pp waves are *not* universal. Instead, type III universal spacetimes admit a recurrent null vector. Type III universal spacetimes admit metric (19) - (22) with $\epsilon = 0$.

Although we have not discussed universal spacetimes of type II here, there are a few examples of universal spacetimes known already [3]; for example, of the type

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + \left(v^2\Lambda + H(u,x,y)\right)\mathrm{d}u^2 + \frac{1}{\Lambda}(\mathrm{d}x^2 + \sinh^2 x \mathrm{d}y^2), \quad \Box H = 0. \tag{39}$$

It is also believed that universal type II spacetimes are Kundt but we will return to the investigation of the type II spacetimes in a later work.

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A Necessary and sufficient conditions for type III spacetimes

Type III spacetimes are defined as spacetimes admitting a frame in which only boost weight -1 and -2 components of the Weyl tensor are present. Thus the Weyl tensor has the form [6, 7]

$$C_{abcd} = 8\Psi'_i \, \ell_{\{a}n_b\ell_c m^{(i)}_{d\,\}} + 4\Psi'_{ijk} \, \ell_{\{a}m^{(i)}_b m^{(j)}_c m^{(k)}_{d\,\}} + 4\Omega'_{ij} \, \ell_{\{a}m^{(i)}_b \ell_c m^{(j)}_{d\,\}}. \tag{40}$$

A.1 The sufficiency proof

We start with a type III analog of proposition 4.1:

Proposition A.1. *For Einstein type III Kundt spacetimes, boost order of $\nabla^{(k)}C$ (covariant derivative of the arbitrary order of the Weyl tensor) is at most -1.*

Proof. For the proof, we need to define balanced scalars as in [18]. Let us say that a scalar η with boost weight b is balanced if $D^{-b}\eta = 0$ for $b < 0$ and $\eta = 0$ for $b \geq 0$ and that a tensor is balanced if all its components are balanced scalars.

For type III Einstein Kundt spacetimes, the Ricci and Bianchi equations (see appendix C) for spin coefficients and Weyl components of various boost weights b imply that

$$b = -2: \quad D^3\kappa'_i = 0, \quad D^2\Omega'_{ij} = 0, \tag{41}$$

$$b = -1: \quad D^2 L_{11} = 0, \quad D^2\rho'_{ij} = 0, \quad D^2\overset{i}{M}_{j1} = 0, \quad D\Psi'_{ijk} = 0, \quad D\Psi'_i = 0, \tag{42}$$

$$b = 0: \quad D\tau_i = 0, \quad D\overset{i}{M}_{jk} = 0. \tag{43}$$

It follows that for a balanced scalar η, scalars $L_{11}\eta$, $\tau_i\eta$, $\kappa'_i\eta$, $\rho'_{ij}\eta$, $\overset{i}{M}_{j1}\,\eta$ and $\overset{i}{M}_{kl}\,\eta$ are also balanced scalars. From commutators given in appendix C, it also follows that $D\eta$, $\delta_i\eta$, $\triangle\eta$ are balanced scalars as well.

Analogously to the type N case, it follows that the covariant derivative of a balanced tensor is a balanced tensor and since type III Weyl tensor for an Einstein type III Kundt spacetime is balanced, proposition A.1 follows. □

A direct consequence of proposition A.1 is

Lemma A.2. *For Ricci-flat type III Kundt spacetimes, a non-trivial rank-2 tensor constructed from the metric, the Weyl tensor and its covariant derivatives of arbitrary order is at most quadratic in the Weyl tensor and its covariant derivatives.*

However, note that rank-2 Weyl polynomial $C_{acde}C_b^{\ cde}$ which is quadratic in C is in general nonvanishing for type III Einstein Kundt spacetimes [15] and thus in general these spacetimes are not universal. Obviously, type III Kundt universal spacetimes will be subject to (14). Furthermore, even when assuming that (14) holds, we find that in the Ricci-flat case, where FKWC basis [26] of rank-2, order-6 Weyl polynomials consists of

$$F_1 \equiv C^{pqrs}C_{pqrs;ab}, \quad F_2 \equiv C^{pqrs}{}_{;a}C_{pqrs;b}, \quad F_3 \equiv C^{pqr}{}_{a;s}C_{pqrb}{}^{;s}, \tag{44}$$

F_1 and F_2 are in general nonvanishing (while F_3 vanishes as a consequence of (14)). We show in sec. A.1.1 that, for $\tau_i = 0$, F_1 and F_2 vanish. Thus in the following, we focus on the $\tau_i = 0$ case leaving the question whether some particular type III Kundt universal metrics with $\tau_i \neq 0$ exist open.

A.1.1 Rank-2 Weyl polynomials quadratic in $\nabla^{(k)}C$ for Einstein type III, $\tau_i = 0$ Kundt spacetimes obeying (14) vanish

As pointed out in sec. 3, for type III Einstein Kundt, $\tau_i = 0$ implies vanishing Λ.

From the commutator of partial derivatives (33) and lemma A.2, it follows that in an expression for a rank-2 Weyl polynomial quadratic in $\nabla^{(k)}C$, RHS of (33) vanishes (since all terms are cubic in C) and thus covariant derivatives in such expressions effectively commute

$$T_{c_1\dots.c_k;ab} \cong T_{c_1\dots.c_k;ba}. \tag{45}$$

This allows us to prove

Lemma A.3. *For Ricci-flat type III, $\tau_i = 0$ Kundt spacetimes, a rank-2 tensor constructed from the metric, the Weyl tensor and its covariant derivatives of arbitrary order quadratic in $\nabla^{(k)}C$ vanishes if it contains a summation within $\nabla^{(k)}C$.*

Proof. i) Summation within C vanishes due to tracelessness of the Weyl tensor. ii) Summation within ∇C can be (using Bianchi identities) expressed in terms of traces of the Weyl tensor and thus it again vanishes. iii) Using Bianchi identities (45) and result of ii) it follows that summation within $\nabla^{(2)}C$ vanishes. Similar argument holds for higher derivatives. □

A direct consequence is

Lemma A.4. *For Ricci-flat type III, $\tau_i = 0$ Kundt spacetimes, let us assume that certain rank-2 polynomial quadratic in $\nabla^{(k)}C$ vanishes. Symbolically we will write $C^{(1)}C^{(2)} = 0$. Then also $C^{(1)}_{;f}C^{(2)}_{;f} = 0$.*

From proposition A.1, we know that $\nabla^{(k)}C$ has only terms of b.w. $\leq$ -1. Now, let us look at boost weight -1 terms after one further differentiation, i.e., b.w -1 terms in $(\nabla^{(k)}C_{abcd})_{;e}$. Such terms arise in one of the three following ways:

i) From (7) - (9) and (12) for $\tau_i = L_{1i} = \rho_{ij} = 0$, we see that differentiations of $\boldsymbol{\ell}'s$ and $\boldsymbol{n}'s$ in the frame decomposition of $\nabla^{(k)}C_{abcd}$ does not lead to b.w. -1 terms, while differentiation of vectors $\boldsymbol{m^{(i)}}$ leads to b.w. -1 terms (via term $\overset{i}{M}_{kl} m^{(k)}_a m^{(l)}_e$ in (9)).

ii) b.w. -1 terms also arise from differentiation of b.w. - 1 frame components (symbolically denoted as $\eta_{(-1)}$) of $\nabla^{(k)}C$ via $\left(\eta_{(-1)}\right)_{;e} \;\to\; \delta_i(\eta_{(-1)}) m^{(i)}_e$.

iii) Finally b.w. -1 terms also arise from differentiation of b.w. - 2 frame components $\eta_{(-2)}$ via $\left(\eta_{(-2)}\right)_{;e} \to D(\eta_{(-2)})n_e$.

It follows, that b.w -1 terms of $(\nabla^{(k)}C_{abcd})_{;e}$ do not contain vector ℓ_e.

Since any rank-2 tensor constructed from the metric, the Weyl tensor and its derivatives has the form $S_{ab} = f\ell_a\ell_b$ and only b.w. -1 terms in $\nabla^{(k)}C$ can contribute to the result, it follows (employing also (45)) that

Lemma A.5. *For Ricci-flat type III, $\tau_i = 0$ Kundt spacetimes, any rank-2 tensor constructed from the metric, the Weyl tensor and its derivatives, quadratic in $\nabla^{(k)}C$ with at least one free derivative index vanishes.*

This immediately implies that F_1 and F_2 given in (44) vanish and since the remaining rank-2 polynomial in order-6 FKWC basis F_3 also vanishes due to (14) it follows that all order-6, rank-2 polynomials quadratic in $\nabla^{(k)}C$ vanish as well.

Taking into account the Weyl tensor symmetries, lemmas A.3, A.4, A.5 and eq. (45), it follows that an existence of a nonvanishing rank-2 tensor constructed from the metric, the Weyl tensor and its derivatives and quadratic in $\nabla^{(k)}C$ implies existence of such nonvanishing rank-2 tensor constructed from at most second derivatives of the Weyl tensor.

Furthermore, the only such potentially nonvanishing term involving second derivatives of the Weyl tensor is

$$C_{agdh;ef}C_b{}^{edf;gh},$$

which, however, vanishes as well due to the Bianchi identities and lemma A.5.

We can thus conclude this section with

Proposition A.6. *For Ricci-flat type III Kundt spacetimes obeying* $C_{acde}C_b^{\ cde} = 0$ *and* $\tau_i = 0$*, all rank-2 tensors constructed from the metric, the Weyl tensor and its covariant derivatives of arbitrary order which are quadratic or of higher order in the Weyl tensor and its derivatives vanish.*

It remain to show that, under assumptions of proposition A.6, rank-2 Weyl polynomials linear in $\nabla^{(k)}C$ vanish as well.

A.1.2 Rank-2 Weyl polynomials linear in $\nabla^{(k)}C$

From proposition A.6 and (33), it follows that when expressing rank-2 polynomials linear in $\nabla^{(k)}C$, covariant derivatives commute. Vanishing of rank-2 Weyl polynomials linear in $\nabla^{(k)}C$ is then a trivial consequence of Bianchi identities and tracelessness of the Weyl tensor.

Together with the results given in sec. A.1 above, this completes the proof of theorem 1.4.

A.2 Necessary conditions

In has been shown in [8] that, under some additional genericity assumptions, the optical matrix of type III Einstein spacetimes has the form (11). In particular, this holds for all five-dimensional type III Einstein spacetimes and all non-twisting spacetimes in arbitrary dimensions. In fact, type III Einstein spacetimes not obeying (11) are not known.

The simplest non-trivial curvature invariant for type III Einstein spacetimes is [18]

$$I_{III} = C^{a_1b_1a_2b_2;e_1}C_{a_1c_1a_2c_2;e_1}C^{d_1c_1d_2c_2;e_2}C_{d_1b_1d_2b_2;e_2}. \tag{46}$$

Assuming (11) holds, the r-dependence of I_{III} was determined in eq. (97) of [25]. It follows that, for non-Kundt spacetimes, I_{III} is clearly non-constant.

Therefore, universal type III Einstein spacetimes obeying (11) belong to the Kundt class.

B Conserved tensors

Let us consider Lagrangians containing the Riemann tensor and its derivatives up to a fixed order p. Then it is shown in [27] that it can be rewritten in the form

$$L = L(g_{ab}, R_{abcd}, \nabla_{a_1}R_{bcde}, ..., \nabla_{(a_1...a_p)}R_{bcde}). \tag{47}$$

By varying the action, one obtains [27]

$$\begin{aligned} -T^{ab} &= \frac{\partial L}{\partial g_{ab}} + E^a_{\ cde}R^{bcde} + 2\nabla_c\nabla_d E^{acdb} + \frac{1}{2}g^{ab}L, \\ E^{bcde} &= \frac{\partial L}{\partial R_{bcde}} - \nabla_{a_1}\frac{\partial L}{\partial \nabla_{a_1}R_{bcde}} + \cdots + (-1)^p\nabla_{(a_1}\cdots\nabla_{a_p)}\frac{\partial L}{\partial \nabla_{(a_1}\cdots\nabla_{a_p)}R_{bcde}}. \end{aligned} \tag{48}$$

Here, the tensor T^{ab} is the associated conserved tensor. By taking the trace, we get

$$-T^a_{\ a} = g_{ab}\frac{\partial L}{\partial g_{ab}} + E_{bcde}R^{bcde} + 2\nabla_c\nabla_d E_a^{\ cda} + \frac{D}{2}L, \tag{49}$$

where D is the dimension of the spacetime.

Definition B.1. Let $\mathrm{CSI_{cons}}$ be the set of metrics with the following property: For all conserved symmetric rank-2 tensors, T_{ab}, then $T^a_{\ a}$ is constant.

Then:

Theorem B.2. *A metric is* $\mathrm{CSI_{cons}}$ *if and only if it is* CSI*; i.e.,* $\mathrm{CSI_{cons}} \Leftrightarrow \mathrm{CSI}$.

Proof. That CSI$\Rightarrow$ $\mathrm{CSI_{cons}}$ is obvious. We thus need to prove that $\mathrm{CSI_{cons}}$$\Rightarrow$ CSI. Assume therefore that a metric is $\mathrm{CSI_{cons}}$.

Consider a polynomial invariant I. Assume that the invariant contains derivatives of the Riemann tensor of orders at most p and by [27] we can assume it is of the form

$$I = I[g_{ab}, R_{abcd}, \nabla_{a_1}R_{bcde}, ..., \nabla_{(a_1...a_p)}R_{bcde}].$$

Let us consider the (infinite) series of Lagrangians $L = I^n$, $n = 1, 2, 3, ...$ For each n, we get, by variation, a conserved tensor $T[n]^a{}_b$. Consider then the trace of these tensors, which are all constants, $-T[n]^a{}_a = c_n$.

We can now use the equations for the variation (49) to obtain an equation for each n. For $n = 1$, we get

$$\frac{D}{2}I + g_{ab}\frac{\partial I}{\partial g_{ab}} + \tilde{E}_{bcde}R^{bcde} + 2\nabla_c\nabla_d\tilde{E}_a{}^{cda} = c_1, \tag{50}$$

where $\tilde{E}_{bcde}$ is E_{bcde} with $L = I$. We note that the sum of the 2nd, 3rd, and 4th terms on the LHS is an invariant itself, which we conveniently define as

$$X_1 \equiv g_{ab}\frac{\partial I}{\partial g_{ab}} + \tilde{E}_{bcde}R^{bcde} + 2\nabla_c\nabla_d\tilde{E}_a{}^{cda}, \tag{51}$$

so that

$$\frac{D}{2}I + X_1 = c_1.$$

Considering $n = 2$, we get

$$\frac{D}{2}I^2 + 2IX_1 + 2X_2 = c_2,$$

where X_1 is as before, and X_2 is also a polynomial invariant. In general, for any n, we can write

$$\frac{D}{2}I^n + nI^{n-1}X_1 + n(n-1)I^{n-2}X_2 + ... + n!X_n = c_n, \tag{52}$$

where X_i are polynomial invariants. This can be seen as follows. The terms in (49) contains derivatives of L of various sorts, symbolically called $\frac{\partial L}{\partial x}$. We then get

$$\begin{aligned}
\frac{\partial I^n}{\partial x} &= nI^{n-1}\frac{\partial I}{\partial x}, \\
\nabla_a\frac{\partial I^n}{\partial x} &= nI^{n-1}\nabla_a\frac{\partial I}{\partial x} + n(n-1)I^{n-2}(\nabla_a I)\frac{\partial I}{\partial x}, \\
\nabla_b\nabla_a\frac{\partial I^n}{\partial x} &= nI^{n-1}\nabla_b\nabla_a\frac{\partial I}{\partial x} + n(n-1)I^{n-2}\left[2\left(\nabla_{(a}I\right)\nabla_{b)}\frac{\partial I}{\partial x} + (\nabla_b\nabla_a I)\frac{\partial I}{\partial x}\right] \\
&\quad + n(n-1)(n-2)I^{n-3}(\nabla_a I)(\nabla_b I)\frac{\partial I}{\partial x}.
\end{aligned} \tag{53}$$

In general, the derivatives will be of the form (symbolically)

$$\nabla^{(k)}\frac{\partial I^n}{\partial x} = nI^{n-1}D_1[I] + n(n-1)I^{n-2}D_2[I] + ... + n(n-1)\cdots(n-k-1)I^{n-k-2}D_{k+1}[I],$$

where $D_i[I]$ are some derivative operators which do not depend on n. Thus, in (49) we collect all terms proportional to I^{n-i}. These all have a common factor $n(n-1)\cdots(n-i+1)$, so in (49) we get a term

$$n(n-1)\cdots(n-i+1)I^{n-i}X_i,$$

where X_i is independent of n. Hence, we arrive at expression (52). Note, however, that if $i > p+3$, then $X_i = 0$; consequently, there is only a finite number of possible non-zero X_i, $1 \leq i \leq p+3$.

We can now write down a set of $n+1 = p+4$ equations in a matrix form

$$\mathbf{Ax} = \mathbf{c},$$

where $\mathbf{x}$ and $\mathbf{c}$ are the column vectors $\mathbf{x} = (D/2, X_1, X_2, ..., X_n)^T$ and $\mathbf{c} = (c_1, c_2, c_3, ..., c_{n+1})^T$, and $\mathbf{A}$ is the matrix

$$\mathbf{A} = \begin{bmatrix}
I & 1 & 0 & 0 & \cdots & 0 \\
I^2 & 2I & 2 & 0 & \cdots & 0 \\
I^3 & 3I^2 & 3\cdot 2I & 3\cdot 2\cdot 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
I^n & nI^{n-1} & n(n-1)I^{n-2} & n(n-1)(n-2)I^{n-3} & \cdots & n! \\
I^{n+1} & (n+1)I^n & (n+1)nI^{n-1} & (n+1)n(n-1)I^{n-2} & \cdots & (n+1)n(n-1)\cdots 2I
\end{bmatrix}. \tag{54}$$

Consider the following elementary row operations (the equations will remain polynomial in nature)

- **Step 1:** Divide row i with i, for all i: R_i/i.
- **Step 2:** Multiply row 1 with appropriate powers of I and subtract row 1 from all other rows: $R_i \mapsto R_i - I^{i-1}R_1$, $\forall i$. This would turn column 2 into $(1,0,0,..,0)^T$.
- **Step 3:** Repeat step 1 and 2 so that column 3 turns into $(0,1,0,...,0)$.
- **Step k:** Repeat above steps so that column k turns into $(0,...,0,1,0,...0)$.

This algorithm ends with a simpler matrix representing the LHS

$$\begin{bmatrix} I & 1\ 0\ 0 \cdots\ 0 \\ -\frac{I^2}{2} & 0\ 1\ 0 \cdots\ 0 \\ \frac{I^3}{3!} & 0\ 0\ 1 \cdots\ 0 \\ \vdots & \vdots\ \vdots\ \vdots\ \ddots\ \vdots \\ \frac{(-1)^{n-1}I^n}{n!} & 0\ 0\ 0 \cdots\ 1 \\ \frac{(-1)^n I^{n+1}}{(n+1)!} & 0\ 0\ 0 \cdots\ 0 \end{bmatrix}; \tag{55}$$

while the components of the RHS are polynomials in I. Hence, row $(n+1)$ represents the equation

$$\frac{(-1)^n}{(n+1)!}\frac{D}{2}I^{n+1} = P_n(I),$$

where $P_n(I)$ is some polynomial with constant coefficients of order $\leq n$. Consequently, I is constant (as well as all X_i's). Since I is arbitrary, we now have $\text{CSI}_{\text{cons}} \Rightarrow$ CSI and the theorem follows. □

Given a CSI spacetime, g_{ab}, we can compute any polynomial invariant $J^i = j_0^i$ (=constant). If this invariant contains derivatives of the Riemann tensor up to order p, then we note that the corresponding conserved tensor obtained by varying $L_i = (J^i - j_0^i)^{m_i}$, $m_i = p+4$, is vanishing for g_{ab} (corresponding to the last row in eq. (54)). Thus if J^i, $i = 1,...,N$ are (all of the) invariants, then the CSI spacetime will be a solution of the class of theories given by

$$L = \sum_{i=1}^{N} a_i (J^i - j_0^i)^{m_i}, \qquad a_i \text{ arbitrary.} \tag{56}$$

Here, we use the above theorem for the universal case where $T_{ab} = \lambda g_{ab}$. However, we should point out that above result applies to a bigger class of spacetimes; for example, the theorem implies that the more general class of spacetimes for which all conserved symmetric rank-2 tensors T_{ab} are covariantly constant; i.e., $T_{ab;c} = 0$, also belong to the CSI class.

C The Ricci and Bianchi equations and commutators for type III and N Kundt Einstein spacetimes in a parallelly propagated frame.

Throughout the paper, we repeatedly use commutators and a subset of the Ricci and Bianchi equations for type III and N Kundt Einstein spacetimes in a parallelly propagated frame. For convenience, we list these equations here. The original, more general form can be found in [6, 8–10].

Ricci equations [9,10]:

$$
\begin{aligned}
DL_{11} &= -L_{1i}\tau_i + \mathcal{R}, && (57)\\
DL_{1i} &= 0, && (58)\\
\triangle L_{1i} - \delta_i L_{11} &= L_{11}(L_{1i} - \tau_i) - \tau_j N_{ji} - L_{1j}(N_{ji} + \overset{j}{M}_{i1}) + \Psi'_i, && (59)\\
D\tau_i &= 0, && (60)\\
D\kappa'_i &= -\rho'_{ij}\tau_j + \Psi'_i, && (61)\\
-\delta_j \tau_i &= -\tau_i\tau_j + \tau_k \overset{k}{M}_{ij} - \mathcal{R}\delta_{ij}, && (62)\\
D\rho'_{ij} &= -\mathcal{R}\delta_{ij}, && (63)\\
D\overset{i}{M}_{j1} &= -\overset{i}{M}_{jk}\tau_k, && (64)\\
D\overset{i}{M}_{jk} &= 0, && (65)\\
& && (66)
\end{aligned}
$$

where

$$\mathcal{R}\delta_{ij} = \tfrac{1}{n-2}(R_{ij} + R_{01}\delta_{ij}) - \tfrac{1}{(n-1)(n-2)} R\delta_{ij} = \tfrac{R}{n(n-1)}\delta_{ij}$$

for Einstein spaces $R_{ab} = (R/n)g_{ab}$.

Bianchi equations [8,10]:

$$
\begin{aligned}
D\Psi'_i &= 0, && (67)\\
D\Psi'_{ijk} &= 0, && (68)\\
D\Omega'_{ij} &= \delta_j \Psi'_i + \Psi'_i(L_{1j} - \tau_j) + \Psi'_{ijs}\tau_s + \Psi'_s \overset{s}{M}_{ij}. && (69)
\end{aligned}
$$

Commutators [6]:

$$
\begin{aligned}
\triangle D - D\triangle &= L_{11}D + \tau_i\delta_i, && (70)\\
\delta_i D - D\delta_i &= L_{1i}D, && (71)\\
\delta_i \triangle - \triangle \delta_i &= \kappa'_i D + (\tau_i - L_{1i}) \triangle + (\rho'_{ji} - \overset{i}{M}_{j1})\delta_j, && (72)\\
\delta_i\delta_j - \delta_j\delta_i &= (\rho'_{ij} - \rho'_{ji})D + (\overset{k}{M}_{ij} - \overset{k}{M}_{ji})\delta_k. && (73)
\end{aligned}
$$

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