

Various concepts of solutions to models of compressible fluid flows

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Compressible Navier-Stokes/Euler system

Field equations

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta \operatorname{div}_x \mathbf{u} \mathbb{I}, \quad \mu \geq 0, \quad \eta \geq 0$$

No-flux/no-slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad [\mathbf{u}]_{\tan}|_{\partial\Omega} = 0$$

Thermodynamics stability

Pressure potential

$$P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} dz$$

Pressure-density state equation

$$p \in C[0, \infty) \cap C^2(0, \infty), \quad p(0) = 0$$

$$\boxed{p'(\varrho) > 0} \text{ for } \varrho > 0, \quad \liminf_{\varrho \rightarrow \infty} p'(\varrho) > 0$$

$$\liminf_{\varrho \rightarrow \infty} \frac{P(\varrho)}{p(\varrho)} > 0$$

Isentropic pressure-density state equation

$$p(\varrho) = a\varrho^\gamma, \quad a > 0, \quad \gamma \geq 1$$

Energy balance - conservation

Energy

$$E = \underbrace{\frac{1}{2}\varrho|\mathbf{u}|^2}_{\text{kinetic energy}} + \underbrace{P(\varrho)}_{\text{elastic energy}}, \quad P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} dz$$

Energy balance equation

$$\partial_t E + \operatorname{div}_x(E\mathbf{u}) + \operatorname{div}_x(p(\varrho)\mathbf{u}) - \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u}) = -\boxed{\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}}$$

Total energy balance

$$\frac{d}{dt} \int_{\Omega} \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) dx + \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} dx \boxed{\leq} 0$$

Classical (strong) solutions

Local existence

Smooth solutions exist on a maximal time interval $(0, T_{\max})$. This is true for both Navier-Stokes and Euler system

Global-in-time solutions for small data

Smooth solutions of the *Navier-Stokes system* exist globally in time provided the initial data are close to an equilibrium solution (**Matsumura and Nishida, Valli and Zajaczkowski, and others**). Solutions of the *Euler system* develop singularities in a finite time no matter how smooth and/or small the initial data are.

Global existence for the 1-D Navier-Stokes system

The Navier-Stokes system in the 1-D geometry admits global-in-time smooth solutions (**Kazhikhov and others**)

Weak solutions

Equation of continuity

$$\left[\int_{\Omega} \varrho \varphi \, dx \right]_{t=0}^T = \int_0^T \int_{\Omega} (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi) \, dx dt$$

for any $\varphi \in C_c^\infty([0, T] \times \bar{\Omega})$

Balance of momentum

$$\begin{aligned} & \left[\int_{\Omega} \varrho \mathbf{u} \cdot \varphi \, dx \right]_{t=0}^T \\ &= \int_0^T \int_{\Omega} (\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho) \operatorname{div}_x \varphi) \, dx dt \\ & \quad - \int_0^T \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \, dx dt \end{aligned}$$

for any $\varphi \in C_c^\infty([0, T] \times \bar{\Omega}; \mathbb{R}^N)$,

$\varphi|_{\partial\Omega} = 0$ for the no-slip condition in the viscous case

Dissipative weak solutions

Energy (entropy) inequality

$$\left[\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) dx \right]_{t=0}^{t=\tau} + \int_0^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} dx dt \leq 0$$

for a.a. $\tau \in (0, T)$

Navier-Stokes system: Weak solutions

Global existence for large data

$$p(\varrho) \approx a\varrho^\gamma, \quad \mu > 0$$

The Navier-Stokes system admits global-in-time weak solutions if:

- $N = 2, \gamma \geq 3/2; N = 3, \gamma \geq 9/5$ **P.L.Lions 1998**
- $N = 2, \gamma > 1, N = 3, \gamma > 3/2$ **EF et al. 2000**
- $N = 2, \gamma \geq 1, N = 3, \gamma \geq 3/2$ **Plotnikov and Vaigant 2014**

Dissipative weak solutions

The weak solutions are not known to be unique. The construction used in the existence theory yields *dissipative* weak solutions. Weak solutions can be obtained as a limit of certain numerical schemes (**Karper**)

Euler system: Weak solutions

Global existence for large data in 1D

The Euler system admits global-in-time weak solutions for any bounded initial data (**DiPerna, Chen et al.**). The weak solutions can be recovered as a vanishing viscosity limit of the Navier-Stokes system (**Chen and Perepelitsa**)

Global existence for large data for $N = 2, 3$

The compressible Euler system admits *infinitely many* global-in-time weak solutions for any smooth initial data (**Chiodaroli, EF** - based on the work of **DeLellis and Székelyhidi**)

Euler system: Dissipative weak solutions

Dissipative weak solutions $N = 2, 3$

- For any ϱ_0 , there exists \mathbf{u}_0 (bounded measurable) such that the Euler system admits infinitely many dissipative weak solutions in a given time interval $(0, T)$ (**Chiodaroli, EF**)
- There is a vast class of initial data for which the Euler system admits infinitely many entropy (dissipative) weak solutions in a given time interval $(0, T)$ (**Chiodaroli, EF**)
- There exist Lipschitz (smooth) initial data for which the Euler system admits infinitely many entropy (dissipative) weak solutions in a given time interval $(0, T)$ (**Chiodaroli, DeLellis, Kreml**)

Relative entropy (energy)

Relative energy functional

$$\begin{aligned} & \mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \\ &= \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) \right) dx \end{aligned}$$

Decomposition

$$\begin{aligned} & \mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \\ &= \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) dx - \int_{\Omega} \varrho \mathbf{u} \cdot \mathbf{U} dx + \int_{\Omega} \frac{1}{2} \varrho |\mathbf{U}|^2 dx \\ & \quad - \int_{\Omega} P'(r) \varrho dx + \int_{\Omega} p(r) dx \end{aligned}$$

Dissipation inequality

Relative energy inequality

$$\begin{aligned} & \left[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \right]_{t=0}^{t=\tau} \\ & + \int_0^\tau \int_\Omega (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, dx \, dt \\ & \leq \int_0^\tau \mathcal{R}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \, dt \end{aligned}$$

Test functions

$$r > 0, \quad \mathbf{U}|_{\partial\Omega} = 0 \quad (\text{or other relevant b.c.})$$

Remainder

$$\begin{aligned} & \int_0^\tau \mathcal{R}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \, dt \\ & \int_\Omega \varrho (\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) \, dx \\ & + \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, dx + \int_\Omega (\rho(r) - \rho(\varrho)) \operatorname{div}_x \mathbf{U} \, dx \\ & + \int_\Omega [(r - \varrho) \partial_t P'(r) + \nabla_x P'(r) \cdot (r \mathbf{U} - \varrho \mathbf{u})] \, dx \end{aligned}$$

Applications

Weak-strong uniqueness

Weak and strong solutions of the compressible Navier-Stokes/Euler system emanating from the same initial data coincide as long as the latter exists (**EF, Jin, Novotný, Sun [2014]**)

Conditional regularity

Weak solution to the Navier-Stokes system with bounded density component emanating from smooth initial data are smooth (**EF, Jin, Novotný, Sun [2014]**)

Singular limits

Rotating fluids

$$\begin{aligned}\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) &= 0 \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon} \varrho \mathbf{b} \times \mathbf{u} + \frac{1}{\varepsilon^{2M}} \nabla_x p(\varrho) \\ &= \varepsilon^R \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^{2F}} \nabla_x G\end{aligned}$$

Path dependent singular limit

$\varepsilon \rightarrow 0$, certain relation between $M, R, F > 0$

- low Mach \Rightarrow compressible \rightarrow incompressible
- high Rossby \Rightarrow 3D \rightarrow 2D
- high Reynolds \Rightarrow viscous \rightarrow inviscid

Convergence to singular limit system

Target problem - Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \operatorname{div}_x \mathbf{v} = 0, x \in R^2$$

Convergence results (EF, Lu, Novotný 2014)

■ Spatial geometry - infinite strip:

$$\Omega = R^2 \times (0, \pi)$$

■ Complete slip (Navier) boundary conditions:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0$$

Limits on domains with variable geometry

Channel like domains

$$\Omega_\varepsilon = \left\{ (\mathbf{x}, z) \mid z \in (0, 1), |\mathbf{x} - \varepsilon \mathbf{X}(z)|^2 < \varepsilon^2 R^2(z) \right\}, |\mathbf{X}(z)| < R(z)$$

Boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_\Sigma = 0, (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_\Sigma = 0$$

$$\Sigma = \partial\Omega \cap \{z \in (0, 1)\}$$

$$\mathbf{u}|_{z=0,1} = 0$$

Target systems

Inviscid limit

$$\begin{aligned}\partial_t(\rho_E A) + \partial_z(\rho_E u_E A) &= 0 \\ \partial_t(\rho_E u_E A) + \partial_z(\rho_E u_E^2 A) + A \partial_z p(\rho_E) &= 0\end{aligned}$$

Viscous limit

$$\begin{aligned}\partial_t(\rho_{NS} A) + \partial_z(\rho_{NS} u_{NS} A) &= 0 \\ \partial_t(\rho_{NS} u_{NS} A) + \partial_z(\rho_{NS} u_{NS}^2 A) + A \partial_z p(\rho_{NS}) \\ = A \nu \partial_z^2 u_{NS} + \nu \partial_z (R'(z)/R(z) u_{NS}), \quad \nu &= \frac{4}{3} \mu + \eta > 0\end{aligned}$$

$$A = R^2$$

Convergence

Korn-Poincaré inequality

$$\int_{\Omega_\varepsilon} |\mathbf{v}|^2 \, dx \leq c_{KP} \int_{\Omega_\varepsilon} |\nabla_x \mathbf{v} + \nabla_x^t \mathbf{v}|^2 \, dx$$

Convergence (Bella, EF, Lewicka, Novotný 2015)

- Convergence to the target Euler system with geometric terms in the inviscid limit
- Convergence to the Navier-Stokes system in the viscous limit provided the bulk viscosity in the primitive system is positive

Navier-Stokes system driven by stochastic forces

Navier–Stokes system with stochastic forcing

$$d\rho + \operatorname{div}_x(\rho \mathbf{u}) dt = 0$$

$$d(\rho \mathbf{u}) + [\operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho)] dt = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) dt + \mathbb{G}(\rho, \rho \mathbf{u}) dW,$$

White–noise forcing

$$\mathbb{G}(\rho, \rho \mathbf{u}) dW = \sum_{k \geq 1} \mathbf{G}_k(\rho, \rho \mathbf{u}) dW_k.$$

Relative energy inequality

Relative energy inequality - (Breit, EF, Hofmanová 2015)

$$\begin{aligned} & - \int_0^T \partial_t \psi \mathcal{E}(\varrho, \mathbf{u} | r, \mathbf{U}) \, dt \\ & + \int_0^T \psi \int_{\Omega} (\mathbb{S}(\nabla \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, dx \, dt \\ & \leq \psi(0) \mathcal{E}(\varrho, \mathbf{u} | r, \mathbf{U})(0) + \int_0^T \psi \, dM_{RE} + \int_0^T \psi \mathcal{R}(\varrho, \mathbf{u} | r, \mathbf{U}) \, dt \end{aligned}$$

$$\psi \in C_c^\infty[0, T] \text{ (deterministic), } \psi \geq 0.$$

Test functions

$$d r = D_t^d r \, dt + D_t^s r \, dW, \quad d \mathbf{U} = D_t^d \mathbf{U} \, dt + D_t^s \mathbf{U} \, dW$$

Stochastic remainder

Remainder

$$\begin{aligned} & \mathcal{R}(\varrho, \mathbf{u} | r, \mathbf{U}) \\ &= \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, dx + \int_{\Omega} \varrho \left(D_t^d \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) (\mathbf{U} - \mathbf{u}) \, dx \\ & \quad + \int_{\Omega} \left((r - \varrho) H''(r) D_t^d r + \nabla_x H'(r) (r \mathbf{U} - \varrho \mathbf{u}) \right) \, dx \\ & - \int_{\Omega} \operatorname{div}_x \mathbf{U} (p(\varrho) - p(r)) \, dx + \frac{1}{2} \sum_{k \geq 1} \int_{\Omega} \varrho \left| \frac{\mathbf{G}_k(\varrho, \varrho \mathbf{u})}{\varrho} - D_t^s \mathbf{U}_k \right|^2 \, dx \\ & \quad + \frac{1}{2} \sum_{k \geq 1} \int_{\Omega} \varrho H'''(r) |D_t^s r_k|^2 \, dx + \frac{1}{2} \sum_{k \geq 1} \int_{\Omega} p''(r) |D_t^s r_k|^2 \, dx \end{aligned}$$

Results for stochastic Navier-Stokes system

Weak-strong uniqueness (Breit, EF, Hofmanová 2015)

- Pathwise weak-strong uniqueness
- Weak-strong uniqueness in law

Inviscid-incompressible limit in the stochastic setting (Breit, EF, Hofmanová 2015)

Convergence to the limit stochastic Euler system for vanishing viscosity and the Mach number. Results for well-prepared data.

Possible extensions

Numerical analysis (Gallouet, Herbin, Maltese, Novotný 2014)

Relative energy inequality for the numerical scheme proposed by K.Karlsen and T. Karper. Error estimates.

Measure-valued solutions

Weak-strong uniqueness for measure-valued solutions (**EF, Gwiazda, Swierczewska-Gwiazda, Wiedemann 2015**)

Preliminaries to measure-valued solutions

Families of integrable solutions

$$[\varrho_n, \mathbf{u}_n] : \underbrace{(0, T) \times \Omega}_{\text{physical space}} \mapsto \underbrace{[0, \infty) \times \mathbb{R}^N}_{\text{phase space}}$$

$$\varrho_n \rightarrow \varrho, \mathbf{u}_n \rightarrow \mathbf{u} \text{ weakly in } L^1((0, T) \times \Omega)$$

Nonlinear compositions - Young measure

$$F(\varrho_n, \mathbf{u}_n) \rightarrow \overline{F(\varrho, \mathbf{u})} \text{ weakly in } L^1((0, T) \times \Omega)$$

\Rightarrow

$$\overline{F(\varrho, \mathbf{u})} = \langle \nu_{t,x}; F(s, \mathbf{v}) \rangle \text{ for a.a. } (t, x)$$

Biting limit

$$\int_0^T \int_{\Omega} |F(\varrho_n, \mathbf{u}_n)| \, dx \, dt \leq c \Rightarrow \langle \nu_{t,x}; F(s, \mathbf{v}) \rangle \in L^1((0, T) \times \Omega)$$

Biting limit decomposition

Bounded integrable compositions

$$\int_0^\tau \int_\Omega |F(\varrho_n, \mathbf{u}_n)| \, dx \, dt \leq c$$



up to a subsequence

$$F(\varrho_n, \mathbf{u}_n) \rightarrow \overline{F(\varrho, \mathbf{u})} \text{ weakly-}^* \text{ in } \mathcal{M}([0, T] \times \overline{\Omega})$$

Biting limit decomposition

$$\overline{F(\varrho, \mathbf{u})} = \underbrace{\overline{F(\varrho, \mathbf{u})} - \langle \nu_{t,x}; F(s, \mathbf{v}) \rangle}_{\text{concentration part}} + \underbrace{\langle \nu_{t,x}; F(s, \mathbf{v}) \rangle}_{\text{oscillatory part}}$$

Measure-valued solutions

Parameterized (Young) measure

$$\nu_{t,x} \in L_{\text{weak}}^{\infty}((0, T) \times \Omega; \mathcal{P}([0, \infty) \times \mathbb{R}^N), [s, \mathbf{v}] \in [0, \infty) \times \mathbb{R}^N)$$

$$\varrho(t, x) = \langle \nu_{t,x}; s \rangle, \quad \mathbf{u} = \langle \nu_{t,x}; \mathbf{v} \rangle$$

Navier-Stokes/Euler, velocity/momentum

$$\text{Navier-Stokes } \mathbf{u} \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^N)),$$

$$\text{Euler } \mathbf{u} \approx \mathbf{m} \approx \varrho \mathbf{u}$$

Initial data

$$\nu_0 = \nu_{0,x}$$

Regular initial data

$$\nu_{0,x} = \delta_{\varrho_0(x), \mathbf{u}_0(x)} \text{ for a.a. } x$$

Field equations

Equation of continuity

$$\left[\int_{\Omega} \langle \nu_{t,x}, \mathbf{s} \rangle \varphi \, dx \right]_{t=0}^{t=\tau}$$
$$= \int_0^{\tau} \int_{\Omega} \langle \nu_{t,x}; \mathbf{s} \rangle \partial_t \varphi + \langle \nu_{t,x}; \mathbf{sv} \rangle \cdot \nabla_x \varphi \, dx \, dt + \langle R_1; \nabla_x \varphi \rangle$$

Momentum balance

$$\left[\int_{\Omega} \langle \nu_{t,x}, \mathbf{sv} \rangle \varphi \, dx \right]_{t=0}^{t=\tau}$$
$$= \int_0^{\tau} \int_{\Omega} \langle \nu_{t,x}; \mathbf{sv} \rangle \cdot \partial_t \varphi + \langle \nu_{t,x}; \mathbf{sv} \otimes \mathbf{v} \rangle : \nabla_x \varphi + \langle \nu_{t,x}; \rho(\mathbf{s}) \rangle \operatorname{div}_x \varphi \, dx \, dt$$
$$- \int_0^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \, dx \, dt + \langle R_2; \nabla_x \varphi \rangle$$

Dissipativity

Energy inequality

$$\left[\int_{\Omega} \left\langle \nu_{\tau,x}; \left(\frac{1}{2} s |\mathbf{v}|^2 + P(s) \right) \right\rangle dx \right]_{t=0}^{t=\tau} + \int_0^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, dx \, dt + \boxed{\mathcal{D}(\tau)} \leq 0$$

Compatibility

$$|R_1[0, \tau] \times \bar{\Omega}| + |R_2[0, \tau] \times \bar{\Omega}| \leq \xi(\tau) \mathcal{D}(\tau), \quad \xi \in L^1(0, T)$$

$$\int_0^{\tau} \int_{\Omega} \langle \nu_{t,x}; |\mathbf{v} - \mathbf{u}|^2 \rangle \, dx \, dt \leq c_P \mathcal{D}(\tau)$$

Truly measure-valued solutions

Truly measure-valued solutions for the Euler system (EF, Chiodaroli, Kreml, Wiedemann)

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded L^p weak solutions to the Euler system.

Do we need measure valued solutions?

Limits of problems with higher order viscosities

Multipolar fluids with complex rheologies (Nečas - Šilhavý)

$$\begin{aligned} & \mathbb{T}(\mathbf{u}, \nabla_x \mathbf{u}, \nabla_x^2 \mathbf{u}, \dots) \\ &= \mathbb{S}(\nabla_x \mathbf{u}) + \delta \sum_{j=1}^{k-1} ((-1)^j \mu_j \Delta^j (\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) + \lambda_j \Delta^j \operatorname{div}_x \mathbf{u} \mathbb{I}) \\ & \quad + \text{non-linear terms} \end{aligned}$$

Limit for $\delta \rightarrow 0$

Limits of numerical solutions

Numerical solutions resulting from Karlsen-Karper and other schemes

Sub-critical parameters

$$p(\varrho) = a\varrho^\gamma, \quad \gamma < \gamma_{\text{critical}}$$

Weak (mv) - strong uniqueness

Theorem - EF, Gwiazda, Swierczewska-Gwiazda, Wiedemann 2015

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

Relative energy (entropy)

Relative energy functional

$$\begin{aligned} & \mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})(\tau) \\ &= \int_{\Omega} \left\langle \nu_{\tau, x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}|^2 + P(s) - P'(r)(s - r) - P(r) \right\rangle dx \\ &= \int_{\Omega} \left\langle \nu_{\tau, x}; \frac{1}{2} s |\mathbf{v}|^2 + P(s) \right\rangle dx - \int_{\Omega} \langle \nu_{\tau, x}; s \mathbf{v} \rangle \cdot \mathbf{U} dx \\ & \quad + \int_{\Omega} \frac{1}{2} \langle \nu_{\tau, x}; s \rangle |\mathbf{U}|^2 dx \\ & \quad - \int_{\Omega} \langle \nu_{\tau, x}; s \rangle P'(r) dx + \int_{\Omega} p(r) dx \end{aligned}$$

Relative energy (entropy) inequality

Relative energy inequality

$$\begin{aligned} & \mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U}) + \int_0^\tau \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, dx \, dt + \mathcal{D}(\tau) \\ & \leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}_0|^2 + P(s) - P'(r_0)(s - r_0) - P(r_0) \right\rangle dx \\ & \quad + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \, dt \end{aligned}$$

Remainder

$$\begin{aligned}
 & \mathcal{R}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \\
 &= - \int_0^T \int_{\Omega} \langle \nu_{t,x}, \mathbf{sv} \rangle \cdot \partial_t \mathbf{U} \, dx \, dt \\
 & - \int_0^T \int_{\Omega} [\langle \nu_{t,x}; \mathbf{sv} \otimes \mathbf{v} \rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(s) \rangle \operatorname{div}_x \mathbf{U}] \, dx \, dt \\
 & + \int_0^T \int_{\Omega} [\langle \nu_{t,x}; s \rangle \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; \mathbf{sv} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U}] \, dx \, dt \\
 & + \int_0^T \int_{\Omega} \left[\left\langle \nu_{t,x}; \left(1 - \frac{s}{r}\right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; \mathbf{sv} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \, dx \, dt \\
 & + \int_0^T \left\langle R_1; \frac{1}{2} \nabla_x (|\mathbf{U}|^2 - P'(r)) \right\rangle \, dt - \int_0^T \langle R_2; \nabla_x \mathbf{U} \rangle \, dt
 \end{aligned}$$

Regularity

Theorem - EF, Gwiazda, Swierczewska-Gwiazda, Wiedemann 2015

Suppose that the initial data are smooth and satisfy the relevant compatibility conditions. Let $\nu_{t,x}$ be a measure-valued solution to the compressible Navier-Stokes system with a dissipation defect \mathcal{D} such that

$$\text{supp } \nu_{t,x} \subset \left\{ (s, \mathbf{v}) \mid 0 \leq s \leq \bar{\varrho}, \mathbf{v} \in R^N \right\}$$

for a.a. $(t, x) \in (0, T) \times \Omega$.

Then $\mathcal{D} = 0$ and

$$\nu_{t,x} = \delta_{\varrho(t,x), \mathbf{u}(t,x)}$$

where ϱ, \mathbf{u} is a smooth solution.

Sketch of the proof

- The Navier-Stokes system admits a local-in-time smooth solution
- The measure-valued solution coincides with the smooth solution on its life-span
- The smooth solution density component remains bounded by $\bar{\rho}$ as long as the solution exists
- Y. Sun, C. Wang, and Z. Zhang [2011]: The strong solution can be extended as long as the density component remains bounded

Corollary

Convergence of numerical solutions

Bounded numerical solutions emanating from smooth data that converge to a measure-valued solution converge, in fact, unconditionally to the unique strong solution