ON THE BOOLEAN FUNCTION GRAPH OF A GRAPH AND ON ITS COMPLEMENT

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(Received April 16, 2004)

Abstract. For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G respectively. The Boolean function graph $B(G, L(G), \mathrm{NINC})$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(G, L(G), \mathrm{NINC})$ are adjacent if and only if they correspond to two adjacent vertices of G, two adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_1(G)$. In this paper, structural properties of $B_1(G)$ and its complement including traversability and eccentricity properties are studied. In addition, solutions for Boolean function graphs that are total graphs, quasi-total graphs and middle graphs are obtained.

Keywords: eccentricity, self-centered graph, middle graph, Boolean function graph

MSC 2000: 05C15

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. *Eccentricity* of a vertex $u \in V(G)$ is defined as $e_G(u) = \max\{d_G(u, v)\colon v \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of a vertex v in G as $e(v)$ and use $d(u, v)$ to denote the distance between two vertices u, v in G respectively. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted $r(G)$ and $\mathrm{diam}(G)$ respectively. When $\mathrm{diam}(G) = r(G)$, G is called a *self-centered* graph with radius r, equivalently G is r-self-centered. A vertex u is said to be an eccentric point of v in a graph G, if $d(u, v) = e(v)$. In general, u is called an *eccentric point*, if it is an eccentric point of some vertex. We also denote the i^{th} neighborhood of v as $N_i(v) = \{u \in V(G)\colon\, d_G(u,v) = i\}$ and denote the cardinality of a set H as $|H|$. If $|N_{e(v)}(v)|$ is m, for each point $v \in V(G)$, then G is called an *m-eccentric point graph.* If $m = 2$, we call the graph G as *bi-eccentric point graph.*

The *closure* of a graph G, denoted $\mathrm{cl}(G)$ is defined to be that super graph of G obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least p until no such pair exists, where p is the number of vertices in G. A vertex and an edge are said to *cover* each other if they are incident. A set of vertices which covers all the edges of a graph G is called a *point cover* for G, while a set of edges which covers all the vertices is a *line cover.* The smallest number of vertices in any point cover for G is called its *point covering number* and is denoted by $\alpha_0(G)$ or α_0. Similarly, $\alpha_1(G)$ or α_1 is the smallest number of edges in any line cover of G and is called its *line covering number.* A set of vertices in G is *independent* if no two of them are adjacent. The largest number of vertices in such a set is called the *point independence number* of G and is denoted by $\beta_0(G)$ or β_0. Analogously, an independent set of edges of G has no two of its edges adjacent and the maximum cardinality of such a set is the *line independence number* $\beta_1(G)$ or β_1.

When a new concept is developed in graph theory, it is often first applied to particular classes of graphs. Afterwards more general graphs are studied. As for every graph (undirected, uniformly weighted) there exists an adjacency $(0,1)$ matrix, we call the general operation a Boolean operation. Boolean operation on a given graph uses the adjacency relation between two vertices or two edges and incidence relationship between vertices and edges and defines new structure from the given graph. This extracts information from the original graph and encodes it into a new structure. If it is possible to decode the given graph from the encoded graph in polynomial time, such operation may be used to analyze various structural properties of the original graph in terms of the Boolean graph. If it is not possible to decode the original graph in polynomial time, then that operation can be used in graph coding or coding of certain grouped signals.

Whitney [9] introduced the concept of the line graph $L(G)$ of a given graph G in 1932. The first characterization of line graphs is due to Krausz. The Middle graph $M(G)$ of a graph G was introduced by Hamada and Yoshimura [5]. Chikkodimath and Sampathkumar [4] also studied it independently and they called it the semi-total graph $T_1(G)$ of a graph G. Characterizations were presented for middle graphs of any graph, trees and complete graphs in [1]. The concept of total graphs was introduced by Behzad [2] in 1966. Sastry and Raju [8] introduced the concept of quasi-total graphs and they solved the graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. This motivates us to define and study other graph operations. Using $L(G)$, G, incidence and non-incidence, complementary operations, complete and totally disconnected structures, one can get thirty-two graph operations. As total graphs, semi-total edge graphs, semi-total vertex graphs and quasi-total graphs and their complements (8 graphs) have already been defined-and-studied, we study all other similar remaining graph operations. This is illustrated below

$$\boxed{G/\overline{G}/K_p/\overline{K}_p} \text{———} \boxed{\text{incident (INC), not incident (NINC)}} \text{———} \boxed{L(G)/\overline{L}(G)/K_q/\overline{K}_q}.$$

Here, $\overline{G}$ and $L(G)$ denote the complement and the line graph of G respectively. K_p is the complete graph on p vertices.

The points and lines of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. The *total graph* $T(G)$ of G has vertex set $V(G) \cup E(G)$ and vertices of $T(G)$ are adjacent whenever they are neighbors in G. The *quasi-total graph* [8] $P(G)$ of G is a graph with vertex set as that of $T(G)$ and two vertices are adjacent if and only if they correspond to two nonadjacent vertices of G or to two adjacent edges of G or to a vertex and an edge incident to it in G. The *middle graph* $M(G)$ of G is the one whose vertex set is as that of $T(G)$ and two vertices are adjacent in $M(G)$ whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge of G incident with it. Clearly, $E(M(G)) = E(T(G)) - E(G)$.

The *Boolean function graph* $B(G, L(G), \text{NINC})$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(G, L(G), \text{NINC})$ are adjacent if and only if they correspond to two adjacent vertices of G, two adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_1(G)$. The terms $G, L(G), \text{NINC}$ in the bracket represent the adjacent vertices of G, adjacent edges of G and vertex and edge not incident in G respectively. In other words, $V(B_1(G)) = V(G) \cup V(L(G))$; and $E(B_1(G)) = [E(T(\overline{G})) - (E(\overline{G}) \cup E(\overline{L}(G)))] \cup E(G) \cup E(L(G))$, where $\overline{G}, L(G)$ and $T(G)$ denote the complement, the line graph and the total graph of G respectively. The vertices of G and $L(G)$ in $B_1(G)$ are referred to as point and line vertices respectively and the line vertex in $B_1(G)$ corresponding to an edge e in G is denoted by e'.

In this paper, we study structural properties of $B_1(G)$ and its complement including traversability and eccentricity properties. Also, we obtain solutions for Boolean function graphs that are total graphs, quasi-total graphs and middle graphs. The definitions and details not furnished in this paper may be found in [6].

2. Prior results

In this section, we list some results, with indicated references, which will be used in the subsequent main results.

Theorem 2.1 [3]. *If G is Hamiltonian, then for every proper subset S of $V(G)$, $\omega(G-S) \leqslant |S|$, where $\omega(G)$ is the number of components of G.*

Theorem 2.2 [3]. *If the closure* $\mathrm{cl}(G)$ *is Hamiltonian, then G is Hamiltonian.*

Corollary 2.3 [3]. *If* $\mathrm{cl}(G)$ *is complete, then G is Hamiltonian.*

Theorem 2.4 [6]. *For any nontrivial connected graph G, $\alpha_0+\beta_0=p=\alpha_1+\beta_1$, where p is the number of vertices in G.*

3. Main results

The following elementary properties of the Boolean function graph $B_1(G)$ of a graph G are immediate. Let G be a (p,q) graph.

Observation. 3.1. *G and $L(G)$ are induced subgraphs of $B_1(G)$.*

3.2. *The number of vertices in $B_1(G)$ is $p+q$ and if $d_i=\deg_G(v_i)$, $v_i\in V(G)$, then the number of edges in $B_1(G)$ is $q(p-2)+\frac{1}{2}\sum\limits_{1\leqslant i\leqslant p} d_i^2$.*

3.3. *The degree of a point vertex in $B_1(G)$ is q and the degree of a line vertex e' in $B_1(G)$ is $\deg_{L(G)}(e')+p-2$. Also if $d^*(e')$ is the degree of a line vertex e' in $B_1(G)$, then $0\leqslant d^*(e')\leqslant p+q-3$. The lower bound is attained if $G\cong K_2$, and the upper bound is attained if $G\cong K_{l,n}$, for $n\geqslant 2$.*

3.4. *$B_1(G)$ is biregular with degree sequence q, $m+p-2$ if and only if $L(G)$ is m-regular, where $m\neq q-p+2$.*

3.5. *$B_1(G)$ is q-regular if and only if $L(G)$ is $q-p+2$ regular. For example, $B_1(C_n)$ is n-regular on $2n$ vertices, for $n\geqslant 3$.*

In the following, we find the graphs G for which $B_1(G)$ is disconnected.

Theorem 3.1. *$B_1(G)$ is disconnected if and only if G is one of the following graphs: nK_1, K_2, $2K_2$ and $K_2 \cup nK_1$, for $n \geqslant 1$.*

Proof. Let G be a (p, q) graph. Assume $B_1(G)$ is disconnected. G is totally disconnected if and only if $B_1(G)$ is totally disconnected.

Case (i): $q = 1$ and $p \geqslant 2$. Ten $G \cong K_2 \cup K_1$, $n \geqslant 0$.

Case (ii): $q = 2$. Then $G \cong 2K_2 \cup nK_1$ or $P_3 \cup mK_1$, for $m, n \geqslant 0$ and $B_1(G)$ is connected except when $G \cong 2K_2$.

Case (iii): $q \geqslant 3$. Then $p \geqslant 3$ and by the construction of $B_1(G)$, each line vertex is adjacent to $p - 2$ point vertices. Since G and $L(G)$ are induced subgraphs of $B_1(G)$, $B_1(G)$ is connected.

From (i), (ii) and (iii), it follows that $B_1(G)$ is disconnected if $G \cong K_2$, $K_2 \cup nK_1$, $2K_2$ and nK_1, for $n \geqslant 1$. The converse is obvious. □

Remark 3.1. $B_1(G)$ contains isolated vertices if and only if G is either K_2 or nK_1, for $n \geqslant 1$.

In the following, we prove that the removal of any single vertex from $B_1(G)$ does not disconnect it, for any connected graph G.

Theorem 3.2. *For any connected graph G with at least three vertices, no vertex of $B_1(G)$ is a cut-vertex.*

Proof. Since G and $L(G)$ are induced subgraphs of $B_1(G)$, the only cut-vertices of $B_1(G)$ may be the cut-vertices of G or $L(G)$. Thus, it is sufficient to show that a cut-vertex of a connected graph G or $L(G)$ is not a cut-vertex of $B_1(G)$.

(a) Let v be a cut-vertex of G. Then $G - v$ is disconnected. Let $G_1, G_2, \ldots, G_t$, $(t \geqslant 2)$ be the components of $G - v$. Then $B_1(G_j)$ is connected, for $1 \leqslant j \leqslant t$.

Case (i): Each component G_j $(1 \leqslant j \leqslant t)$ contains at least one edge. Since v is a cut-vertex of G, v is adjacent to at least one of the vertices in each of the components G_j of $G - v$. Let $v_j \in G_j$ and $e_j = (v, v_j) \in E(G)$, $j = 1, 2, \ldots, t$. Since G_j contains at least one edge, the line vertex e'_j in $B_l(G) - v$ corresponding to the edge e_j is adjacent to at least one point vertex in G_j $(1 \leqslant j \leqslant t)$ and the subgraph of $B_1(G) - v$ induced by the vertices $e'_1, e'_2, \ldots, e'_t$ is complete in $B_1(G) - v$, since $L(G)$ is an induced subgraph of $B_1(G)$. Hence, $B_1(G) - v$ is connected and, thus, v is not a cut-vertex of $B_1(G)$.

Case (ii): At least one (not all) of the components G_j of $G - v$ is K_1. Let $v_j \in G_j$ and $v_k \in V(K_1)$ $(i \neq j)$ with $e_j = (v, v_j)$, $e_k = (v, v_k) \in E(G)$. Then the line vertices e'_j and e'_k in $B_1(G) - v$ are mutually adjacent and are adjacent to v_k and v_j, respectively, and hence $B_1(G) - v$ is connected. Thus, v is not a cut-vertex of $B_1(G)$.

Case (iii): $G - v \cong mK_1$, for $m \geqslant 2$. Then v in G is adjacent to each of the vertices in mK_1 and the line vertices in $B_1(G) - v$ corresponding to these edges are adjacent to each other and each line vertex is adjacent to $m-1$ vertices in mK_1. Hence, $B_1(G) - v$ is connected. Thus, v is not a cut-vertex of $B_1(G)$.

(b) Let e' be a cut-vertex of $L(G)$. Since each line vertex in $B_1(G) - v$ is adjacent to $p-2$ point vertices, $B_1(G) - e'$ is connected and hence e' is not a cut-vertex of $B_1(G)$. This completes the proof. □

Corollary 3.2.1. *Let G be any graph such that $B_1(G)$ is disconnected. Then $B_1(G)$ contains a cut-vertex if $G \cong K_2 \cup nK_1$, for $n \geqslant 2$.*

Theorem 3.3. *For any (p, q) graph G with $p \geqslant 3$, the girth of $B_1(G)$ is either 3 or 5.*

Proof. If G contains triangles, it is obvious that the girth of $B_1(G)$ is 3, since G is an induced subgraph of $B_1(G)$. Assume G is triangle-free.

Case (i): $\beta_1(G) \geqslant 2$. Then there exist at least two independent edges in G, say $e_{12} = (v_1, v_2)$ and $e_{34} = (v_3, v_4)$, where $v_1, v_2, v_3, v_4 \in V(G)$. If e'_{12} and e'_{34} are the line vertices in $B_1(G)$ corresponding to the edges e_{12} and e_{34} respectively, then $v_1, v_2, v_3, v_4, e'_{12}, e'_{34} \in V(B_1(G))$ and the subgraph of $B_1(G)$ induced by the vertices v_1, v_2, e'_{34} is a triangle in $B_1(G)$. Thus, the girth of $B_1(G)$ is 3.

Case (ii): $\beta_1(G) = 1$. Then $G \cong K_{1,n}$, $n \geqslant 2$. If $G \cong K_{1,n}$, $n \geqslant 3$, since $L(G)$ is an induced subgraph of $B_1(G)$, $B_1(G)$ contains C_3 and hence the girth of $B_1(G)$ is 3. If $G \cong K_{1,2}$, then $B_1(G) \cong C_5$, and the girth of $B_1(G)$ is 5.

From (i) and (ii), it follows that the girth of $B_1(G)$ is 3 or 5. □

A connected graph G is said to be *geodetic*, if a unique shortest path joins any two of its vertices. In the following theorem, we find the graph G for which $B_1(G)$ is geodetic.

Theorem 3.4. *Let G be any graph such that $B_1(G)$ is connected. Then $B_1(G)$ is geodetic if and only if $G \cong P_3$; a path on 3 vertices.*

Proof. Since $B_1(G)$ is connected, G cannot be one of the graphs nK_1, K_2, $2K_2$, $K_2 \cup nK_1$, for $n \geqslant 1$ by Theorem 3.1. Assume $B_1(G)$ is geodetic. If G contains triangles, then $B_1(G)$ contains C_4 as an induced subgraph and hence is not geodetic. Assume G is triangle-free. If $\beta_1(G) \geqslant 2$, then $B_1(G)$ contains C_4, $K_4 - e$ or C_6 as an induced subgraph and hence is not geodetic. Let $\beta_1(G) = 1$. If G contains $K_{1,2} \cup K_1$ or $K_{1,3}$ as an induced subgraph, then $B_1(G)$ contains either C_6 with one chord or C_4 as an induced subgraph and hence is not geodetic. Thus, it follows that $G \cong P_3$. Conversely, if $G \cong P_3$, then $B_1(G) \cong C_5$ and hence is geodetic. □

Next, we characterize the graphs G for which $B_1(G)$ contains $K_{1,3}$ as an induced subgraph.

Theorem 3.5. *For any graph G, $B_1(G)$ contains $K_{1,3}$ as an induced subgraph if and only if G contains either $K_{1,3}$ or $K_2 \cup 3K_1$ as an induced subgraph.*

P r o o f. Assume $B_1(G)$ contains $K_{1,3}$ as an induced subgraph.

(i) If all the vertices of $K_{1,3}$ in $B_1(G)$ are point vertices, then G contains $K_{1,3}$ as an induced subgraph.

(ii) If the center vertex of $K_{1,3}$ in $B_1(G)$ is a point vertex (line vertex) and the remaining vertices are line vertices (point vertices), then G contains $3K_2 \cup K_1$ ($K_2 \cup 3K_1$) as an induced subgraph. The remaining cases are not possible.

Hence, G contains $K_{1,3}$ or $K_2 \cup 3K_1$ as an induced subgraph.

The converse is obvious. $\square$

R e m a r k 3.2. Theorem 3.5 reveals that

(i) If G contains $K_{1,3}$ or $K_2 \cup 3K_1$ as an induced graph, then $B_1(G)$ cannot be the line graph of any graph.

(ii) If G is a graph with diameter at least 7 or is triangle-free with at lest one vertex of degree greater than or equal to 3, then $B_1(G)$ is not the line graph of any graph.

Theorem 3.6. *For any connected graph G other than P_4 with $\beta_1(G) \geqslant 2$, every vertex of $B_1(G)$ lies on a triangle.*

P r o o f. Assume $\beta_1(G) \geqslant 2$. Let v be a point vertex in $B_1(G)$. Then $v \in V(G)$. If v lies on a triangle in G, then since G is an induced subgraph of $B_1(G)$, v lies on a triangle in $B_1(G)$. If not, let u be a vertex in G adjacent to v and e be an edge in G not incident with both u and v. If e' is the line vertex in $B_1(G)$ corresponding to the edge e, then v lies on the triangle uve' in $B_1(G)$. Hence, each point vertex lies on a triangle in $B_1(G)$. Let x' be a line vertex in $B_1(G)$ and x be the corresponding edge in G. If x lies on a triangle or in a $K_{1,3}$ in G, then since $L(G)$ is an induced subgraph of $B_1(G)$, x' lies on a triangle in $B_1(G)$. If not, then there exists an edge, say, $x_1 = (u_1, v_1)$, in G not adjacent to x in G, where $u_1, v_1 \in V(G)$, since $\beta_1(G) \geqslant 2$ and $G \neq P_4$. Then u_1v_1x' is a triangle in $B_1(G)$. Hence, each line vertex of $B_1(G)$ lies on a triangle. $\square$

R e m a r k 3.3. If $G \cong P_4$, a path on 4 vertices, then there exists a line vertex in $B_1(G)$ not lying in any triangle.

An edge $e = (u, v)$ is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

Theorem 3.7. *For any graph G, if $B_1(G)$ has a dominating edge, then the end vertices of this edge correspond to a point vertex and a line vertex.*

Proof. Let x be a dominating edge in $B_1(G)$. If $x = (u, v)$, where u, v are both point vertices, then $u, v \in V(G)$ and $x \in E(G)$. Then the line vertex x' in $B_1(G)$ corresponding to the edge x is adjacent neither to u, nor to v in $B_1(G)$. Hence, the end vertices of the dominating edge x cannot be both point vertices. If $x = (e'_1, e'_2)$, where e'_1 and e'_2 are both line vertices in $B_1(G)$, then the respective edges e_1 and e_2 in G are adjacent and the point vertex in $B_1(G)$ corresponding to the common vertex of e_1 and e_2 is not adjacent to any of the end vertices e'_1, e'_2. Hence, the end vertices of the dominating edge x cannot be both line vertices. Thus, the end vertices of the dominating edge in $B_1(G)$ correspond to a point vertex and a line vertex. $\square$

Now, we establish a criterion for $B_1(G)$ to be Eulerian. For simplicity, we use $d_1(u)$, $e_1(v)$ and $d_1(u, v)$ to denote the degree of a vertex u, the eccentricity of a vertex v and the distance between the vertices u and v in $B_1(G)$ respectively.

Theorem 3.8. *Let G be any (p, q) graph such that $B_1(G)$ is connected. Then $B_1(G)$ is Eulerian if and only if one of the following holds*

(i) *p and q are even and G or each component of G is Eulerian;*

(ii) *q is even and each vertex in G is of odd degree; and*

(iii) *q is even, p is odd and the degrees of adjacent vertices of G are of different parities.*

Proof. Assume $B_1(G)$ is Eulerian. Then each vertex in $B_1(G)$ is of even degree. If w is a point vertex in $B_1(G)$, then $d_1(w) = q$ must be even, where $d_1(w)$ is the degree of w in $B_1(G)$. Let e' be a line vertex in $B_1(G)$ and e be the corresponding edge in G. Assume $e = (u, v)$, where $u, v \in V(G)$. Then $d_1(e')$ being even implies that $\deg_G(u) + \deg_G(v) + p - 4$ is even. Then it is clear that one of the following holds

(i) p and q are even and G or each component of G is Eulerian;

(ii) q is even and each vertex of G is of odd degree (hence p is even); and

(iii) q is even, p is odd and the degrees of adjacent vertices of G are of different parities.

Conversely, if G is one of the above graphs, then it is clear that the degree of each vertex in $B_1(G)$ is even and hence $B_1(G)$ is Eulerian. $\square$

In the following, we prove that $B_1(G)$ is Hamiltonian by proving that the closure of $B_1(G)$ is either complete or Hamiltonian.

Theorem 3.9. *If G is a connected (p, q) graph with at least 3 vertices, then $B_1(G)$ is Hamiltonian.*

Proof. If $G \cong P_3$, then $B_1(G) \cong C_5$ and is Hamiltonian. Assume G is not isomorphic to P_3. Since G is a connected graph, $L(G)$ is connected and hence $\deg_{L(G)}(e_i') \geqslant 1$, for all e_i' in $V(L(G))$. The theorem is proved by finding the closure of $B_1(G)$. Let $e = (u, v) \in E(G)$ be such that $\deg_{L(G)}(e') \geqslant 2$. Then $e', u, v \in V(B_1(G))$ and (e', u), $(e', v) \notin E(B_1(G))$. Now, $d_1(e') = \deg_{L(G)}(e') + p - 2 \geqslant p$ and $d_1(u) = d_1(v) = q$. Hence, the sum of the degrees of e', u (or v) in $B_1(G)$ is greater than or equal to $p + q$ and therefore e' can be made adjacent with u and v in $B_1(G)$. Similarly, all the line vertices x' with $\deg_{L(G)}(x') \geqslant 2$ can be joined with their incident point vertices in $B_1(G)$. Let the resulting graph be $B_1'(G)$. In $B_1'(G)$, $\deg(u) \geqslant q + 1$ and $\deg(e') \geqslant p + 2$. Now choose an edge $e_1 = (u_1, v_1)$ in G with $\deg_{L(G)}(e_1') = 1$ such that v_1 is a pendant vertex. Then $u_1, v_1, e_1' \in V(B_1(G))$ and $d_1(e_1') = p - 1$ and the sum of degrees of e_1' and u_1 in $B_1'(G)$ is at least $p + q$, where e_1' is the vertex in $L(G)$ corresponding to the edge e_1. Then e_1' and u_1 can be made adjacent in $B_1'(G)$ and then e_1' and v_1 can be made adjacent in $B_1'(G)$. Hence, all the line vertices in $B_1(G)$ can be joined with their incident point vertices. Let the resulting graph be $B_1''(G)$. In $B_1''(G)$, $\deg(y') = \deg_{L(G)}(y') + p$, for every line vertex y' in $B_1(G)$ and $\deg(u) = \deg_G(u) + q$, for every point vertex u in $B_1(G)$.

Case (i): $p \geqslant q$. Let e_i' and e_j' be two nonadjacent line vertices in $B_1''(G)$. The sum of their degrees is at least $p + q$ and e_i' and e_j' can be made adjacent in $B_1''(G)$. Hence, all the nonadjacent line vertices in $B_1''(G)$ can be joined. Since G is connected, $q \geqslant p - 1$. Let v_i and v_j be two nonadjacent vertices in G and hence in $B_1''(G)$. In $B_1''(G)$, $\deg(v_i) = \deg_G(v_i) + q$ and $\deg(v_j) = \deg_G(v_j) + q$. The sum of degrees of v_i and v_j in $B_1''(G)$ is greater than or equal to $p + q$ and hence all the nonadjacent point vertices in $B_1''(G)$ can be made adjacent and the resulting graph is complete and hence Hamiltonian. That is, the closure of $B_1(G)$ is Hamiltonian. Thus, $B_1(G)$ is Hamiltonian by Corollary 2.3.

Case (ii): $p < q$. Let v_i and v_j be two nonadjacent vertices in G. The sum of degrees of v_i and v_j in $B_1''(G)$ is $\deg_G(v_i) + \deg_G(v_j) + 2q > p + q$. Thus, all the nonadjacent v_i and v_j can be made adjacent in $B_1''(G)$. That is, the subgraph of $B_1''(G)$ induced by all the point vertices is complete. Since G is connected, $L(G)$ is connected. Hence, a path (not necessarily induced) on $q - p + 1$ vertices can be found in $L(G)$. Let the path be $e_p', e_{p+1}', \ldots, e_q'$. Then $v_1 e_1' v_2 e_2' \ldots v_p e_p' e_{p+1}' \ldots e_q' v_1$ is a Hamiltonian cycle in the closure of $B_1(G)$ and this closure is Hamiltonian. Hence, $B_1(G)$ is Hamiltonian by Theorem 2.2. □

Corollary 3.9.1. *Let G be any disconnected graph such that none of the components of G is K_2 or K_1. Then $B_1(G)$ is Hamiltonian.*

Proof. Let $G_1, G_2, \ldots, G_n$, $(n \geqslant 2)$ be the components of G such that $G_i \ncong K_2$ and $G \ncong K_1$, $i = 1, 2, \ldots, n$. By Theorem 3.9, $B_1(G_i)$ $(i = 1, 2, \ldots, n)$ is Hamiltonian. Also each point vertex (line vertex) in $B_1(G_i)$ is adjacent to all the line vertices (point vertices) in the remaining $B_1(G_j)$ $(i \neq j)$. Hence, Hamiltonian cycle in each $B_1(G_i)$ can be joined to form a Hamiltonian cycle of $B_1(G)$. Thus, $B_1(G)$ is Hamiltonian. $\square$

Proposition 3.1. *$B_1(nK_2)$, $n \geqslant 3$, is Hamiltonian.*

Proof. Let $v_1, v_2, \ldots, v_{n-1}, v_{2n}$ be the vertices of nK_2 with $\langle\{v_{2k-1}, v_k\}\rangle \cong K_2$ and $e_{2k-1,2k} = (v_{2k-1}, v_{2k})$, for $1 \leqslant k \leqslant n$. If $e'_{2k-1,2k}$ is the line vertex corresponding to the edge $e_{2k-1,2k}$, for $1 \leqslant k \leqslant n$, then $v_1 v_2 e'_{2n-1,2n} v_3 v_4 e'_{12} v_5 v_6 \ldots v_{2n-1}$ $v_{2n} e'_{2n-3,2n-2} v_1$ is a Hamiltonian cycle in $B_1(nK_2)$ and hence $B_1(nK_2)$ is Hamiltonian, for $n \geqslant 3$. $\square$

Proposition 3.2. *$B_1(G \cup mK_2)$, $m \geqslant 3$, is Hamiltonian, where G is any connected (p, q) graph.*

Proof. Case (i): $p \geqslant q$. Then the closure of $B_1(G)$ is complete by Case (i) of Theorem 3.9. Construct a Hamiltonian path in the closure of $B_1(G)$ with a point vertex as the initial vertex and a line vertex as the terminal vertex. In $B_1(mK_2)$, construct a Hamiltonian path starting with a point vertex and ending with a line vertex. This is possible, since $B_1(mK_2)$ is Hamiltonian by Proposition 3.1. These two Hamiltonian paths can be combined to form a Hamiltonian cycle in the closure of $B_1(G \cup mK_2)$ and hence the closure of $B_1(G \cup mK_2)$ is Hamiltonian.

Case (ii): $p < q$. Then the closure of $B_1(G)$ is Hamiltonian by Case (ii) of Theorem 3.9, and there exists a Hamiltonian path in the closure of $B_1(G)$ starting with a point vertex and ending with a line vertex. As in Case (i), a Hamiltonian cycle can be formed in the closure of $B_1(G \cup mK_2)$.

From Case (i) and Case (ii), it follows that $B_1(G \cup mK_2)$ is Hamiltonian. $\square$

Proposition 3.3. *For any connected (p, q) graph G with $p \geqslant q$, $B_1(G \cup (q - 1)K_1 \cup mK_2)$, $m \geqslant 3$, is Hamiltonian.*

Proof. Since $p \geqslant q$, the closure of $B_1(G)$ is complete. In $B_1(mK_2)$, a Hamiltonian path can be constructed starting with a point vertex and ending with a line vertex as in Proposition 3.1. A Hamiltonian cycle in the closure of $B_1(G \cup (q-1)K_1 \cup mK_2)$ can be formed in the following way. First, place the p point vertices and the q line vertices belonging to the closure of $B_1(G)$ and then place the Hamiltonian path in $B_1(mK_2)$. The point vertices corresponding to the vertices in $(q-1)K_1$ can be placed in between the q line vertices, thereby forming a Hamiltonian cycle in the closure of $B_1(G \cup (q-1)K_1 \cup mK_2)$. Thus, $B_1(G \cup (q-1)K_1 \cup mK_2)$ is Hamiltonian. $\square$

Proposition 3.4. *For any connected (p,q) graph with $q \geqslant p$, $B_1(G \cup (q-p)K_1 \cup mK_2)$, $m \geqslant 3$, is Hamiltonian.*

P r o o f. Since $q \geqslant p$, there exists a Hamiltonian cycle in the closure of $B_1(G)$ by Case (ii) of Theorem 3.9. Let $v_1, v_2, \ldots, v_p$ be the vertices and $e_1, e_2, \ldots, e_q$ be the edges of G. Then there exists a Hamiltonian cycle $v_1 e_1' v_2 e_2' \ldots v_p e_p' e_{p+1}' \ldots e_q' v_1$ in the closure of $B_1(G)$, where e_i' is the line vertex corresponding to the edge e_i, for $i = 1, 2, \ldots, q$. In this cycle, the point vertices in $B_1((q-p)K_1)$ can be placed in between the line vertices e_i' and e_{i+1}', $i = p,\ p+1, \ldots, q-1$. In $B_1(mK_2)$, there exists a Hamiltonian path and this can be placed in between e_q' and v_1. Thus, a Hamiltonian cycle can be constructed in the closure of $B_1(G \cup (q-p)K_1 \cup mK_2)$ and $B_1(G \cup (q-p)K_1 \cup mK_2)$ is Hamiltonian. $\square$

Proposition 3.5. $B_1(mK_2 \cup nK_1)$ $(m \geqslant 2, n \geqslant 1)$ *is not Hamiltonian.*

P r o o f. Let S be the set of line vertices in $B_1(G)$ corresponding to the edges in mK_2. Then the number of components in $B_1(G) - S$ is greater than the number of vertices in S and hence $B_1(mK_2 \cup nK_1)$ is not Hamiltonian by Theorem 2.1. $\square$

Next, we find the eccentricity properties of $B_1(G)$ and observe that radius of $B_1(G)$ is at most three. In the following theorem, we prove that all the vertices of $B_1(G)$ have eccentricity 2 if G is a connected graph.

Theorem 3.10. *For any connected graph G with at least three vertices, $B_1(G)$ is self-centered with radius 2.*

P r o o f. Let u, v be two point vertices in $B_1(G)$. Then $u, v \in V(G)$. If $d_G(u,v) \leqslant 2$, then $d_1^*(u,v) \leqslant 2$, since G is an induced subgraph of $B_1(G)$. If $d_G(u,v) \geqslant 3$, then there exists at least one edge e (say) not incident with both u and v and hence $d_1(u,v) = 2$, since $ue'v$ is geodetic in $B_1(G)$, where e' is the line vertex corresponding to e. Thus, the distance between any two point vertices in $B_1(G)$ is at most 2. Let e_1', e_2' be two line vertices in $B_1(G)$ and e_1, e_2 be the corresponding edges in G. If e_1 and e_2 are adjacent edges in G, then $d_1(e_1', e_2') = 1$, since $L(G)$ is an induced subgraph of $B_1(G)$. If e_1 and e_2 are nonadjacent edges in G, then either there exists an edge e in G adjacent to both e_1 and e_2 or there exists a vertex not incident with both e_1 and e_2. Then in both cases $d_1(e_1', e_2') = 2$. Hence, the distance between any two line vertices in $B_1(G)$ is at most 2. Let v, e' be a point and a line vertex in $B_1(G)$ respectively and e be the edge in G corresponding to e'. If $e \in E(G)$ is not incident with $v \in V(G)$, then $d_1(v, e') = 1$. Let $e \in E(G)$ be incident with v and let $e = (u, v)$, where $u \in V(G)$. Since G is connected, there exists an edge incident with either u or v. Therefore, there is a geodesic path of length 2 in $B_1(G)$ connecting v and e'. Hence, $d_1(v, e') = 2$. From the above argument it follows that the eccentricity of a point vertex and a line vertex in $B_1(G)$ are each equal to 2. Hence, $B_1(G)$ is self-centered with radius 2. $\square$

The graphs G for which $B_1(G)$ is self-centered with radius 2 can also be characterized when G is disconnected.

Theorem 3.11. *Let G be a disconnected graph such that $B_1(G)$ is connected. Then $B_1(G)$ is self-centered with radius 2 if and only if one of the following holds*

(i) *None of the components of G is K_2;*

(ii) *$G \ncong K_{1,n} \cup K_{1,m}$, where $n, m \geqslant 2$; and*

(iii) *$G \ncong K_{1,n} \cup mK_1$, where $n \geqslant 2$ and $m \geqslant 1$.*

Proof. Since $B_1(G)$ is connected, G cannot be any of the graphs mK_1 $(m \geqslant 1)$ $K_2 \cup nK_1$ $(n \geqslant 0)$ and $2K_2$. Assume $B_1(G)$ is self-centered with radius 2. Let $G_1, G_2, \ldots, G_t$ $(t \geqslant 2)$ be the components of G such that $G_1 \cong K_2$ (say). Let $v \in V(K_2)$ and let e' be the line vertex in $B_1(G)$ corresponding to the edge in K_2. Then $d_1(v, e') = 3$, which is a contradiction. Hence, none of the components of G is K_2. If $G = K_{1,n} \cup K_{1,m}$ $(n, m \geqslant 2)$, then the distance between the point vertices in $B_1(G)$ corresponding to the center vertices of $K_{1,n}$ and $K_{1,m}$ is 3. Hence, $G \ncong K_{1,n} \cup K_{1,m}$, for $n, m \geqslant 2$. Similarly, if $G = K_{1,n} \cup mK_1$, where $n \geqslant 2$ and $m \geqslant 1$, then the distance between the point vertices in $B_1(G)$ corresponding to the center vertex of $K_{1,n}$ and the vertex in K_1 is 3. Hence, $G \ncong K_{1,n} \cup mK_1$, for $n \geqslant 2$ and $m \geqslant 1$. Conversely, assume none of the components of G is K_2, $G \ncong K_{1,n} \cup K_{1,m}$, for n, $m \geqslant 2$, and $G \ncong K_{1,n} \cup mK_1$, for $n \geqslant 2$ and $m \geqslant 1$. Let u, v be two point vertices in $B_1(G)$. By the assumption, as in Theorem 3.10, $d_1(u, v) \leqslant 2$. If e'_1 and e'_2 are two line vertices in $B_1(G)$, then $d_1(e'_1, e'_2) \leqslant 2$. Let v, e' be a point and a line vertex in $B_1(G)$ respectively and e be the edge in G corresponding to e'. If $e \in E(G)$ is not incident with $v \in V(G)$, then $d_1(v, e') = 1$. Let $e \in E(G)$ be incident with $v \in V(G)$. Since none of the components of G is K_2, at least one of the components of G contains two adjacent edges. Hence, $d_1(v, e') = 2$. From the above argument, it follows that $B_1(G)$ is self-centered with radius 2. $\square$

Corollary 3.11.1. *Let $G \ncong nK_2 \cup mK_1$, for $n \geqslant 2$ and $m \geqslant 0$. Then $B_1(G)$ is bieccentric with radius 2 if and only if one of the components of G is K_2, $G = K_{1,n} \cup K_{1,m}$, for $m, n \geqslant 2$ or $G = K_{1,n} \cup mK_1$, for $n \geqslant 2$ and $m \geqslant 1$.*

Proof. Follows from the proof of Theorem 3.11. $\square$

Proposition 3.6. *If $G \cong nK_2 \cup mK_1$, for $n \geqslant 2$ and $m \geqslant 1$, then $B_1(G)$ is bieccentric with radius* 2.

Proof. The point vertices in $B_1(G)$ corresponding to the isolated vertices in G have eccentricity 2 and the remaining vertices have eccentricity 3. □

Corollary 3.11.2. *$B_1(G)$ is self-centered with radius 3 if and only if $G \cong nK_2$, for $n \geqslant 3$.*

In the following, solutions for Boolean function graphs that are total graphs, quasi-total graphs and middle graphs are obtained.

Theorem 3.12. *Let G be any (p, q) graph wit $q \geqslant 1$. Then $B_1(G) \cong T(G)$, the total graph of G, if and only if G is either C_4 or $2K_2$.*

Proof. $|E(B_1(G))| = |E(G)| + |E(L(G))| + q(p-2)$ and $|E(T(G))| = |E(G)| + |E(L(G))| + 2q$. Hence, $B_1(G) \cong T(G)$ implies $q(p-2) = 2q$. That is, $p = 4$ and G is one of the graphs $K_2 \cup 2K_1, 2K_2, P_4, C_4, P_3 \cup K_1,\ K_4 - e$ and K_4. Among these graphs, $B_1(G)$ is isomorphic to $T(G)$ if G is either C_4 or $2K_2$. The converse is obvious. □

Theorem 3.13. *Let B be any connected (p, q) graph such that $p \geqslant 3$. Then the quasi total graph $P(G)$ and $B_1(G)$ are nonisomorphic.*

Proof. Assume $P(G)$ and $B_1(G)$ are isomorphic. Then the numbers of edges in $P(G)$ and $B_1(G)$ are equal. That is, $|E(\overline{G})| + |E(L(G))| + 2q = |E(G)| + |E(L(G))| + q(p-2)$ which implies

$$p(p-1) = 2q(p-2). \tag{1}$$

Case (i): G is unicyclic. Then $p = q$ and from (1), $p = q = 3$ and hence $G \cong C_3$. But $B_1(C_3)$ is not isomorphic to $P(C_3)$.

Case (ii): G is a tree. Then $q = p - 1$ and from (1), $p = 4$, $q = 3$ and $G \cong P_4$ or $K_{1,3}$. But in both cases, $B_1(G)$ is not isomorphic to $P(G)$.

Case (iii): $q > p$. Then from (1), $p < 3$, which is a contradiction to our assumption that $p \geqslant 3$.

By Case (i), Case (ii) and Case (iii), it follows that $P(G)$ and $B_1(G)$ are nonisomorphic graphs. □

Theorem 3.14. *For any graph G with $q \geqslant 1$, the middle graph $M(G)$ and $B_1(G)$ are nonisomorphic graphs.*

P r o o f. The number of edges in $M(G)$ and $B_1(G)$ are equal if $p = 3$. Hence, G is one of the graphs $K_2 \cup K_1, K_{1,2}$ and C_3. For these graphs, $M(G)$ and $B_1(G)$ are nonisomorphic. $\square$

4. Complement of $B_1(G)$

The *complement of the Boolean function graph* $B_1(G)$ of G is the graph with vertex set $V(G) \cup E(G)$ and two vertices in the complement of $B_1(G)$ are adjacent if and only if they correspond to two nonadjacent vertices of G, two nonadjacent edges of G or to a vertex and an edge incident to it in G. For brevity, this graph is denoted by $\overline{B}_1(G)$.

As in $B_1(G)$, the following elementary properties of the complement $\overline{B}_1(G)$ of the Boolean function graph $B_1(G)$ of a graph G are immediate. Let G be a (p, q) graph.

Observation.

4.1. *$\overline{G}$ and $\overline{L}(G)$ are disjoint induced subgraphs of $\overline{B}_1(G)$.*

4.2. *The degree of a point vertex in $\overline{B}_1(G)$ is $p-1$; and the degree of a line vertex e' in $\overline{B}_1(G)$ is $q + 1 - \deg_{L(G)}(e')$.*

4.3. *$\overline{B}_1(G)$ is a connected graph, for any graph G.*

4.4. *$\overline{B}_1(G)$ is biregular if and only if $L(G)$ is k-regular, where $k \neq q - p + 2$.*

4.5. *$\overline{B}_1(G)$ is $(p-1)$-regular if and only if $L(G)$ is $(q - p + 2)$-regular.*

In the following, we prove that the removal of any single vertex from $\overline{B}_1(G)$ does not disconnect it, for any graph G.

Theorem 4.1. *For any graph G, no vertex of $\overline{B}_1(G)$ is a cut-vertex.*

P r o o f. Let $\overline{B}_1(G)$ have a cut-vertex. This cut-vertex is either a point vertex or a line vertex.

C a s e (i): A point vertex in $\overline{B}_1(G)$ is a cut-vertex. Let the point vertex $v \in V(\overline{B}_1(G))$ be a cut-vertex of $\overline{B}_1(G)$. Then $v \in V(G)$.

S u b c a s e (i): v is a cut-vertex of G. Consider the graph $\langle \overline{B}_1(G) - \{v\} \rangle_G$, the subgraph of $\overline{B}_1(G)$ induced by the vertices of $V(G) - \{v\}$. The line vertices in $\overline{B}_1(G)$ corresponding to the edges in $V(G) - \{v\}$ are adjacent to at least one of the vertices in $\langle \overline{B}_1(G) - \{v\} \rangle_G$ and hence in $\overline{B}_1(G)$, and therefore $\overline{B}_1(G) - \{v\}$ is not disconnected. Hence, v is not a cut-vertex of $\overline{B}_1(G)$.

Subcase (ii): v is not a cut vertex of G. If v is a cut-vertex of $\overline{B}_1(G)$, then $\overline{B}_1(G)-\{v\}$ is disconnected. Let G_1 and G_2 be any two components of $\overline{B}_1(G)-\{v\}$. As there is sharing of edges between G_1 and G_2 in G, the line vertices corresponding to these edges connect these two disconnected components and hence $\overline{B}_1(G)-\{v\}$ is connected. Therefore, v cannot be a cut-vertex of $\overline{B}_1(G)$.

Case (ii): A line vertex in $\overline{B}_1(G)$ is a cut-vertex. Let a line vertex e' be a cut-vertex of $\overline{B}_1(G)$ and let e be the corresponding edge in G.

Subcase (i): e is a cut-edge of G. Then e' is not a cut-vertex of $\overline{B}_1(G)$ and hence the proof of this case is similar to that of subcase (i) of case (i).

Subcase (ii): e is not a cut-edge of G. If $\overline{B}_1(G)-\{e'\}$ is disconnected, then there must be sharing of at least two edges from a component of $\overline{B}_1(G)-\{e'\}$ to others. Hence, $\overline{B}_1(G)-\{e'\}$ is connected and e' is not a cut-vertex of $\overline{B}_1(G)$.

By Case (i) and Case (ii), it follows that $\overline{B}_1(G)$ contains no cut-vertices. □

Theorem 4.2. *For any graph G having at least four vertices, the girth of $\overline{B}_1(G)$ is 3, 4 or 5.*

Proof. Since $\overline{G}$ and $\overline{L}(G)$ are induced subgraphs of $\overline{B}_1(G)$, $\overline{B}_1(G)$ contains triangles if and only if at least one of $\overline{G}$, $\overline{L}(G)$ contains triangles. In this case, the girth of $\overline{B}_1(G)$ is 3. Assume neither $\overline{G}$ nor $\overline{L}(G)$ contains triangles and hence $\beta_0(G)\leqslant 2$ and $\beta_1(G)\leqslant 2$. If $K_2\cup K_1$ is an induced subgraph of G, then $\overline{B}_1(G)$ contains C_4 and this is the smallest cycle. If G contains P_3 as an induced subgraph, then $\overline{B}_1(G)$ contains C_5 and this is the smallest cycle in $\overline{B}_1(G)$. Hence, the girth of $\overline{B}_1(G)$ is 3, 4 or 5. □

Remark 4.1. If $G\cong K_2\cup K_1$, P_3 or C_3, then $\overline{B}_1(G)\cong C_4$, C_5 or C_6 respectively.

In the following, we find the graphs G for which $\overline{B}_1(G)$ is geodetic.

Theorem 4.3. *If $r(\overline{G})\geqslant 2$, then $\overline{B}_1(G)$ is not geodetic.*

Proof. Assume $r(\overline{G})\geqslant 2$. Then there exists a vertex $v\in V(G)$ such that $e(v)=2$ in $\overline{G}$. Let $P(v,u)\colon vwu$ be a path of length 2 in $\overline{G}$. Then v and u are adjacent in $\overline{G}$. Let $e=(v,u)\in E(G)$ and let e' be the corresponding line vertex in $\overline{B}_1(G)$. The subgraph of $\overline{B}_1(G)$ induced by the vertices v,w,u,e' forms C_4 in $\overline{B}_1(G)$ and hence $\overline{B}_1(G)$ is not geodetic. □

Theorem 4.4. *$\overline{B}_1(G)$ is geodetic if and only if G is one of the graphs K_4, $K_{1,n}$, for $n\geqslant 1$.*

Proof. Case (i): $\beta_1(G)\geqslant 2$. If G contains $K_2\cup K_1$, C_4 or $K_4-\{e\}$ as an induced subgraph, then $\overline{B}_1(G)$ contains C_4 as an induced subgraph and hence $\overline{B}_1(G)$

is not geodetic. Similarly, if G contains $P_3 \cup K_2$ as a subgraph, then $\overline{B}_1(G)$ is also not geodetic. Hence, $G \cong K_4$.

Case (ii): $\beta(G) = 1$. Then $G \cong K_{1,n}$, for $n \geqslant 1$, or C_3. If $G \cong C_3$, then $\overline{B}_1(G) \cong C_6$, which is not geodetic. Hence, $G \cong K_{1,n}$, for $n \geqslant 1$. Conversely, if $G \cong K_4$ or $K_{1,n}$, for $n \geqslant 1$, then there exists a unique shortest path between every pair of vertices in $\overline{B}_1(G)$ and hence is geodetic. □

Now we characterize the graphs G for which $\overline{B}_1(G)$ is Eulerian.

Theorem 4.5. *Let G be any (p, q) graph such that p and q are odd. Then $\overline{B}_1(G)$ is Eulerian if and only if the degrees of adjacent vertices of G are of the same parities.*

Proof. Assume $\overline{B}_1(G)$ is Eulerian. The degree of a line vertex e' in $\overline{B}_1(G)$ is equal to $q + 1 - \deg_{L(G)}(e')$. Let $e = (u, v)$ be the edge in G corresponding to the line vertex e' in $\overline{B}_1(G)$, where $u, v \in V(G)$. Then the degree of e' in $\overline{B}_1(G)$ is $(q+3)-(\deg_G(u)+\deg_G(v))$, since $\deg_{L(G)}(e') = (\deg_G(u)+\deg_G(v))-2$. Since the degree of e' in $\overline{B}_1(G)$ is even and p and q are odd, the degrees of adjacent vertices are of the same parities. Conversely, assume p and q are odd and the degrees of adjacent vertices are of the same parities. Then the degrees of all the vertices in $\overline{B}_1(G)$ are even and hence $\overline{B}_1(G)$ is Eulerian. □

Theorem 4.6. *$\overline{B}_1(K_n)$, where $n \geqslant 3$ is odd, is Hamiltonian.*

Proof. $\overline{B}_1(K_n)$ has n point vertices and $\frac{1}{2}n(n-1)$ line vertices. Since n is odd, $\frac{1}{2}n(n-1)$ is divisible by n. Separate the line vertices into n groups, each group containing $\frac{1}{2}(n-1)$ line vertices such that the corresponding edges in K_n are mutually independent. Then form a Hamiltonian cycle in $\overline{B}_1(K_n)$, by placing n point vertices in between each group of line vertices. Hence $\overline{B}_1(K_n)$ is Hamiltonian, where $n \geqslant 3$ is odd. □

This is illustrated by an example.

Example 4.1. Consider K_5. Let v_1, v_2, v_3, v_4, v_5 be the vertices and $e_{ij} = (v_i, v_j)$, $i < j$ and $i, j = 1, 2, \ldots, 5$, be the edges of K_5. Let e'_{ij} be the corresponding line vertices in $\overline{B}_1(K_5)$. Then $v_1, v_2, v_3, v_4, v_5 \in V(\overline{B}_1(K_5))$. Separate the line vertices into five groups $e'_{15}e'_{24}$; $e'_{12}e'_{35}$; $e'_{23}e'_{14}$; $e'_{34}e'_{25}$ and $e'_{45}e'_{13}$, each containing two line vertices. Then the five point vertices are placed in between each group so as to form a Hamiltonian cycle $v_1e'_{15}e'_{24}v_2e'_{12}e'_{35}v_3e'_{23}e'_{14}v_4e'_{34}e'_{25}v_5e'_{45}e'_{13}v_1$ in $\overline{B}_1(K_5)$, and hence $\overline{B}_1(K_5)$ is Hamiltonian.

Remark 4.2. $\overline{B}_1(K_n)$, where $n \geqslant 4$ is even, is not Hamiltonian.

A non-Hamiltonian graph G having the property that for any vertex v of G, $G - v$ is Hamiltonian, is called a hypo-Hamiltonian graph. The Petersen graph is the smallest graph with this property. We now prove that $\overline{B}_1(K_n)$ that $B_1(K_n)$ is hypo-Hamiltonian, for $n \geqslant 4$ and n even.

Theorem 4.7. *$\overline{B}_1(K_n)$, where $n \geqslant 4$ is even, is hypo-Hamiltonian.*

Proof. $\overline{B}_1(K_n)$, where $n \geqslant 4$ is even, is not Hamiltonian. Let v be a point vertex in $\overline{B}_1(K_n)$. $\overline{B}_1(K_n) - \{v\}$ has $n-1$ point vertices and $\frac{1}{2}n(n-1)$ line vertices. As in Theorem 4.6, separate the line vertices into $n-1$ groups, each group containing $\frac{1}{2}n$ vertices, such that the corresponding edges in K_n are independent. Then a Hamiltonian cycle in $\overline{B}_1(K_n) - \{v\}$ can be formed by placing $n-1$ point vertices in between each group of line vertices. Let e' be a line vertex in $\overline{B}_1(K_n)$. Then $\overline{B}_1(K_n) - \{e'\}$ contains n point vertices and $\frac{1}{2}(n-2)(n-3)$ line vertices. Separate the line vertices into $\frac{1}{2}n+1$ groups each containing $\frac{1}{2}n-1$ line vertices, and $\frac{1}{2}n-1$ groups each containing $\frac{1}{2}n$ line vertices, such that the corresponding edges in G are independent.

Then a Hamiltonian cycle in $\overline{B}_1(K_n) - \{e'\}$ can be formed as above. Hence, $\overline{B}_1(K_n) - \{x\}$ is Hamiltonian, for all $x \in \overline{B}_1(K_n)$, and, hence, $\overline{B}_1(K_n)$ is hypo-Hamiltonian, where $n \geqslant 4$ is even. In the following example, it is shown that $\overline{B}_1(K_6)$ is hypo-Hamiltonian.

Example 4.2. Consider K_6. Let v_1, v_2, v_3, v_4, v_5 and v_6 be the vertices and $e_{ij} = (v_i, v_j)$, $i < j$ and $i, j = 1, 2, \ldots, 6$ be the edges of K_6. Let e'_{ij} be the corresponding line vertices in $\overline{B}_1(K_6)$. Then $v_1, v_2, v_3, v_4, v_5, v_6 \in V(\overline{B}_1(K_6))$. Consider $\overline{B}_1(K_6) - v_6$. Separate the line vertices into 5 groups: $e'_{16}e'_{45}e'_{23}$; $e'_{25}e'_{46}e'_{13}$; $e'_{35}e'_{26}e'_{14}$; $e'_{24}e'_{36}e'_{15}$ and $e'_{56}e'_{34}e'_{12}$, each group containing three vertices. Then a Hamiltonian cycle $v_1e'_{16}e'_{45}e'_{23}v_2e'_{25}e'_{46}e'_{13}v_3e'_{35}e'_{26}e'_{14}v_4e'_{24}e'_{36}e'_{15}v_5e'_{56}e'_{34}e'_{12}v_1$ can be formed in $\overline{B}_1(K_6) - v_6$. Consider $\overline{B}_1(K_6) - e'_{12}$. Separate the 14 line vertices into 4 groups, each containing 2 line vertices, and 2 groups, each containing 3 line vertices: $e'_{15}e'_{24}$; $e'_{26}e'_{35}$; $e'_{36}e'_{14}$; $e'_{34}e'_{56}$; $e'_{45}e'_{23}e'_{16}$ and $e'_{46}e'_{25}e'_{13}$. Then $v_1e'_{15}e'_{24}v_2e'_{26}e'_{35}v_3e'_{36}e'_{14}v_4e'_{34}e'_{56}v_5e'_{45}e'_{23}e'_{16}v_6e'_{46}e'_{25}e'_{13}v_1$ is a Hamiltonian cycle in $\overline{B}_1(K_6) - e'_{12}$. Similarly, it can be shown that $\overline{B}_1(K_6) - v$ contains a Hamiltonian cycle, for any v in $\overline{B}_1(K_6)$. Thus, $\overline{B}_1(K_6)$ is hypo-Hamiltonian.

Theorem 4.8. *If both $\overline{G}$ and $\overline{L}(G)$ are Hamiltonian, then $\overline{B}_1(G)$ is Hamiltonian.*

Proof. Let $v_1v_2\ldots v_nv_1$; $e_1e_2\ldots e_me_1$ be Hamiltonian cycles in $\overline{G}$ and $\overline{L}(G)$ respectively. Then $v_1e_1e_2\ldots e_mv_2\ldots v_nv_1$ is Hamiltonian cycle in $\overline{B}_1(G)$, where the edges in G corresponding to the line vertices e_1 and e_m are incident with v_1 and v_2 respectively. $\square$

Example 4.3.
(i) $\overline{B}_1(K_{1,n})$, $n \geqslant 3$, is not Hamiltonian, whereas $\overline{B}_1(K_{1,n}+e)$ is Hamiltonian.
(ii) $\overline{B}_1(P_n)$, $\overline{B}_1(C_n)$, $\overline{B}_1(C_n^+)$, for $n \geqslant 3$, and $\overline{B}_1(W_m)$, for $m \geqslant 4$, are Hamiltonian.

In the following, eccentricity properties of $\overline{B}_1(G)$ are discussed. Here, for simplicity, the distance between the vertices u and v and the eccentricity of a vertex w in $\overline{B}_1(G)$ are denoted by $\bar{d}^*(u,v)$ and $\bar{e}^*(w)$ respectively.

Theorem 4.9. *For any graph G, the diameter of $\overline{B}_1(G)$ is less than or equal to three.*

Proof. (i) Let u, v be any two point vertices in $\overline{B}_1(G)$. Then $u, v \in V(G)$ and

$$\begin{aligned}\bar{d}^*(u,v) &= 2, \text{ if } (u,v) \in E(G); \text{ and}\\ &= 1, \text{ if } (u,v) \notin E(G).\end{aligned}$$

(ii) Let v, e' be a point vertex and a line vertex in $\overline{B}_1(G)$ respectively and e be the edge in G corresponding to e'. Then $v \in V(G)$ and $e \in E(G)$ and $\bar{d}^*(v,e') = 1$, if $e \in E(G)$ is incident with $v \in V(G)$. Let $e \in E(G)$ be not incident with $v \in V(G)$ and let $e = (u,w)$, where $u, w \in V(G)$.
(a) If v is not adjacent to at least one of the vertices u, w in G, then $\bar{d}^*(v,e') = 2$.
(b) If $\langle\{u,v,w\}\rangle \cong C_3$ in G and $\deg_G(v) \geqslant 3$, then also $\bar{d}^*(v,e') = 2$.
(c) If $\langle\{u,v,w\}\rangle \cong C_3$ in G and $\deg_G(v) = 2$, then $\bar{d}^*(v,e') = 3$.
(iii) Let e_1' and e_2' be two line vertices in $\overline{B}_1(G)$ and e_1 and e_2 be the corresponding edges in G. Then

$$\begin{aligned}\bar{d}^*(e_1',e_2') &= 1, \text{ if } e_1 \text{ and } e_2 \text{ are independent edges in } G\\ &= 2, \text{ if } e_1 \text{ and } e_2 \text{ are adjacent edges in } G.\end{aligned}$$

From (i), (ii) and (iii), it follows that the distance between any two vertices in $\overline{B}_1(G)$ is at most three and hence $\operatorname{diam}(\overline{B}_1(G)) \leqslant 3$. □

In the following, we find a necessary and sufficient condition for $\overline{B}_1(G)$ to be self-centered with radius 2.

Theorem 4.10. *Let G be any graph with at least two edges. Then $\overline{B}_1(G)$ is self-centered with radius 2 if and only if either G is triangle-free or each vertex lying on a triangle in G has degree at least three.*

Proof. Let $\overline{B}_1(G)$ be self-centered with radius 2. If the given conditions are not true, then G contains a triangle in which the degree of at least one vertex is 2.

Let v be a vertex in G lying on any C_3 induced by the vertices v, u, w such that $\deg_G(v) = 2$. Let $e = (u, w)$ and e' be the corresponding line vertex in $\overline{B}_1(G)$. Then $\overline{d}^*(v, e') = 3$, which is a contradiction. Conversely, assume that either G is triangle free or each vertex lying on a triangle has degree at least three. Then by Theorem 4.9, the eccentricity of each vertex in $\overline{B}_1(G)$ is 2. Hence, $\overline{B}_1(G)$ is self-centered with radius 2. □

Theorem 4.11. *Let $G \cong nC_3$, $n \geqslant 1$. Then $\overline{B}_1(G)$ is bieccentric with radius 2 and diameter 3 if and only if G contains triangles such that the degree of at least one vertex of the triangle is 2.*

Proof. Let $\overline{B}_1(G)$ be bieccentric with radius 2 and diameter 3. Assume G is triangle-free or G contains triangles such that the degrees of all the vertices of the triangles are at least three. Then by Theorem 4.10, $\overline{B}_1(G)$ is self-centered with radius 2, which is a contradiction. Conversely, assume G is a graph give as in the theorem. Then the point vertices corresponding to the vertices of the triangles having degree 2 in G and the line vertices corresponding to the edges of the triangles not incident with the above vertices in G have eccentricity 3 in $\overline{B}_1(G)$ and the remaining vertices in $\overline{B}_1(G)$ have eccentricity 2. Hence, $\overline{B}_1(G)$ is bieccentric with radius 2 and diameter 3.

Remark 4.3.
(i) $\overline{B}_1(G)$ is self-centered with radius 3 if and only if $G \cong nC_3$, $n \geqslant 1$.
(ii) If v is point vertex in $\overline{B}_1(G)$ with $\bar{e}^*(v) = 3$, then the eccentric point of v is a line vertex in $\overline{B}_1(G)$.

Notation 4.1 [6]. For any real number x, $[x]$ denotes the greatest integer not exceeding x and $\{x\} = -[-x]$, the smallest integer not less than x.

In the following, covering and independence numbers of $\overline{B}_1(G)$ are determined.

Theorem 4.12. *Let G be any (p, q) graph having no isolated vertices. Then $\alpha_0(\overline{B}_1(G)) \leqslant \min\{p + \alpha_0(\overline{L}(G)), q + \alpha_0(\overline{G})\}$.*

Proof. Let A be a minimal point cover for $\overline{L}(G)$. Then the set $V(G) \cup A$ is a point cover for $\overline{B}_1(G)$. Similarly, if B is a minimal point cover for $\overline{G}$, then $V(\overline{L}(G)) \cup B$ is a point cover for $\overline{B}_1(G)$. Hence, $\alpha_0(\overline{B}_1(G)) \leqslant \min\{p + \alpha_0(\overline{L}(G)), q + \alpha_0(\overline{G})\}$. □

Remark 4.4. If $G \cong K_1 + K_2 + 2K_1$, then $\alpha_0(\overline{B}_1(G)) = p + \alpha_0(\overline{L}(G)) = 8$ and if $r(G) = 1$, then $\alpha_0(\overline{B}_1(G)) = p + \alpha_0(\overline{L}(G))$.

Example 4.4.
(i) $\alpha_0(\overline{B}_1(P_n)) = n$, if $n \geqslant 3$.
(ii) $\alpha_0(\overline{B}_1(C_n)) = 3$, if $n = 3$; and $= n+1$, if $n \geqslant 4$.
(iii) $\alpha_0(\overline{B}_1(K_n)) = \frac{1}{2}n(n-1)$, if $n \geqslant 3$.
(iv) $\alpha_0(\overline{B}_1(K_{1,n})) = n+1$, if $n \geqslant 2$.

Theorem 4.13. $\beta_0(\overline{B}_1(G)) \geqslant \max\{\beta_0(\overline{G}), \beta_0(\overline{L}(G))\} = \max\{\omega(G), \omega(L(G))\}$, *where $\omega(G)$ is the clique number of G, the maximum number of mutually adjacent vertices.*

Proof. Using $\alpha_0(\overline{B}_1(G)) + \beta_0(\overline{B}_1(G)) = p + q$, the proposition follows. □

Theorem 4.14. *Let $G \ncong nK_1$. Then* $\alpha_1(\overline{B}_1(G)) \leqslant \min\{\max(p,q), \alpha_1(\overline{G}) + \alpha_1(\overline{L}(G))\}$.

Proof. Since $\overline{G}$ and $\overline{L}(G)$ are induced subgraphs of $\overline{B}_1(G)$, $\alpha_1(\overline{B}_1(G)) \leqslant \alpha_1(\overline{G}) + \alpha_1(\overline{L}(G))$. Let $v_1, v_2, \ldots, v_p$ be the p vertices and $e_{ij} = (u_j, v_j)$, $i \neq j$ be the q edges in G and e'_{ij} be the corresponding line vertices. Then the set of edges of the form (v_i, e'_{ij}) in $\overline{B}_1(G)$ is a line cover for $\overline{B}_1(G)$. Thus, $\alpha_1(\overline{B}_1(G)) \leqslant \min\{\max(p,q), \alpha_1(\overline{G}) + \alpha_1(\overline{L}(G))\}$. □

Remark 4.5.
(i) If $G \cong P_4 \cup K_2$, then $\alpha_1(\overline{B}_1(G)) = \alpha_1(\overline{G}) + \alpha_1(\overline{L}(G))$.
(ii) Using $\alpha_1(\overline{B}(G)) + \beta_1(\overline{B}_1(G)) = p + q$, $\beta_1(\overline{B}_1(G))$ can be determined.

Example 4.5.
(i) $\alpha(\overline{B}_1(P_n)) = n$, if $n \geqslant 3$.
(ii) $\alpha_1(\overline{B}(C_n)) = n$, if $n \geqslant 3$.
(iii) $\alpha_1(\overline{B}_1(K_n)) = \{\frac{1}{4}n(n+1), \}$, if $n \geqslant 3$.
(iv) $\alpha_1(\overline{B}_1(K_{1,n})) = n+1$, if $n \geqslant 3$.

In the following, the chromatic number χ of $\overline{B}_1(G)$ is determined.

Theorem 4.15. *If G is a disconnected graph containing exactly two components such that each is a clique, then $\chi(\overline{B}_1(G)) = 2$.*

Proof. Let G be a disconnected graph with exactly two components G_1 and G_2 such that $\langle G\rangle$ is a clique, for $i = 1, 2$. Then $\overline{G}$ is bipartite with the bipartition $[G_1, G_2]$. That is, $V(\overline{G})$ can be partitioned into two independent sets $V(G_1)$ and $V(G_2)$. Thus, $\overline{G}$ is 2-colorable. Let the colors used for vertices in $V(G_i) \subseteq V(\overline{G})$ be i, $i = 1, 2$. The color the line vertices corresponding to the edges in $\langle G_1\rangle \subseteq V(G)$ by the color 2 and in $\langle G_2\rangle \subseteq V(G)$ by the color 1. Hence, $\chi(\overline{B}_1(G)) = 2$. □

R e m a r k 4.6. If G is disconnected with k components such that each component is a clique, then $\chi(\overline{B}_1(G)) = k$.

Theorem 4.16. *If G is a (p, q) graph, then $\chi(\overline{B}_1(G)) \leqslant \min\{p, q\}$, where $p, q \geqslant 3$.*

P r o o f. Let $v_1, v_2, \ldots, v_p$ be the p vertices of G.

C a s e (i): $p \leqslant q$. Color the point vertices $v_1, v_2, \ldots, v_p$ in $\overline{B}_1(G)$ by the colors $1, 2, \ldots, p$ respectively. Let $e_{ij} = (v_i, v_j) \in E(G)$ and e'_{ij} be the corresponding line vertex. Then color the vertex e'_{ij} by a color other than i and j. This is possible, since $p \leqslant q$. Hence, $\overline{B}_1(G)$ is p-colorable.

C a s e (ii): $p \geqslant q$. Color the q line vertices by q colors. Let e'_{ij} be a line vertex and $e_{ij} = (v_i, v_j)$ be the corresponding edge in G. Then color the vertices v_i, v_j by the same color other than the color given to e'_{ij}. Hence, $\overline{B}_1(G)$ is q colorable. By Case (i) and Case (ii), it follows that $\chi(\overline{B}_1(G)) \leqslant \min\{p, q\}$.

R e m a r k 4.7.

(i) If $G \cong P_3 \cup K_2$, then $\chi(\overline{B}_1(G)) = q$; and

(ii) If $G \cong K_1 + K_2 + K_2$, then $\chi(\overline{B}_1(G)) = p$.

E x a m p l e 4.6.

(i)
$$\begin{aligned} \chi(\overline{B}_1(P_n)) &= 3, \quad \text{if } n = 3, 4; \\ &= \{n/2\}, \ \text{if } n \geqslant 5. \end{aligned}$$

(ii)
$$\begin{aligned} \chi(\overline{B}_1(C_n)) &= 2, \quad \text{if } n = 3; \\ &= 3, \quad \text{if } n = 4; \\ &= \{n/2\}, \ \text{if } n \geqslant 5. \end{aligned}$$

(iii)
$$\begin{aligned} \chi(\overline{B}_1(K_{1,n})) &= 3, \quad \text{if } n = 2; \\ &= n, \quad \text{if } n \geqslant 3. \end{aligned}$$

(iv)
$$\chi(\overline{B}_1(K_n)) = n - 1, \quad \text{if } n \geqslant 3.$$

For $v \in V(G)$, the *neighborhood* $N(v)$ of v is the set of all vertices adjacent to v in G. The set $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v. A subset S of $V(G)$ is a *neighborhood set* (*n-set*) of G if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph of G induced by $N[v]$. The *neighborhood number* $n_0(G)$ of G is the minimum cardinality of an n-set of G, see [7].

Theorem 4.17. *Let $G \ncong nK_1$. Then $n_0(\overline{B}_1(G)) \leqslant \min\{p+n_0(\overline{L}(G)), q+n_0(\overline{G})\}$.*

Proof. $V(G)$ together with an n-set of $\overline{L}(G)$ or $V(L(G))$ together with an n-set of $\overline{G}$ is an n-set of $\overline{B}_1(G)$. Hence, the proposition follows. □

Remark 4.8. If $G \cong P_3 \cup 3K_1$ or $K_2 \cup K_1$, then $n_0(\overline{B}_1(G)) = q + n_0(\overline{G})$.

Example 4.7.
(i) $n_0(\overline{B}_1(P_n)) = n$, if $n \geqslant 3$.
(ii) $n_0(\overline{B}_1(C_n)) = n+1$, if $n \geqslant 4$.
(iii) $n_0(\overline{B}_1(K_n)) = \frac{1}{2}n(n-1)$, if $n \geqslant 3$.
(iv) $n_0(\overline{B}_1(K_{1,n})) = n+1$, if $n \geqslant 3$.

Acknowledgement. Thanks are due to the referee for the modified version of this paper.

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