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MORE GENERAL CREDIBILITY MODELS

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Abstract. This communication gives some extensions of the original Bühlmann model. The paper is devoted to semi-linear credibility, where one examines functions of the random variables representing claim amounts, rather than the claim amounts themselves. The main purpose of semi-linear credibility theory is the estimation of $\mu_0(\theta) = E[f_0(X_{t+1})|\theta]$ (the net premium for a contract with risk parameter θ) by a linear combination of given functions of the observable variables: $\underline{X}' = (X_1, X_2, \ldots, X_t)$. So the estimators mainly considered here are linear combinations of several functions $f_1, f_2, \ldots, f_n$ of the observable random variables. The approximation to $\mu_0(\theta)$ based on prescribed approximating functions $f_1, f_2, \ldots, f_n$ leads to the optimal non-homogeneous linearized estimator for the semi-linear credibility model. Also we discuss the case when taking $f_p = f$ for all p to find the optimal function f. It should be noted that the approximation to $\mu_0(\theta)$ based on a unique optimal approximating function f is always better than the one in the semi-linear credibility model based on prescribed approximating functions: $f_1, f_2, \ldots, f_n$. The usefulness of the latter approximation is that it is easy to apply, since it is sufficient to know estimates for the structure parameters appearing in the credibility factors. Therefore we give some unbiased estimators for the structure parameters. For this purpose we embed the contract in a collective of contracts, all providing independent information on the structure distribution. We close this paper by giving the semi-linear hierarchical model used in the applications chapter.

Keywords: contracts, unbiased estimators, structure parameters, approximating functions, semi-linear credibility theory, unique optimal function, parameter estimation, hierarchical semi-linear credibility theory

MSC 2000: 62P05

Introduction

In this paper we first give the semi-linear credibility model (see Section 1), which involves only one isolated contract. Our problem (from Section 1) is the estimation of $\mu_0(\theta) = E[f_0(X_{t+1})|\theta]$ (the net premium for a contract with risk parameter θ) by a linear combination of given functions $f_1, f_2, \ldots, f_n$ of the observable variables

$\underline{X}' = (X_1, X_2, \ldots, X_t)$. So our problem (from Section 1) is the determination of the linear combination of 1 and the random variables $f_p(X_r)$, $p = \overline{1,n}$, $r = \overline{1,t}$ closest to $\mu_0(\theta) = E[f_0(X_{t+1})|\theta]$ in the MSE sense, where θ is the risk parameter. The solution of this problem

$$\mathbb{E}\left\{\left[\mu_0(\theta) - \alpha_0 - \sum_{p=1}^{n}\sum_{r=1}^{t}\alpha_{pr} f_p(X_r)\right]^2\right\}, \qquad \text{where: } \alpha = (\alpha_{pr})_{p,r},$$

is the optimal non-homogeneous linearized estimator (i.e. the semi-linear credibility result). In Section 2 we discuss the case when taking $f_p = f$ for all p we are to find the unique optimal function f. It should be noted that the approximation of $\mu_0(\theta)$ based on a unique optimal approximating function f is always better than the one in the semi-linear credibility model based on prescribed approximating functions $f_1, f_2, \ldots, f_n$. The usefulness of the latter approximation is that it is easy to apply, since it is sufficient to know estimates for the structure parameters a_{pq}, b_{pq} (with $p, q = \overline{0,n}$) appearing in the credibility factors z_p (where $p = \overline{1,n}$). To obtain estimates for these structure parameters from the semi-linear credibility model, in Section 3 we embed the contract in a collective of contracts, all providing independent information on the structure distribution. We close this paper by giving the semi-linear hierarchical model used in the applications chapter (see Section 4).

1. The approximation to $\mu_0(\theta)$ based on prescribed approximating functions $f_1, f_2, \ldots, f_n$

In this section we consider one contract with an unknown and fixed risk parameter θ during a period of t years. The yearly claim amounts are denoted by $X_1, \ldots, X_t$. The risk parameter θ is supposed to be drawn from some structure distribution function $U(\cdot)$. It is assumed that for a given θ, the claims are conditionally independent and identically distributed (conditionally i.i.d.) with a known common distribution function $F_{X|\theta}(x, \theta)$. The random variables $X_1, \ldots, X_t$ are observable, and the random variable X_{t+1} is considered as not (yet) observable. We assume that $f_p(X_r)$, $p = \overline{0,n}$, $r = \overline{1,t+1}$ have finite variance. For f_0, we take the function of X_{t+1} we want to forecast.

We use the notation

$$\mu_p(\theta) = E[f_p(X_r)|\theta], \quad (p = \overline{0,n};\ r = \overline{1,t+1}). \tag{1.1}$$

This expression does not depend on r.

We define the following structure parameters:

$$m_p = E[\mu_p(\theta)] = E\{E[f_p(X_r)|\theta]\} = E[f_p(X_r)], \tag{1.2}$$

$$a_{pq} = E\{\mathrm{Cov}[f_p(X_r), f_q(X_r)|\theta]\}, \tag{1.3}$$

$$b_{pq} = \mathrm{Cov}[\mu_p(\theta), \mu_q(\theta)], \tag{1.4}$$

$$c_{pq} = \mathrm{Cov}[f_p(X_r), f_q(X_r)], \tag{1.5}$$

$$d_{pq} = \mathrm{Cov}[f_p(X_r), \mu_q(\theta)] \tag{1.6}$$

for $p, q = \overline{0, n} \wedge r = \overline{1, t+1}$. These expressions do not depend on $r = \overline{1, t+1}$. The structure parameters are connected by the relations

$$c_{pq} = a_{pq} + b_{pq}, \tag{1.7}$$

$$d_{pq} = b_{pq} \tag{1.8}$$

for $p, q = \overline{0, n}$. This follows from the covariance relations obtained in the probability theory where they are very well-known. Just as in the case of linear combinations of the observable variables themselves, we can also obtain non-homogeneous credibility estimates, taking as estimators the class of linear combinations of given functions of the observable variables, as shown in the following theorem.

Theorem 1.1 (Optimal non-homogeneous linearized estimators). *The linear combination of 1 and the random variables $f_p(X_r)$, $p = \overline{1, n}$; $r = \overline{1, t}$ closest to $\mu_0(\theta) = E[f_0(X_{t+1})|\theta]$ and to $f_0(X_{t+1})$ in the least squares sense equals*

$$M = \sum_{p=1}^{n} z_p \sum_{r=1}^{t} \frac{1}{t} f_p(X_r) + m_0 - \sum_{p=1}^{n} z_p m_p, \tag{1.9}$$

where the credibility factors $z_1, z_2, \ldots, z_n$ are a solution to the linear system of equations

$$\sum_{p=1}^{n} [c_{pq} + (t-1)d_{pq}] z_p = t d_{0q} \quad (q = \overline{1, n}), \tag{1.10}$$

or to the equivalent linear system of equations

$$\sum_{p=1}^{n} (a_{pq} + t b_{pq}) z_p = t b_{0q} \quad (q = \overline{1, n}). \tag{1.11}$$

Proof. We have to examine the solution of the problem

$$\mathbb{E}\left\{\left[\mu_0(\theta) - \alpha_0 - \sum_{p=1}^{n} \sum_{r=1}^{t} \alpha_{pr} f_p(X_r)\right]^2\right\}. \tag{1.12}$$

Taking the derivative with respect to α_0 gives

$$E[\mu_0(\theta)] - \sum_{p=1}^{n}\sum_{r=1}^{t} \alpha_{pr} E[f_p(X_r)] = \alpha_0, \text{ or } \alpha_0 = m_0 - \sum_{p=1}^{n}\sum_{r=1}^{t} \alpha_{pr} m_p.$$

Inserting this expression for α_0 into (1.12) leads to the problem

$$\text{(1.13)} \qquad \operatorname*{Min}_{\alpha} E\left\{\left[\mu_0(\theta) - m_0 - \sum_{p=1}^{n}\sum_{r=1}^{t} \alpha_{pr}(f_p(X_r) - m_p)\right]^2\right\}.$$

If we put the derivatives with respect to $\alpha_{qr'}$ equal to zero, we get the following system of equations ($q = \overline{1,n}$; $r' = \overline{1,t}$):

$$\text{(1.14)} \qquad \operatorname{Cov}[\mu_0(\theta), f_q(X_{r'})] = \sum_{p=1}^{n}\sum_{r=1}^{t} \alpha_{pr} \operatorname{Cov}[f_p(X_r), f_q(X_{r'})].$$

Because of the identical distribution in time $\alpha_{p1} = \alpha_{p2} = \ldots = \alpha_{pt} = \alpha_p$, so using the covariance results for $q = \overline{1,n}$ this system of equations can be written as

$$\text{(1.15)} \qquad b_{0q} = \sum_{p=1}^{n} \alpha_p[c_{pq} + (t-1)d_{pq}].$$

Now (1.15) and (1.13) lead to (1.9) with $\alpha_p = z_p/t$, $p = \overline{1,n}$.

2. The approximation of $\mu_0(\theta)$ based on a unique optimal approximating function f

The estimator M for $\mu_0(\theta)$ of Theorem 1.1 can be represented as

$$\text{(2.1)} \qquad M = f(X_1) + \ldots + f(X_t),$$

where

$$f(x) = \frac{1}{t}\sum_{p=1}^{n} z_p f_p(x) + \frac{1}{t} m_0 - \frac{1}{t}\sum_{p=1}^{n} z_p m_p.$$

Let us forget now this structure of f and look for any function f such that (2.1) is closest to $\mu_0(\theta)$. If only functions f such that $f(X_1)$ has finite variance are considered, then the optimal approximating function f results from the following theorem.

Theorem 2.1 (Optimal approximating function). *$f(X_1) + \ldots + f(X_t)$ is closest to $\mu_0(\theta)$ and to $f_0(X_{t+1})$ in the least squares sense, if and only if f is a solution of the equation*

$$f(X_1) + (t-1)E[f(X_2)|X_1] - E[f_0(X_2)|X_1] = 0 \tag{2.2}$$

Proof. We have to solve the minimization problem

$$\operatorname*{Min}_g E\left\{[f_0(X_{t+1}) - g(X_1) - \ldots - g(X_t)]^2\right\}. \tag{2.3}$$

Supposing that f denotes the solution to this problem, we consider $g(X) = f(X) + \alpha h(X)$, with $h(\cdot)$ arbitrary, like in variational calculus. Let

$$\varphi(\alpha) = E\{[f_0(X_{t+1}) - f(X_1) - \ldots - f(X_t) - \alpha h(X_1) - \ldots - \alpha h(X_t)]^2\}. \tag{2.4}$$

Clearly, for f to be optimal we have $\varphi'(0) = 0$, so for every choice of h the identity

$$E\{[f_0(X_{t+1}) - f(X_1) - \ldots - f(X_t)][h(X_1) + \ldots + h(X_t)]\} = 0 \tag{2.5}$$

must hold. This can be rewritten as

$$E[tf_0(X_2)h(X_1) - tf(X_1)h(X_1) - t(t-1)f(X_2)h(X_1)] = 0, \tag{2.6}$$

or

$$E[h(X_1)\{-f(X_1) - (t-1)E[f(X_2)|X_1] + E[f_0(X_2)|X_1]\}] = 0. \tag{2.7}$$

Because this equation has to be satisfied for every choice of the function h the expression in brackets in (2.7) must be identically equal to zero, which proves (2.2).

An application of Theorem 2.1. If $X_1, \ldots, X_{t+1}$ can only take the values $0, 1, \ldots, n$ and $p_{qr} = P[X_1 = q, X_2 = r]$ for $q, r = \overline{0, n}$, then $f(X_1) + \ldots + f(X_t)$ is closest to $\mu_0(\theta)$ and to $f_0(X_{t+1})$ in the least squares sense, if and only if for $q = \overline{0, n}$, $f(q)$ is a solution of the linear system

$$f(q)\sum_{r=0}^{n} p_{qr} + (t-1)\sum_{r=0}^{n} f(r)p_{qr} = \sum_{r=0}^{n} f_0(r)p_{qr}. \tag{2.8}$$

Indeed,

$$f(X_1):\begin{pmatrix} f(q) \\ P(X_1=q)\end{pmatrix}=\begin{pmatrix} f(q) \\ \sum\limits_{r=0}^{n} p_{qr}\end{pmatrix},\quad q=\overline{0,n};$$

$$E[f(X_2)|X_1]=\sum_{r=0}^{n} f(r)P(X_2=r|X_1=q)=\sum_{r=0}^{n} f(r)\frac{p_{qr}}{\sum\limits_{r=0}^{n} p_{qr}};$$

$$E[f_0(X_2)|X_1]=\sum_{r=0}^{n} f_0(r)P(X_2=r|X_1=q)=\sum_{r=0}^{n} f_0(r)\frac{p_{qr}}{\sum\limits_{r=0}^{n} p_{qr}}.$$

Inserting these expressions for $f(X_1)$, $E[f(X_2)|X_1]$ and $E[f_0(X_2)|X_1]$ into (2.2) leads to (2.8).

3. Parameter estimation

It should be noted that the approximation of $\mu_0(\theta)$ based on a unique optimal approximating function f is always better than the one obtained in Section 1 based on prescribed approximating functions $f_1, f_2, \dots, f_n$. The usefulness of the latter approximation is that it is easy to apply, since it is sufficient to know estimates for the structure parameters a_{pq}, b_{pq} (with $p,q=\overline{0,n}$) appearing in the credibility factors z_p (where $p=\overline{1,n}$). For this reason we give some unbiased estimators for the structure parameters. For this purpose we consider k contracts, $j=\overline{1,k}$, and k ($\geqslant 2$) independent and identically distributed vectors $(\theta_j, \underline{X}'_j)=(\theta_j, X_{j1}, \dots, X_{jt})$, $j=\overline{1,k}$. The contract indexed j is a random vector consisting of a random structure parameter θ_j and observations $X_{j1},\dots,X_{jt}$, where $j=\overline{1,k}$. For every contract $j=\overline{1,k}$ and for θ_j fixed, the variables $X_{j1},\dots,X_{jt}$ are conditionally independent and identically distributed.

Theorem 3.1 (Unbiased estimators for the structure parameters)**.** *Let*

$$\hat{m}_p=\frac{1}{kt}X^p_{..}=\frac{1}{kt}\sum_{j=1}^{k}\sum_{r=1}^{t} f_p(X_{jr}), \tag{3.1}$$

$$\hat{a}_{pq}=\frac{1}{k(t-1)}\sum_{j=1}^{k}\sum_{r=1}^{t}\Big(X^p_{jr}-\frac{1}{t}X^p_{j.}\Big)\Big(X^q_{jr}-\frac{1}{t}X^q_{j.}\Big), \tag{3.2}$$

$$\hat{b}_{pq}=\frac{1}{k-1}\sum_{j=1}^{k}\Big(\frac{1}{t}X^p_{j.}-\frac{1}{kt}X^p_{..}\Big)\Big(\frac{1}{t}X^q_{j.}-\frac{1}{kt}X^q_{..}\Big)-\frac{\hat{a}_{pq}}{t}, \tag{3.3}$$

then $E(\hat{m}_p) = m_p$, $E(\hat{a}_{pq}) = a_{pq}$, $E(\hat{b}_{pq}) = b_{pq}$, *where* $X^p_{j.} = \sum\limits_{r=1}^{t} X^p_{jr}$, $X^q_{j.} = \sum\limits_{r=1}^{t} X^q_{jr}$, $X^p_{..} = \sum\limits_{j=1}^{k}\sum\limits_{r=1}^{t} X^p_{jr}$, $X^q_{..} = \sum\limits_{j=1}^{k}\sum\limits_{r=1}^{t} X^q_{jr}$ *with* $X^p_{jr} = f_p(X_{jr})$ *(*$j = \overline{1,k}$ *and* $r = \overline{1,t}$*),* $X^q_{jr} = f_q(X_{jr})$ *(*$j = \overline{1,k}$ *and* $r = \overline{1,t}$*) for* $p,q = \overline{0,n}$ *such that* $p < q$*.*

P r o o f. Note that the usual definitions of the structure parameters apply, with θ_j replacing θ and X_{jr} replacing X_r as follows:

$$
\begin{aligned}
&E(\hat{m}_p) = \frac{1}{kt}\sum_{j,r} E[f_p(X_{jr})] = \frac{1}{kt}\sum_{j,r} m_p = \frac{kt}{kt}m_p = m_p;\\
&E(\hat{a}_{pq}) = \frac{1}{k(t-1)}\sum_{j,r}\Big[\operatorname{Cov}(X^p_{jr}, X^q_{jr}) + E(X^p_{jr})E(X^q_{jr}) - \operatorname{Cov}\Big(X^p_{jr}, \frac{1}{t}X^q_{j.}\Big)\\
&- E(X^p_{jr})E\Big(\frac{1}{t}X^q_{j.}\Big) - \operatorname{Cov}\Big(\frac{1}{t}X^p_{j.}, X^q_{jr}\Big) - E\Big(\frac{1}{t}X^p_{j.}\Big)E(X^q_{jr}) + \operatorname{Cov}\Big(\frac{1}{t}X^p_{j.}, \frac{1}{t}X^q_{j.}\Big)\\
&+ E\Big(\frac{1}{t}X^p_{j.}\Big)E\Big(\frac{1}{t}X^q_{j.}\Big)\Big] = \frac{1}{k(t-1)}\cdot\sum_{j,r}\Big[(a_{pq}+b_{pq}) + m_p m_q - \Big(\frac{1}{t}a_{pq}+b_{pq}\Big)\\
&- m_p m_q - \Big(\frac{1}{t}a_{pq}+b_{pq}\Big) - m_p m_q + \Big(\frac{1}{t}a_{pq}+b_{pq}\Big) + m_p m_q\Big]\\
&= \frac{1}{k(t-1)}\sum_{j,r}(a_{pq}+b_{pq} - \frac{1}{t}a_{pq} - b_{pq}) = \frac{1}{k(t-1)}kt\frac{(t-1)}{t}a_{pq} = a_{pq};\\
&E(\hat{b}_{pq}) = \frac{1}{k-1}\sum_{j}\Big[\operatorname{Cov}(\frac{1}{t}X^p_{j.}, \frac{1}{t}X^q_{j.}) + E(\frac{1}{t}X^p_{j.})E(\frac{1}{t}X^q_{j.}) - \operatorname{Cov}\Big(\frac{1}{t}X^p_{j.}, \frac{1}{kt}X^q_{..}\Big)\\
&- E(\frac{1}{t}X^p_{j.})E\Big(\frac{1}{kt}X^q_{..}\Big) - \operatorname{Cov}\Big(\frac{1}{kt}X^p_{..}, \frac{1}{t}X^q_{j.}\Big) - E\Big(\frac{1}{kt}X^p_{..}\Big)E\Big(\frac{1}{t}X^q_{j.}\Big)\\
&+ \operatorname{Cov}\Big(\frac{1}{kt}X^p_{..}, \frac{1}{kt}X^q_{..}\Big) + E\Big(\frac{1}{kt}X^p_{..}\Big)E\Big(\frac{1}{kt}X^q_{..}\Big)\Big] - \frac{a_{pq}}{t}\\
&= \frac{1}{k-1}\cdot\sum_{j}\Big[\Big(\frac{1}{t}a_{pq}+b_{pq}\Big) + m_p m_q - \Big(\frac{1}{kt}a_{pq} + \frac{1}{k}b_{pq}\Big)\\
&- m_p m_q - \Big(\frac{1}{kt}a_{pq} + \frac{1}{k}b_{pq}\Big) - m_p m_q + \Big(\frac{1}{kt}a_{pq} + \frac{1}{k}b_{pq}\Big) + m_p m_q\Big] - \frac{a_{pq}}{t}\\
&= \frac{1}{k-1}\sum_{j}\Big(\frac{1}{t}a_{pq} + b_{pq} - \frac{1}{kt}a_{pq} - \frac{1}{k}b_{pq}\Big) - \frac{a_{pq}}{t}\\
&= \frac{1}{k-1}k\frac{k-1}{k}b_{pq} + \frac{1}{k-1}k\frac{k-1}{kt}a_{pq} - \frac{a_{pq}}{t} = b_{pq} + \frac{a_{pq}}{t} - \frac{a_{pq}}{t} = b_{pq}.
\end{aligned}
$$

We close this paper by giving the *semi-linear hierarchical model* used in the applications chapter. Similarly to Jewell's hierarchical model we consider a portfolio of contracts which can be broken up into P sectors, each sector p consisting of k_p groups of contracts. Instead of estimating $X_{p,j,t+1}, \mu(\theta_p, \theta_{p_j}) = E[X_{p,j,t+1}|\theta_p, \theta_{p_j}]$ (the pure net risk premium of the contract (p, j)), $\nu(\theta_p) = E[X_{p,j,t+1}|\theta_p]$ (the pure net risk premium of the sector p), we now estimate $f_0(X_{p,j,t+1})$, $\mu_0(\theta_p, \theta_{p_j}) = E[f_0(X_{p,j,t+1})|\theta_p, \theta_{p_j}]$ (the pure net risk premium of the contract (p, j)), $\nu_0(\theta_p) = E[f_0(X_{p,j,t+1})|\theta_p]$ (the pure net risk premium of the sector p), where $p = \overline{1, P}$ and $j = \overline{1, k_p}$. In the semi-linear credibility theory the following class of estimators is considered: $\alpha_0 + \sum\limits_{p=1}^{n} \sum\limits_{q=1}^{P} \sum\limits_{i=1}^{k_q} \sum\limits_{r=1}^{t} \alpha_{pqir} f_p(X_{qir})$, where $f_1(\cdot), \ldots, f_n(\cdot)$ are functions given in advance. Let us consider the case of one given function f_1 in order to approximate $f_0(X_{p,j,t+1})$ or $\nu_0(\theta_p)$ and $\mu_0(\theta_p, \theta_{p_j})$. We formulate the following theorem:

Theorem 4.1 (Hierarchical semi-linear credibility)**.** *Using the same notation as introduced for the hierarchical model of Jewell and denoting $X^0_{pjs} = f_0(X_{pjs})$ and $X^1_{pjs} = f_1(X_{pjs})$ one obtains the following least squares estimates for the pure net risk premiums:*

$$\hat{\nu}_0(\theta_p) = (m_0 - z_p m_1) + z_p X^1_{pzw}, \;\; \hat{\mu}_0(\theta_p, \theta_{pj}) = (m_0 - z_{pj} m_1) + z_{pj} X^1_{pjw} \tag{3.1}$$

where

$$X^1_{pjw} = \sum_{r=1}^{t} \frac{w_{pjr}}{w_{pj.}} X^1_{pjr}, \quad X^1_{pzw} = \sum_{j=1}^{k_p} \frac{z_{pj}}{z_{p.}} X^1_{pjw},$$

$$z_{pj} = \frac{w_{pj.} d_{01}}{c_{11} + (w_{pj.} - 1) d_{11}}$$

(the credibility factor on contract level) with $d_{01} = \mathrm{Cov}(X^0_{pjr}, X^1_{pjr'})$, $d_{11} = \mathrm{Cov}(X^1_{pjr}, X^1_{pjr'})$, $r \neq r'$, $c_{11} = \mathrm{Cov}(X^1_{pjr}, X^1_{pjr}) = \mathrm{Var}(X^1_{pjr})$ *and*

$$z_p = \frac{z_{p.} D_{01}}{C_{11} + (z_{p.} - 1) D_{11}}$$

(the credibility factor at sector level) with $D_{01} = \mathrm{Cov}(X^0_{pjw}, X^1_{pj'w})$, $D_{11} = \mathrm{Cov}(X^1_{pjw}, X^1_{pj'w})$, $j \neq j'$, $C_{11} = \mathrm{Cov}(X^1_{pjw}, X^1_{pjw}) = \mathrm{Var}(X^1_{pjw})$.

Remark 4.1. The linear combination of 1 and the random variables X^1_{pjr} ($p = \overline{1, P}$, $j = \overline{1, k_p}$, $r = \overline{1, t}$) closest to $f_0(X_{p,j,t+1})$ and to $\nu_0(\theta_p)$ in the least squares sense equals $\hat{\nu}_0(\theta_p)$, and the linear combination of 1 and the random variables X^1_{pjr} ($p = \overline{1, P}$, $j = \overline{1, k_p}$, $r = \overline{1, t}$) closest to $\mu_0(\theta_p, \theta_{p_j})$ in the least squares sense equals $\hat{\mu}_0(\theta_p, \theta_{pj})$.

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