INSTITUTE OF MATHEMATICS

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On a periodic problem for second-order Duffing type equations

Alexander Lomtatidze

Jiří Šremr

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ON A PERIODIC PROBLEM FOR SECOND-ORDER DUFFING TYPE EQUATIONS

ALEXANDER LOMTATIDZE AND JIŘÍ ŠREMR

Abstract. Sufficient and necessary conditions are found for the existence of a positive periodic solution to the Duffing type equation

$$u'' = p(t)u - q(t, u)u.$$

The results obtained are compared with facts well known for the autonomous Duffing equation

$$y'' - ay + by^3 = 0.$$

Uniqueness of solutions and possible generalisations are discussed, as well.

1. Introduction

In the paper, we are intersted in the question on the existence of a **positive** solution to the periodic problem

$$u'' = p(t)u - q(t, u)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{1.1}$$

where $p \in L([0, \omega])$ and $q\colon [0, \omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Under a solution to problem (1.1), as usually, we understand a function $u\colon [0, \omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies given equation almost everywhere, and verifies periodic conditions. Equation in (1.1) is a natural generalisation of the equation

$$y'' - ay + by^3 = 0, \tag{1.2}$$

where $a, b \in \mathbb{R}$. This equation is the central topic of the monograph [1] by Duffing published in 1918 and still bears his name today. Consider a free undamped oscillator consisting of the mass body with the weight m and two linear springs with the characteristic k and the non-deformed length ℓ (see Fig. 1) whose equation of motion has the form

$$y'' + \frac{2k}{m}\, y \left(1 - \frac{\ell}{\sqrt{(\ell - d)^2 + y^2}} \right) = 0. \tag{1.3}$$

Equation (1.2) with $a, b > 0$ appears when approximating the non-linearity in (1.3) by Taylor's polynomial of the third order with the centre at 0. It can be also interpreted as the equation of motion of a free undamped oscillator with a spring whose restoring force is given as a third-order polynomial. The phase portrait of (1.2) with $a, b > 0$ can be easily determined and it is illustrated on Fig. 2.

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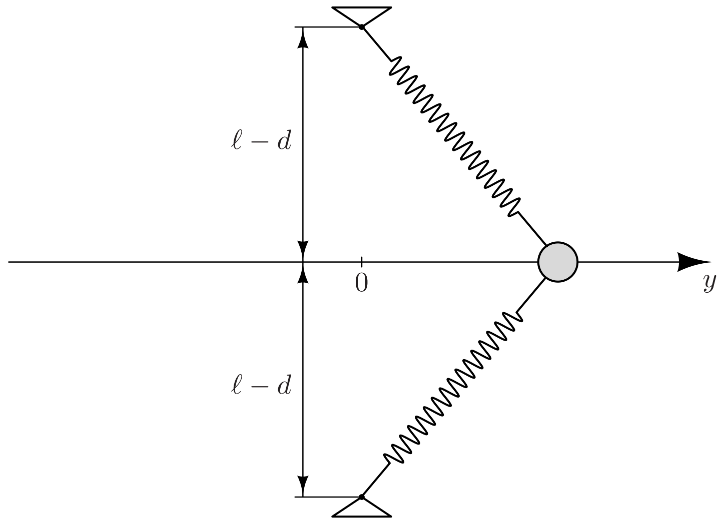

FIGURE 1. Free undamped transversal oscillator.

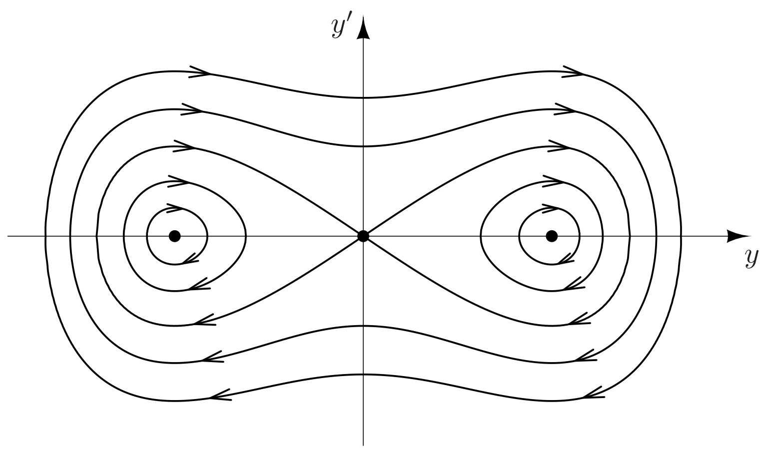

FIGURE 2. Phase portrait of equation (1.2) with $a, b > 0$.

Definition 1.1. A solution u to problem (1.1) is referred as a sign-constant solution if there exists $i \in \{0, 1\}$ such that

$$(-1)^i u(t) \geq 0 \quad \text{for } t \in [0, \omega],$$

and a sign-changing solution otherwise.

Let us summarize some well-known facts concerning periodic solutions to equation (1.2) (see, e. g., [6, 7]).

Proposition 1.2. *The following statements hold:*

(1) *For any $a \leq 0$ and $b > 0$, equation* (1.2) *has a unique equilibrium $y = 0$ and every non-trivial periodic solution to* (1.2) *changes its sign.*

(2) *For any $a, b > 0$, equation* (1.2) *has exactly three equilibria $y = 0$, $y = \sqrt{\frac{a}{b}}$, and $y = -\sqrt{\frac{a}{b}}$, positive and negative non-constant periodic solutions, and periodic sign-changing solutions with various periods.*
(3) *For any $a, b > 0$ and $T \leq \frac{\pi\sqrt{2}}{\sqrt{a}}$, equation* (1.2) *has exactly two non-trivial T-periodic sign-constant solutions.*
(4) *For any $a, b > 0$ and $T > \frac{\pi\sqrt{2}}{\sqrt{a}}$, equation* (1.2) *has a positive (resp. negative) non-constant periodic solution with the minimal period T.*

In the present paper, we generalise above assertions to a non-autonomous case and an arbitrary power of the super-linearity in (1.2) (see Corollary 2.11 and Remark 2.12). Therefore, we fix $\omega > 0$ and consider the non-autonomous periodic problem

$$u'' = p(t)u - h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{1.4}$$

where $p, h \in L([0, \omega])$ and $\lambda > 1$. Since our results do not depend on the value of the power λ, it is nature to generalise (1.4) to problem (1.1). In spite of the autonomous case, only a few results dealing with question on the existence of periodic solutions to the non-autonomous Duffing type equations with a super-linear non-linearity is known (see, e. g., [2, 8, 9, 12–14] and references therein). Below, we establish effective conditions for the existence of a **positive** periodic solution to (1.1) and their consequences for non-autonomous Duffing equation in (1.4), which can be easily compared with the facts well known in the autonomous case (1.2). At last, we discuss possible extensions for a more general problem than (1.4), namely, for the periodic problem with two super-linear terms

$$u'' = p(t)u - h(t)|u|^{\lambda} \operatorname{sgn} u + f(t)|u|^{\mu} \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{1.5}$$

where $p, h, f \in L([0, \omega])$ and $\lambda, \mu > 1$. It is worth mentioning that Duffing type equations with two or more super-linear terms appear when approximating the non-linearity in the equation of oscillator (1.3) by Taylor's polynomials of higher orders than 3.

The following notation is used throughout the paper:

– $\mathbb{N}$ and $\mathbb{R}$ are the sets of natural and real numbers, respectively. For any $x \in \mathbb{R}$, we put $[x]_+ = \frac{1}{2}(|x| + x)$ and $[x]_- = \frac{1}{2}(|x| - x)$.
– $C(I)$ denotes the linear space of continuous real functions defined on the interval $I \subseteq \mathbb{R}$. For any $u \in C([a, b])$, we put $\|u\|_C = \max\{|u(t)| : t \in [a, b]\}$.
– $AC^1([a, b])$ is the set of functions $u\colon [a, b] \to \mathbb{R}$ which are absolutely continuous together with their first derivatives.
– $AC_\ell([a, b])$ (resp. $AC_u([a, b])$) is the set of absolutely continuous functions $u\colon [a, b] \to \mathbb{R}$ such that u' admits the representation $u'(t) = \gamma(t) + \sigma(t)$ for a. e. $t \in [a, b]$, where $\gamma\colon [a, b] \to \mathbb{R}$ is absolutely continuous and $\sigma\colon [a, b] \to \mathbb{R}$ is a non-decreasing (resp. non-increasing) function whose derivative is equal to zero almost everywhere on $[a, b]$.
– $L([0, \omega])$ denotes the Banach space of Lebesgue integrable functions $p\colon [0, \omega] \to \mathbb{R}$ equipped with the norm $\|p\|_L = \int_0^\omega |p(s)| \mathrm{d}s$. The symbol $\operatorname{Int} A$ stands for the interior of the set $A \subset L([0, \omega])$.

Definition 1.3 ([11, Definition 0.1])**.** We say that the function $p \in L([0, \omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if for any function $u \in AC^1([0, \omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a. e. } t \in [0, \omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega),$$

the inequality

$$u(t) \geq 0 \quad \text{for } t \in [0,\omega] \quad (\text{resp.} \quad u(t) \leq 0 \quad \text{for } t \in [0,\omega])$$

holds.

Remark 1.4. Efficient conditions for p to belong to each of the sets $\mathcal{V}^+(\omega)$ and $\mathcal{V}^-(\omega)$ are given in [11].

2. Main results

In this part, we formulate all the results, their proofs are given later in Section 4. Let us introduce the hypothesis:

$$\left.\begin{aligned} &q(t,x) \geq q_0(t,x) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \geq 0, \\ &q_0\colon [0,\omega] \times [0,+\infty[\to \mathbb{R} \text{ is a Carathéodory function}, \\ &q_0(t,\cdot)\colon [0,+\infty[\to \mathbb{R} \text{ is non-decreasing for a. e. } t \in [0,\omega]. \end{aligned}\right\} \tag{H_1}$$

Theorem 2.1. *Let $p \in \mathcal{V}^-(\omega)$, $q(\cdot,0) \equiv 0$, and hypothesis (H_1) be fulfilled. Let, moreover, there exist a function $\alpha \in AC_\ell([0,\omega])$ satisfying*

$$\alpha(t) > 0 \quad \text{for } t \in [0,\omega], \tag{2.1}$$

$$\alpha''(t) \geq p(t)\alpha(t) - q(t,\alpha(t))\alpha(t) \quad \text{for a. e. } t \in [0,\omega], \tag{2.2}$$

$$\alpha(0) = \alpha(\omega), \qquad \alpha'(0) \geq \alpha'(\omega). \tag{2.3}$$

Then problem (1.1) *has at least one positive solution u such that*

$$u(t_u) \leq \alpha(t_u) \quad \text{for some } t_u \in [0,\omega]. \tag{2.4}$$

Corollary 2.2. *Let $q(\cdot,0) \equiv 0$ and hypothesis (H_1) be satisfied. Let, moreover,*

$$p \in \mathcal{V}^-(\omega) \tag{2.5}$$

and at least one of the following conditions be fulfilled:

(a) *There exists $c > 0$ such that*

$$p(t) \leq q(t,c) \quad \text{for a. e. } t \in [0,\omega]. \tag{2.6}$$

(b) *There exists $r > 0$ such that $p - q_0(\cdot,r) \in \operatorname{Int}\mathcal{V}^+(\omega)$.*

Then problem (1.1) *has at least one positive solution.*

Now we give an effective condition guaranteeing that the assumption (b) of Corollary 2.2 is satisfied.

Corollary 2.3. *Let $q(\cdot,0) \equiv 0$, hypothesis (H_1) be satisfied, condition* (2.5) *hold, and*

$$\lim_{x\to+\infty} \int_0^\omega q_0(s,x)\mathrm{d}s = +\infty. \tag{2.7}$$

Then problem (1.1) *has at least one positive solution.*

Remark 2.4. By using Lebesgue's domination theorem, one can show that for the function q_0 appearing in hypothesis (H_1), condition (2.7) is satisfied if there exists $E \subseteq [0,\omega]$ such that $\operatorname{meas} E > 0$ and the equality

$$\lim_{x\to+\infty} q_0(t,x) = +\infty \quad \text{for every } t \in E \tag{2.8}$$

holds.

Under the hypothesis

$$\left.\begin{array}{l}\text{for every } b > a > 0 \text{ there exists } h_{ab} \in L([0,\omega]) \text{ such that}\\ h_{ab}(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \quad h_{ab} \not\equiv 0,\\ q(t,x) \geq h_{ab}(t) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in [a,b],\end{array}\right\} \qquad (H_2)$$

the assumption $p \in \mathcal{V}^-(\omega)$ in the above results is also necessary as follows from the next proposition.

Proposition 2.5. *Let hypothesis* (H_2) *hold. If problem* (1.1) *has a positive solution then condition* (2.5) *is satisfied.*

Finally, we give a statement guaranteeing that any pair of positive solutions to problem (1.1) has to intersect. Introduce the hypothesis:

$$\left.\begin{array}{l}\text{For every } d > c > 0 \text{ and } e > 0, \text{ there exists } h_{cde} \in L([0,\omega]) \text{ such that}\\ h_{cde}(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \; h_{cde} \not\equiv 0,\\ q(t,x+e) - q(t,x) \geq h_{cde}(t) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in [c,d].\end{array}\right\} \qquad (H_3)$$

Proposition 2.6. *Let hypothesis* (H_3) *hold and*

$$q(t,0) \geq 0 \quad \textit{for a. e. } t \in [0,\omega]. \tag{2.9}$$

Let, moreover, u *and* v *be distinct positive solutions to problem* (1.1)*. Then there exist* $t_1, t_2 \in [0,\omega]$ *such that*

$$u(t_1) < v(t_1), \qquad u(t_2) > v(t_2). \tag{2.10}$$

If q in (1.1) is a function with separated variables, we arrive at the problem

$$u'' = p(t)u - h(t)\varphi(u)u; \quad u(0) = u(\omega), \; u'(0) = u'(\omega), \tag{2.11}$$

where $p, h \in L([0,\omega])$ and $\varphi \in C(\mathbb{R})$. This problem covers a rather wide class of problems arising in applications and serves us as a model problem to illustrate the results stated above.

Theorem 2.7. *Let* $p \in \mathcal{V}^-(\omega)$*,* $\varphi(0) = 0$*, and*

$$h(t) \geq 0 \quad \textit{for a. e. } t \in [0,\omega], \qquad h \not\equiv 0. \tag{2.12}$$

Let, moreover, at least one of the following conditions be fulfilled:

(i) *The inequality*

$$\liminf_{x\to+\infty} \varphi(x) > -\infty \tag{2.13}$$

holds and there exists $c > 0$ *such that*

$$p(t) \leq h(t)\varphi(c) \quad \textit{for a. e. } t \in [0,\omega]. \tag{2.14}$$

(ii) *The equality*

$$\lim_{x\to+\infty} \varphi(x) = +\infty \tag{2.15}$$

holds.

Then problem (2.11) *has at least one positive solution. If in addition, the function* φ *is increasing on* $[0,+\infty[$ *and* u *and* v *are distinct positive solutions to problem* (2.11) *then inequalities* (2.10) *hold with some* $t_1, t_2 \in [0,\omega]$*.*

Remark 2.8. If $\varphi(x) > 0$ for $x > 0$ then the assumption $p \in \mathcal{V}^-(\omega)$ in Theorem 2.7 is also necessary for the existence of a positive solution to problem (2.11) (see Proposition 2.5 with $q(t,x) := h(t)\varphi(x)$).

Remark 2.9. It follows from results obtained in [12] that problem (2.11) has a positive solution provided that condition (2.12) holds,

$$p(t) \geq 0 \quad \text{for } t \in [0, \omega], \quad p \not\equiv 0 \tag{2.16}$$

and

$$\varphi(0) = 0, \quad \varphi(x) \geq 0 \quad \text{for } x > 0, \quad \lim_{x \to +\infty} \varphi(x) = +\infty. \tag{2.17}$$

Observe that condition (2.16) yields $p \in \mathcal{V}^-(\omega)$ (see, e. g., Lemma 3.7 with $g(t) := p(t)$ and $\gamma(t) := 1$). Therefore, in Theorem 2.7(ii), condition (2.16) is weakened to the assumption $p \in \mathcal{V}^-(\omega)$ and condition (2.17) is relaxed to $\varphi(0) = 0$ and (2.15).

Now we derive corollaries for a non-autonomous Duffing equation and compare the results with facts well known in the autonomous case. We first give an existence and uniqueness result for a particular case of (1.4), where $p \equiv h$.

Proposition 2.10. *Let* $\lambda > 1$,

$$p(t) \geq 0 \quad \textit{for a. e. } t \in [0, \omega], \qquad p \not\equiv 0, \tag{2.18}$$

and

$$\int_0^\omega p(s)\mathrm{d}s \leq \frac{16\lambda^*}{\omega}\,, \tag{2.19}$$

where

$$\lambda^* := \begin{cases} \left\lfloor \frac{1}{\lambda - 1} \right\rfloor & \textit{for } \lambda \in]1, 2], \\ \frac{1}{\lceil \lambda - 1 \rceil} & \textit{for } \lambda > 2 \end{cases} \tag{2.20}$$

in which $\lfloor \cdot \rfloor$ *and* $\lceil \cdot \rceil$ *denote the floor function and ceiling function, respectively. Then the constant function*

$$u(t) := 1 \quad \textit{for } t \in [0, \omega]$$

is a unique positive solution to the problem

$$u'' = p(t)\big(1 - |u|^{\lambda - 1}\big)u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega). \tag{2.21}$$

Corollary 2.11. *Let* $\lambda > 1$ *and condition* (2.12) *hold. Then the following assertions hold:*

1. *Problem* (1.4) *has at least one positive (resp. negative) solution if and only if* $p \in \mathcal{V}^-(\omega)$.
2. *If* u *and* v *are distinct positive (resp. negative) solutions to problem* (1.4) *then inequalities* (2.10) *are satisfied with some* $t_1, t_2 \in [0, \omega]$.
3. *If* $p \in \mathcal{V}^-(\omega)$ *and*

$$\mathrm{e}^{-1+\sqrt{1+\frac{\omega}{4}\int_0^\omega p(s)\mathrm{d}s}} \left(-1 + \sqrt{1 + \frac{\omega}{4} \int_0^\omega p(s)\mathrm{d}s} \right) \leq 8\lambda^*, \tag{2.22}$$

where the number λ^* *is defined by formula* (2.20), *then problem* (1.4) *has a unique positive (resp. negative) solution.*

Remark 2.12. It is clear that the Duffing equation (1.2) is a particular case of the equation in (1.4), where $\lambda := 3$ and

$$p(t) := a, \quad h(t) := b \quad \text{for } t \in [0, \omega]. \tag{2.23}$$

One can easily derive that, in this case, $p \in \mathcal{V}^-(\omega)$ if and only if $a > 0$. Hence, it follows from Corollary 2.11(1) that for any $a \leq 0$ and $b, \omega > 0$, equation (1.2) has no non-trivial sign-constant ω-periodic solution. This is in a compliance with assertion

(1) of Proposition 1.2. On the other hand, Corollary 2.11(1) also yields that for any $a, b, \omega > 0$, equation (1.2) has at least one positive (resp. negative) ω-periodic solution. This is in a compliance with assertion (2) of Proposition 1.2. Finally, it follows from Proposition 2.10 that if $0 < a \leq \frac{8}{\omega^2}$ and $b > 0$, then the equilibrium $\sqrt{\frac{a}{b}}$ is a unique positive ω-periodic solution to equation (1.2). Therefore, if $a, b > 0$ and y is a periodic solution to equation (1.2) corresponding to a closed orbit on Fig. 2, then the minimal period T of y satisfies the estimate

$$T > \frac{2\sqrt{2}}{\sqrt{a}}\,.$$

This estimate was derived from the result dealing with a non-autonomous equation and thus, it is not surprising that it can be improved in the autonomous case (see assertion (3) of Proposition 1.2). Consequently, Corollary 2.11 naturally extends the basic facts concerning periodic solutions to the Duffing equation (1.2) to the non-autonomous case.

Finally, we consider problem (1.5), where two super-linear terms are involved. Clearly, if u is a solution to problem (1.5) then the function $-u$ is its solution, as well. Therefore, the following statements follow from Corollary 2.3.

Theorem 2.13. *Let $\lambda > \mu > 1$, relation* (2.12) *hold, and there exists $c > 0$ such that*

$$[f(t)]_+ \leq ch(t) \quad \textit{for a. e. } t \in [0, \omega]. \tag{2.24}$$

If, moreover, condition (2.5) *is satisfied then problem* (1.5) *has at least three solutions (positive, negative, and trivial).*

Remark 2.14. It follows from [3, Theorem 0.1 and Remark 1] that the problem

$$u'' = p(t)u - h(t)u^3 + f(t)u^2; \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{2.25}$$

has at least one nontrivial solution provided that $p, h, f\colon [0, \omega] \to \mathbb{R}$ are continuous functions such that

$$p(t) > 0, \quad h(t) > 0 \quad \text{for } t \in [0, \omega]. \tag{2.26}$$

In Theorem 2.13, a stronger assertion is claimed under weaker assumptions then (2.26), because Theorem 2.13 guarantees the existence of a positive solution to problem (2.25).

Theorem 2.15. *Let $\lambda > \mu > 1$,*

$$h(t) > 0 \quad \textit{for a. e. } t \in [0, \omega], \tag{2.27}$$

and

$$[f]_+^{\frac{\lambda-1}{\lambda-\mu}} h^{-\frac{\mu-1}{\lambda-\mu}} \in L([0, \omega]). \tag{2.28}$$

If, moreover, condition (2.5) *is satisfied then problem* (1.5) *has at least three solutions (positive, negative, and trivial).*

Remark 2.16. If

$$f(t) \leq 0 \quad \text{for a. e. } t \in [0, \omega]$$

then both conditions (2.24) and (2.28) are satisfied and, moreover, it follows from Proposition 2.5 that assumption (2.5) is necessary in Theorems 2.13 and 2.15 for the existence of a positive solution to problem (1.5).

3. Auxiliary statements

We first recall some results proved in [11].

Lemma 3.1 ([11, Proposition 10.8]). *If $p \in \mathcal{V}^-(\omega)$ then $\int_0^\omega p(s)\mathrm{d}s > 0$.*

Lemma 3.2 ([11, Proposition 10.1]). *The set $\mathcal{V}^-(\omega)$ is open in $L([0,\omega])$.*

Definition 3.3 ([11, Definition 0.4]). We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{D}(\omega)$ if the problem

$$u'' = \widetilde{p}(t)u; \quad u(a) = 0, \ u(b) = 0 \tag{3.1}$$

has no non-trivial solution for any $a < b$ satisfying $b - a < \omega$, where $\widetilde{p}$ is the ω-periodic extension of the function p to the whole real axis.

Lemma 3.4. *$\mathcal{V}^-(\omega) \subset \operatorname{Int}\mathcal{D}(\omega)$.*

Proof. It follows from Propositions 2.1, 10.5, and 10.6 established in [11]. ☐

Lemma 3.5 ([11, Proposition 2.5]). *Let $g\colon \mathbb{R} \to \mathbb{R}$ be an ω-periodic function such that $g \in \mathcal{D}(\omega)$. Then for any $t_1 < t_2$ and $w \in AC^1([t_1,t_2])$ satisfying $t_2 - t_1 < \omega$ and*

$$w''(t) \geq g(t)w(t) \quad \text{for a. e. } t \in [t_1,t_2], \qquad w(t_1) \leq 0, \qquad w(t_2) \leq 0,$$

the inequality

$$w(t) \leq 0 \quad \text{for } t \in [t_1,t_2] \tag{3.2}$$

holds.

Lemma 3.6 ([11, Proposition 2.2]). *Let $p \in L([0,\omega])$. Then the inclusion $p \in \operatorname{Int}\mathcal{D}(\omega)$ holds if and only if problem (3.1) has no non-trivial solution for any $a < b$ satisfying $b - a \leq \omega$, where $\widetilde{p}$ is the ω-periodic extension of the function p to the whole real axis.*

Lemma 3.7 ([11, Theorem 8.3]). *Let $g \in L([0,\omega])$. Then the inclusion $g \in \mathcal{V}^-(\omega)$ holds if and only if there exists a positive function $\gamma \in AC^1([0,\omega])$ satisfying*

$$\gamma''(t) \leq g(t)\gamma(t) \quad \text{for a. e. } t \in [0,\omega], \quad \gamma(0) \geq \gamma(\omega), \quad \frac{\gamma'(\omega)}{\gamma(\omega)} \geq \frac{\gamma'(0)}{\gamma(0)}\,,$$

and

$$\gamma(0) - \gamma(\omega) + \frac{\gamma'(\omega)}{\gamma(\omega)} - \frac{\gamma'(0)}{\gamma(0)} + \operatorname{meas}\{t \in [0,\omega] : \gamma''(t) < g(t)\gamma(t)\} > 0.$$

Lemma 3.8 ([11, Theorem 9.1′]). *Let $g \in L([0,\omega])$. Then the inclusion $g \in \operatorname{Int}\mathcal{V}^+(\omega)$ holds if and only if $g \in \operatorname{Int}\mathcal{D}(\omega)$ and there exists a positive function $\gamma \in AC^1([0,\omega])$ satisfying*

$$\gamma''(t) \geq g(t)\gamma(t) \quad \text{for a. e. } t \in [0,\omega], \quad \gamma(0) = \gamma(\omega), \quad \gamma'(0) \geq \gamma'(\omega),$$

and

$$\gamma'(0) - \gamma'(\omega) + \operatorname{meas}\{t \in [0,\omega] : \gamma''(t) > g(t)\gamma(t)\} > 0.$$

Lemma 3.9 ([11, Theorem 16.4]). *Let $g \in \operatorname{Int}\mathcal{V}^+(\omega)$. Then there exist $\nu, \Delta > 0$ such that for any non-negative function $f \in L([0,\omega])$, the problem*

$$u'' = g(t)u + f(t); \quad u(0) = u(\omega), \ u'(0) = u'(\omega) \tag{3.3}$$

has a unique solution u and this solution satisfies the relation

$$\nu \int_0^\omega |f(s)|\mathrm{d}s \leq u(t) \leq \Delta \int_0^\omega |f(s)|\mathrm{d}s \quad \textit{for } t \in [0,\omega]. \tag{3.4}$$

Lemma 3.10 ([11, Theorem 16.2])**.** *Let $g \in \mathcal{V}^-(\omega)$. Then there exist $\nu, \Delta > 0$ such that for any non-positive function $f \in L([0,\omega])$, problem* (3.3) *has a unique solution u and this solution satisfies relation* (3.4)*.*

Lemma 3.11. *Let $n \in \mathbb{N}$. Then*

$$\sum_{k=0}^{n-1} x^{\frac{k}{n}} \geq n x^{\frac{n-1}{2n}} \quad \textit{for } x > 0.$$

Proof. To prove the lemma, it is sufficient to show that for any $n \in \mathbb{N}$, the inequality

$$\sum_{k=0}^{n-1} z^k \geq n z^{\frac{n-1}{2}} \quad \text{for } z > 0 \tag{3.5}$$

is fulfilled. For $n = 1$, the inequality (3.5) holds. Assume that the inequality (3.5) is satisfied for $n := m$. We show that (3.5) remains true for $n := m+1$. It is clear that

$$\sum_{k=0}^{m} z^k = z^m + \sum_{k=0}^{m-1} z^k \geq z^m + m z^{\frac{m-1}{2}} = z^{\frac{m-1}{2}}\left(z^{\frac{m+1}{2}} + m\right) \quad \text{for } z > 0. \tag{3.6}$$

Put $\ell(z) := z^{\frac{m+1}{2}} + m - (m+1)z^{\frac{1}{2}}$ for $z \geq 0$. Then $\ell(0) = m$, $\ell(1) = 0$, and

$$\ell'(z) = \frac{m+1}{2}\, z^{-\frac{1}{2}}\left(z^{\frac{m}{2}} - 1\right) \quad \text{for } z > 0.$$

Hence, $\ell(z) \geq 0$ for $z \geq 0$. Now it follows from inequality (3.6) that

$$\sum_{k=0}^{m} z^k \geq (m+1) z^{\frac{m}{2}} \quad \text{for } z > 0,$$

i. e., (3.5) holds for $n := m+1$. □

Lemma 3.12. *Let $p \in L([0,\omega])$ and u be a nontrivial solution to the problem*

$$u'' = p(t)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{3.7}$$

with at least two zeros in the interval $[0,\omega]$. Then

$$\int_0^\omega [p(s)]_- \mathrm{d}s > \frac{16}{\omega}\,. \tag{3.8}$$

Proof. Extend the functions u and p periodically to the whole real axis and denote them by the same symbols. Then there exists $a \in [0,\omega[$ such that the function u has at least three zeros in the interval $[a, a+\omega]$. It follows from [5, Corollary 5.2] (and its proof) that

$$\int_a^{a+\omega} [p(s)]_- \mathrm{d}s > \frac{16}{\omega}$$

and thus, the inequality (3.8) holds. □

Now along with problem (3.3), we consider the sequence of the problems

$$u'' = g_n(t)u + f_n(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{3.3$_n$}$$

where $g_n, f_n \in L([0,\omega])$, $n \in \mathbb{N}$. A simple application of the Arzelà-Ascoli theorem leads to the following statement.

Lemma 3.13. *Let*

$$\lim_{n\to+\infty} \|g_n - g\|_L = 0, \qquad \lim_{n\to+\infty} \|f_n - f\|_L = 0.$$

Let, moreover, for any $n \in \mathbb{N}$, u_n be a solution to problem (3.3$_n$) *and the sequence $\{\|u_n\|_C\}_{n=1}^{+\infty}$ be bounded. Then there exists a subsequence $\{u_{n_k}\}_{k=1}^{+\infty}$ of $\{u_n\}_{n=1}^{+\infty}$ such that*

$$\lim_{k\to+\infty} u_{n_k}^{(i)} = u^{(i)}(t) \quad \text{uniformly on } [0,\omega],\ i = 0,1,$$

where u is a solution to problem (3.3)*.*

Furthermore, we recall a classical result concerning the solvability of the periodic problem

$$u'' = f(t,u); \quad u(a) = u(b),\ u'(a) = u'(b), \tag{3.9}$$

where $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function (see, e. g., [4]).

Lemma 3.14. *Let there exist functions $\alpha \in AC_\ell([a,b])$ and $\beta \in AC_u([a,b])$ satisfying*

$$\begin{gathered}\alpha(t) \le \beta(t) \quad \text{for } t \in [a,b],\\ \alpha''(t) \ge f(t,\alpha(t)) \quad \text{for a. e. } t\in[a,b], \quad \alpha(a) = \alpha(b), \quad \alpha'(a) \ge \alpha'(b),\end{gathered} \tag{3.10}$$

and

$$\beta''(t) \le f(t,\beta(t)) \quad \text{for a. e. } t\in[a,b], \quad \beta(a) = \beta(b), \quad \beta'(a) \le \beta'(b). \tag{3.11}$$

Then problem (3.9) *has at least one solution u such that*

$$\alpha(t) \le u(t) \le \beta(t) \quad \text{for } t \in [a,b].$$

Moreover, the following existence result is known.

Lemma 3.15 ([10, Theorem 1.1 and Remark 1.2])**.** *Let there exist $p \in \operatorname{Int}\mathcal{D}(\omega)$ and a Carathéodory function $g\colon [0,\omega]\times[0,+\infty[\,\to[0,+\infty[$ such that*

$$f(t,x)\operatorname{sgn} x \ge p(t)|x| - g(t,|x|) \quad \text{for a. e. } t\in[0,\omega] \text{ and all } x\in\mathbb{R}$$

and

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^\omega g(s,x)\mathrm{d}s = 0.$$

Let, moreover, there exist functions $\alpha \in AC_\ell([0,\omega])$ and $\beta \in AC_u([0,\omega])$ satisfying relations (3.10) *and* (3.11) *with $a := 0$, $b := \omega$. Then problem* (3.9) *with $a := 0$, $b := \omega$ has a solution u such that*

$$\min\{\alpha(t_u),\beta(t_u)\} \le u(t_u) \le \max\{\alpha(t_u),\beta(t_u)\}$$

for some $t_u \in [0,\omega]$.

Lemma 3.16. *Let $\lambda > 1$, $\ell, h \in L([0,\omega])$, and $u \in AC^1([0,\omega])$ be such that*

$$h(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \tag{3.12}$$

$$u(t) > 0 \quad \text{for } t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega), \tag{3.13}$$

and

$$u''(t) \leq \ell(t)u(t) - h(t)u^{\lambda}(t) \quad \text{for a. e. } t \in [0,\omega]. \tag{3.14}$$

Then

$$\int_0^\omega \ell(s)\mathrm{d}s \geq 0 \tag{3.15}$$

and

$$M \leq m\,\mathrm{e}^{\sqrt{\frac{\omega}{4}\int_0^\omega \ell(s)\mathrm{d}s}}, \tag{3.16}$$

where

$$M := \max\{u(t) : t \in [0,\omega]\}, \qquad m := \min\{u(t) : t \in [0,\omega]\}. \tag{3.17}$$

Moreover, the function u satisfies the estimate

$$\begin{aligned} &\int_0^\omega \frac{1}{u^2(s)}\,\mathrm{d}s \int_0^\omega h(s)u^{\lambda+1}(s)\mathrm{d}s \\ &\qquad \leq 2\,\mathrm{e}^{-1+\sqrt{1+\frac{\omega}{4}\int_0^\omega p(s)\mathrm{d}s}} \left(-1 + \sqrt{1 + \frac{\omega}{4}\int_0^\omega p(s)\mathrm{d}s}\right). \end{aligned} \tag{3.18}$$

Proof. First observe that, in view of assumption (3.13), we have

$$\int_0^\omega \frac{u''(s)}{u(s)}\,\mathrm{d}s = \int_0^\omega \left(\frac{u'(s)}{u(s)}\right)^2 \mathrm{d}s$$

which together with (3.13) and (3.14) yields that

$$\int_0^\omega h(s)u^{\lambda-1}(s)\mathrm{d}s \leq \int_0^\omega \ell(s)\mathrm{d}s - \int_0^\omega \left(\frac{u'(s)}{u(s)}\right)^2 \mathrm{d}s. \tag{3.19}$$

Taking (3.12) into account, it follows from (3.19) that inequality (3.15) is fulfilled.

Let the numbers M and m be defined by formulae (3.17). Extend the functions u, ℓ, and h periodically to the whole real axis and denote them by the same symbols. Clearly, there exist $t_m \in [0,\omega[$ and $t_M \in]t_m, t_m+\omega[$ such that $u(t_m) = m$ and $u(t_M) = M$. Then, by Hölder's inequality, one gets

$$\ln^2 \frac{M}{m} = \left(\int_{t_m}^{t_M} \frac{u'(s)}{u(s)}\,\mathrm{d}s\right)^2 \leq (t_M - t_m)\int_{t_m}^{t_M} \left(\frac{u'(s)}{u(s)}\right)^2 \mathrm{d}s$$

and

$$\ln^2 \frac{M}{m} \leq \left(\int_{t_M}^{t_m+\omega} \left|\frac{u'(s)}{u(s)}\right| \mathrm{d}s\right)^2 \leq (t_m + \omega - t_M)\int_{t_M}^{t_m+\omega} \left(\frac{u'(s)}{u(s)}\right)^2 \mathrm{d}s$$

which, by virtue of the inequality $4xy \leq (x+y)^2$ for $x, y \in \mathbb{R}$, implies

$$\ln^4 \frac{M}{m} \leq \frac{\omega^2}{16}\left[\int_0^\omega \left(\frac{u'(s)}{u(s)}\right)^2 \mathrm{d}s\right]^2.$$

Consequently, it follows from (3.19) that

$$\int_0^\omega h(s)u^{\lambda-1}(s)\mathrm{d}s \leq \int_0^\omega \ell(s)\mathrm{d}s - \frac{4}{\omega}\ln^2 \frac{M}{m}.$$

Therefore, in view of (3.12) and (3.17), we get

$$\begin{aligned}\int_0^\omega \frac{1}{u^2(s)}\,\mathrm{d}s \int_0^\omega h(s)u^{\lambda+1}(s)\mathrm{d}s &\leq \omega\left(\frac{M}{m}\right)^2 \int_0^\omega h(s)u^{\lambda-1}(s)\mathrm{d}s \\ &\leq \omega\left(\frac{M}{m}\right)^2\left(\int_0^\omega \ell(s)\mathrm{d}s - \frac{4}{\omega}\ln^2\frac{M}{m}\right).\end{aligned} \tag{3.20}$$

Now observe that the latter relation immediately yields inequality (3.16). Moreover, it follows from (3.20) that

$$\int_0^\omega \frac{1}{u^2(s)}\,\mathrm{d}s \int_0^\omega h(s)u^{\lambda+1}(s)\mathrm{d}s \leq 4\max\big\{f(z) : z\in[0,z_0]\big\},$$

where

$$f(z) := \mathrm{e}^{2z}\left(z_0^2 - z^2\right) \quad \text{for } z\in\mathbb{R}, \qquad z_0 := \sqrt{\frac{\omega}{4}\int_0^\omega \ell(s)\mathrm{d}s}.$$

It can be easily verified by direct calculation that

$$f(z) \leq f\Big(-\tfrac{1}{2} + \sqrt{\tfrac{1}{4}+z_0^2}\,\Big) \quad \text{for } z\in[0,z_0]$$

and thus, we have

$$\int_0^\omega \frac{1}{u^2(s)}\,\mathrm{d}s \int_0^\omega h(s)u^{\lambda+1}(s)\mathrm{d}s \leq 4f\Big(-\tfrac{1}{2} + \sqrt{\tfrac{1}{4}+z_0^2}\,\Big)$$

which yields the desired estimate (3.18). □

Lemma 3.17. *Let $p\in\mathcal{V}^-(\omega)$, hypothesis (H_1) hold, and there exist functions $\alpha\in AC_\ell([0,\omega])$ and $\beta\in AC_u([0,\omega])$ satisfying relations* (2.1), (2.2), (2.3), *and*

$$\beta(t) > 0 \quad \textit{for } t\in[0,\omega], \tag{3.21}$$

$$\beta''(t) \leq p(t)\beta(t) - q(t,\beta(t))\beta(t) \quad \textit{for a. e. } t\in[0,\omega], \tag{3.22}$$

$$\beta(0)=\beta(\omega), \quad \beta'(0)\leq\beta'(\omega). \tag{3.23}$$

Then problem (1.1) *has a positive solution u such that*

$$u(t_u) \leq \max\{\alpha(t_u),\beta(t_u)\} \quad \textit{for some } t_u\in[0,\omega]. \tag{3.24}$$

Proof. Put

$$\delta := \max\big\{\|\alpha\|_C, \|\beta\|_C\big\}\,\mathrm{e}^{\sqrt{\frac{\omega}{4}\left(\|p\|_L + \|q_0(\cdot,0)\|_L\right)}}$$

and

$$\chi(t,x) := [x+\beta(t)]_+ - [x-\delta]_+ - \beta(t) \quad \text{for } t\in[0,\omega],\ x\in\mathbb{R}. \tag{3.25}$$

Then obviously,

$$|\chi(t,x)| \leq \delta \quad \text{for } t\in[0,\omega],\ x\in\mathbb{R} \tag{3.26}$$

and there is a non-negative function $q_\delta\in L([0,\omega])$ such that

$$|q(t,x)| \leq q_\delta(t) \quad \text{for a. e. } t\in[0,\omega] \text{ and all } x\in\mathbb{R},\ |x|\leq\delta. \tag{3.27}$$

Consider the periodic problem

$$u'' = p(t)u - q(t,|\chi(t,u)|)\chi(t,u), \tag{3.28}$$

$$u(0)=u(\omega), \quad u'(0)=u'(\omega). \tag{3.29}$$

Since $\|\alpha\|_C \le \delta$ and $\|\beta\|_C \le \delta$, it follows from relations (2.1), (2.2), (3.21), and (3.22) that

$$\alpha''(t) \ge p(t)\alpha(t) - q(t, |\chi(t,\alpha(t))|)\chi(t,\alpha(t)) \quad \text{for a. e. } t \in [0,\omega]$$

and

$$\beta''(t) \le p(t)\beta(t) - q(t, |\chi(t,\beta(t))|)\chi(t,\beta(t)) \quad \text{for a. e. } t \in [0,\omega].$$

Therefore, by virtue of (3.26), (3.27), and Lemma 3.4, all the assumptions of Lemma 3.15 with $f(t,x) := p(t)x - q(t, |\chi(t,x)|)\chi(t,x)$ and $g(t,x) := \delta q_\delta(t)$ are satisfied and thus, problem (3.28), (3.29) has a solution u satisfying

$$0 < u(t_u) \le \max\{\alpha(t_u), \beta(t_u)\} \quad \text{for some } t_u \in [0,\omega]. \tag{3.30}$$

We first show that

$$u(t) \ge -\beta(t) \quad \text{for } t \in [0,\omega]. \tag{3.31}$$

Indeed, assume on the contrary that (3.31) is violated. Extend the functions p, u, β, $q(\cdot,x)$, and $\chi(\cdot,x)$ periodically to the whole real axis and denote them by the same symbols. Then, in view of the first inequality in (3.30), there exist $t_1, t_2 \in \mathbb{R}$ such that $0 < t_2 - t_1 < \omega$ and

$$u(t) < -\beta(t) \quad \text{for } t \in]t_1, t_2[\,, \quad u(t_1) = -\beta(t_1), \quad u(t_2) = -\beta(t_2), \tag{3.32}$$

whence we get, in particular, that $\chi(t, u(t)) = -\beta(t)$ for $t \in [t_1, t_2]$. Therefore, equality (3.28) yields

$$u''(t) = p(t)u(t) + q(t, \beta(t))\beta(t) \quad \text{for a. e. } t \in [t_1, t_2].$$

The latter relation and (3.22) result in

$$(u(t) + \beta(t))'' \le p(t)(u(t) + \beta(t)) \quad \text{for a. e. } t \in [t_1, t_2].$$

Consequently, by virtue of Lemmas 3.4 and 3.5 with $g(t) := p(t)$, we get

$$u(t) + \beta(t) \ge 0 \quad \text{for } t \in [t_1, t_2]$$

which is in a contradiction with (3.32). The contradiction obtained proves that inequality (3.31) holds.

It is clear that either

$$u(t) \ge 0 \quad \text{for } t \in [0,\omega], \tag{3.33}$$

or

$$\text{there exists } t_0 \in [0,\omega] \text{ such that } u(t_0) < 0. \tag{3.34}$$

First suppose that (3.33) is satisfied. We show that

$$u(t) > 0 \quad \text{for } t \in [0,\omega].$$

Indeed, assume on the contrary that there is $t_* \in [0,\omega]$ such that $u(t_*) = 0$. Then, by virtue of (3.30) and (3.33), there exist $t_1, t_2 \in [0,\omega]$ such that $t_1 < t_2$, $t_* \in [t_1, t_2]$, and

$$0 \le u(t) \le \delta \quad \text{for } t \in [t_1, t_2], \qquad u \not\equiv 0 \quad \text{on } [t_1, t_2]. \tag{3.35}$$

Hence, $u'(t_*) = 0$ and the function u is a solution to the initial value problem

$$w'' = \big(p(t) - q(t, u(t))\big)w; \quad w(t_*) = 0,\ w'(t_*) = 0$$

on the interval $[t_1, t_2]$. However it means that $u \equiv 0$ on $[t_1, t_2]$ which is in a contradiction with (3.35). The contradiction obtained proves that u is a positive solution to problem (3.28), (3.29). It remains to show that

$$u(t) \le \delta \quad \text{for } t \in [0,\omega]. \tag{3.36}$$

Obviously, we have $0 < \chi(t, u(t)) \leq u(t)$ for $t \in [0, \omega]$ and thus, taking hypothesis (H_1) into account, from (3.28) we get

$$u''(t) \leq p(t)u(t) - q_0(t, 0)\chi(t, u(t)) \leq \big(p(t) + |q_0(t, 0)|\big)u(t) \quad \text{for a. e. } t \in [0, \omega].$$

Consequently, in view of (3.29) and (3.30), it follows from Lemma 3.16 with $\ell(t) := p(t) + |q_0(t, 0)|$ and $h(t) := 0$ that

$$\max\{u(t) : t \in [0, \omega]\} \leq \max\big\{\|\alpha\|_C, \|\beta\|_C\big\}\, \mathrm{e}^{\sqrt{\frac{\omega}{4}\left(\|p\|_L + \|q_0(\cdot,0)\|_L\right)}},$$

i. e., inequality (3.36) holds. Therefore, we have $\chi(t, u(t)) = u(t)$ for $t \in [0, \omega]$ and thus, u is a positive solution to problem (1.1) satisfying relation (3.24).

Now assume that (3.34) is fulfilled. Extend the functions p, u, β, $q(\cdot, x)$, and $\chi(\cdot, x)$ periodically to the whole real axis and denote them by the same symbols. Then, in view of (3.30), there exist $a, t_1, t_2 \in \mathbb{R}$ such that $a < t_1 < t_0 < t_2 < a + \omega$ and

$$u(t) < 0 \quad \text{for } t \in \,]t_1, t_2[\,, \quad u(t_1) = 0, \quad u(t_2) = 0. \tag{3.37}$$

Put

$$\alpha_0(t) := \begin{cases} 0 & \text{for } t \in [a, t_1] \cup [t_2, a + \omega], \\ -u(t) & \text{for } t \in \,]t_1, t_2[\,. \end{cases}$$

It is not difficult to verify that $\alpha_0 \in AC_\ell([a, a + \omega])$,

$$\alpha_0(a) = \alpha_0(a + \omega), \quad \alpha_0'(a) = \alpha_0'(a + \omega). \tag{3.38}$$

In view of (3.31), it is clear that

$$0 \leq \alpha_0(t) \leq \beta(t) \quad \text{for } t \in [a, a + \omega]. \tag{3.39}$$

Moreover, from equality (3.28) we get

$$\alpha_0''(t) = p(t)\alpha_0(t) - q(t, \alpha_0(t))\alpha_0(t) \quad \text{for a. e. } t \in [a, a + \omega]. \tag{3.40}$$

Therefore, by virtue of relations (3.22), (3.23), (3.38), (3.39), and (3.40), it follows from Lemma 3.14 with $f(t, x) := p(t)x - q(t, x)x$, $\alpha(t) := \alpha_0(t)$, and $b := a + \omega$ that there exists a function $u_0 \in AC^1([a, a + \omega])$ satisfying

$$u_0''(t) = p(t)u_0(t) - q(t, u_0(t))u_0(t) \quad \text{for a. e. } t \in [a, a + \omega],$$
$$u_0(a) = u_0(a + \omega), \quad u_0'(a) = u_0'(a + \omega),$$

and

$$\alpha_0(t) \leq u_0(t) \leq \beta(t) \quad \text{for } t \in [a, a + \omega]. \tag{3.41}$$

If we extend the function u_0 periodically to the whole real axis and denote it by the same symbol, in view of (3.41), we easily conclude that the restriction of u_0 to the interval $[0, \omega]$ is a solution to problem (1.1) such that

$$0 \leq u_0(t) \leq \beta(t) \quad \text{for } t \in [0, \omega], \qquad u_0 \not\equiv 0. \tag{3.42}$$

Suppose now that there exists $t^* \in [0, \omega]$ such that $u_0(t^*) = 0$. Then u_0 is obviously a solution to the initial value problem

$$w'' = \big(p(t) - q(t, u_0(t))\big)w; \quad w(t^*) = 0, \; w'(t^*) = 0$$

on the interval $[0, \omega]$. However it means that $u_0 \equiv 0$ which is in a contradiction with (3.42). Consequently, in view of (3.42), the function $u := u_0$ is a positive solution to problem (1.1) satisfying (3.24) (e. g., $t_u := 0$). $\square$

Proposition 3.18. *Let $p \in \mathcal{V}^-(\omega)$ and $f\colon [0,\omega]\times\,]0,+\infty[\,\to \mathbb{R}$ be a locally Carathéodory function*[1] *such that*

$$\text{the function } f(t,\cdot)\colon\,]0,+\infty[\,\to \mathbb{R} \text{ is non-decreasing for a. e. } t\in[0,\omega], \tag{3.43}$$

$$\lim_{x\to 0+} f(t,x) \le 0 \quad \text{for a. e. } t\in[0,\omega], \tag{3.44}$$

and

$$\lim_{x\to+\infty}\int_0^\omega f(s,x)\mathrm{d}s = +\infty. \tag{3.45}$$

Then there exists $r>0$ such that $p - f(\cdot,r) \in \operatorname{Int}\mathcal{V}^+(\omega)$.

Proof. For any $\nu>0$, we put

$$p_\nu(t) := p(t) - f(t,\nu) \quad \text{for a. e. } t\in[0,\omega].$$

Let

$$A := \{\nu>0 : p_\nu \notin \mathcal{V}^-(\omega)\}.$$

According to assumption (3.45), there exists $R>0$ such that

$$\int_0^\omega f(s,R)\mathrm{d}s \ge \int_0^\omega p(s)\mathrm{d}s.$$

Therefore, by virtue of Lemma 3.1, we get $p - f(\cdot,R)\notin\mathcal{V}^-(\omega)$ and consequently, $A\neq\varnothing$. Let

$$\nu^* := \inf A.$$

Now we show that

$$\nu^* > 0. \tag{3.46}$$

Indeed, put

$$g_k(t) := \max\left\{-1, f(t,\tfrac{1}{k})\right\} \quad \text{for a. e. } t\in[0,\omega],\ k\in\mathbb{N}.$$

Then, in view of assumption (3.43), we have

$$-1 \le g_{k+1}(t) \le g_k(t) \quad \text{for a. e. } t\in[,\omega],\ k\in\mathbb{N}. \tag{3.47}$$

Let

$$g(t) := \lim_{k\to+\infty} g_k(t) \quad \text{for a. e. } t\in[0,\omega]. \tag{3.48}$$

By virtue of (3.44), it is clear that $g\in L([0,\omega])$ and

$$g(t)\le 0 \quad \text{for a. e. } t\in[0,\omega].$$

Since $p\in\mathcal{V}^-(\omega)$, it follows from Lemma 3.7 that $p-g\in\mathcal{V}^-(\omega)$. Moreover, according to (3.47) and (3.48), we have

$$\lim_{k\to+\infty}\int_0^\omega |g_k(s)-g(s)|\mathrm{d}s = 0.$$

The set $\mathcal{V}^-(\omega)$ is open (see Lemma 3.2) and thus, there exists $k_0\in\mathbb{N}$ such that

$$p - g_{k_0} \in \mathcal{V}^-(\omega). \tag{3.49}$$

On the other hand, we have

$$f(t,x) \le f(t,\tfrac{1}{k_0}) \le g_{k_0}(t) \quad \text{for a. e. } t\in[0,\omega] \text{ and all } x\in\left]0,\tfrac{1}{k_0}\right].$$

[1]It means that for any $[x_1,x_2]\subset\,]0,+\infty[$, the restriction of f to the set $[0,\omega]\times[x_1,x_2]$ is a Carathéodory function.

Therefore, in view above-proved inclusion (3.49), from Lemma 3.7 we get $p - f(\cdot, x) \in \mathcal{V}^-(\omega)$ for every $x \in \left]0, \frac{1}{k_0}\right]$. Consequently, inequality (3.46) holds and

$$\nu^* \in A, \tag{3.50}$$

because the set $\mathcal{V}^-(\omega)$ is open (see Lemma 3.2).

Now let $\{\nu_n\}_{n=1}^{+\infty} \subseteq]0, \nu^*[$ be an increasing sequence such that

$$\lim_{n\to+\infty} \nu_n = \nu^*.$$

Clearly, $p_{\nu_n} \in \mathcal{V}^-(\omega)$ for $n \in \mathbb{N}$. By virtue of Lemma 3.10 with $g(t) := p_{\nu_n}(t)$ and $f(t) := -1$, for any $n \in \mathbb{N}$, the problem

$$u'' = p_{\nu_n}(t)u - 1; \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{3.51}$$

has a unique solution u_n and this solution satisfies

$$u_n(t) > 0 \quad \text{for } t \in [0, \omega]. \tag{3.52}$$

Observe that from (3.43) we get

$$p_{\nu_{n+1}}(t) \le p_{\nu_n}(t) \quad \text{for a. e. } t \in [0, \omega],\ n \in \mathbb{N} \tag{3.53}$$

and, moreover,

$$\lim_{n\to+\infty} \|p_{\nu_n} - p_{\nu^*}\|_L = 0 \tag{3.54}$$

because the function f is continuous in the second argument. Therefore, in view of inequalities (3.52) and (3.53), it follows from (3.51) that

$$(u_n(t) - u_{n+1}(t))'' \ge p_{\nu_{n+1}}(t)(u_n(t) - u_{n+1}(t)) \quad \text{for a. e. } t \in [0, \omega],\ n \in \mathbb{N}.$$

However, the inclusion $p_{\nu_{n+1}} \in \mathcal{V}^-(\omega)$ holds for every $n \in \mathbb{N}$ and thus, the latter inequality yields that

$$u_n(t) \le u_{n+1}(t) \quad \text{for } t \in [0, \omega],\ n \in \mathbb{N}. \tag{3.55}$$

It follows from (3.52) and (3.55) that the sequence $\{\|u_n\|_C\}_{n=1}^{+\infty}$ is non-decreasing. We show that

$$\lim_{n\to+\infty} \|u_n\|_C = +\infty. \tag{3.56}$$

Indeed, if the sequence $\{\|u_n\|_C\}_{n=1}^{+\infty}$ has a finite limit then, by virtue of (3.54), (3.55), and Lemma 3.13 (with $g_n(t) := p_{\nu_n}(t)$, $g(t) := p_{\nu^*}(t)$, $f_n(t) := -1$, $f(t) := -1$), we obtain

$$\lim_{n\to+\infty} u_n^{(i)}(t) = u_0^{(i)}(t) \quad \text{uniformly on } [0, \omega],\ i = 0, 1, \tag{3.57}$$

where $u_0 \in AC^1([0, \omega])$ satisfies

$$\begin{gathered} u_0''(t) = p_{\nu^*}(t)u_0(t) - 1 \quad \text{for a. e. } t \in [0, \omega], \\ u_0(0) = u_0(\omega), \quad u_0'(0) = u_0'(\omega). \end{gathered} \tag{3.58}$$

Moreover, it follows from (3.55) and (3.57) that

$$u_0(t) \ge u_1(t) > 0 \quad \text{for } t \in [0, \omega].$$

Therefore, from (3.58) and Lemma 3.7 with $g(t) := p_{\nu^*}(t)$ we get the inclusion $p_{\nu^*} \in \mathcal{V}^-(\omega)$ which contradicts condition (3.50). The contradiction obtained proves that relation (3.56) holds.

Put

$$v_n(t) := \frac{u_n(t)}{\|u_n\|_C}, \quad f_n(t) := \frac{1}{\|u_n\|_C} \quad \text{for } t \in [0, \omega],\ n \in \mathbb{N}.$$

Then for any $n \in \mathbb{N}$, the function v_n is a solution to the problem

$$v'' = p_{\nu_n}(t)v - f_n(t); \quad v(0) = v(\omega), \ v'(0) = v'(\omega).$$

It is clear that

$$\|v_n\|_C = 1 \quad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to+\infty} \|f_n\|_L = 0. \tag{3.59}$$

Therefore, by virtue of (3.54), (3.59), and Lemma 3.13 (with $g_n(t) := p_{\nu_n}(t)$, $g(t) := p_{\nu^*}(t)$, $f(t) := 0$), we can assume without loss of generality that

$$\lim_{n\to+\infty} v_n^{(i)}(t) = v_0^{(i)}(t) \quad \text{uniformly on } [0,\omega], \ i = 0, 1, \tag{3.60}$$

where $v_0 \in AC^1([0,\omega])$ satisfies

$$v_0''(t) = p_{\nu^*}(t)v_0(t) \quad \text{for a. e. } t \in [0,\omega], \quad v_0(0) = v_0(\omega), \quad v_0'(0) = v_0'(\omega). \tag{3.61}$$

Moreover, it follows from (3.52), (3.59), and (3.60) that

$$v_0(t) \geq 0 \quad \text{for } t \in [0,\omega], \qquad v_0 \not\equiv 0.$$

Consequently, we have

$$v_0(t) > 0 \quad \text{for } t \in [0,\omega]. \tag{3.62}$$

If we extend the function v_0 periodically to the whole real axis, in view of Sturm's separation theorem and Lemma 3.6 with $p(t) := p_{\nu^*}(t)$, we get

$$p_{\nu^*} \in \operatorname{Int} \mathcal{D}(\omega). \tag{3.63}$$

Finally, we put

$$B := \left\{ \nu \geq \nu^* : \int_0^\omega f(s,\nu)\mathrm{d}s = \int_0^\omega f(s,\nu^*)\mathrm{d}s \right\}$$

and

$$\nu_0 := \sup B.$$

Clearly, assumption (3.45) yields that $\nu^* \leq \nu_0 < +\infty$. Moreover, since the Carathéodory function f satisfies condition (3.43), we have $\nu_0 \in B$,

$$f(t,\nu^*) = f(t,\nu_0) \quad \text{for a. e. } t \in [0,\omega], \tag{3.64}$$

and

$$\int_0^\omega f(s,\nu)\mathrm{d}s > \int_0^\omega f(s,\nu_0)\mathrm{d}s \quad \text{for } \nu > \nu_0.$$

Consequently, $p_{\nu^*} \equiv p_{\nu_0}$ and, in view of inclusion (3.63), there exists $\delta > 0$ such that

$$p_{\nu_0+\delta} \in \operatorname{Int} \mathcal{D}(\omega) \tag{3.65}$$

and

$$f(t,\nu_0+\delta) - f(t,\nu_0) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \quad f(\cdot,\nu_0+\delta) \not\equiv f(\cdot,\nu_0). \tag{3.66}$$

On the other hand, by virtue of (3.62), (3.64), and (3.66), it follows from (3.61) that

$$\begin{aligned} v_0''(t) &= \big(p(t) - f(t,\nu^*)\big)v_0(t) \\ &= p_{\nu_0+\delta}(t)v_0(t) + \big(f(t,\nu_0+\delta) - f(t,\nu_0)\big)v_0(t) \\ &\geq p_{\nu_0+\delta}(t)v_0(t) \end{aligned}$$

for a. e. $t \in [0,\omega]$ and

$$\operatorname{meas}\{t \in [0,\omega] : v_0''(t) > p_{\nu_0+\delta}(t)v_0(t)\} > 0.$$

Consequently, by virtue of inclusion (3.65) and Lemma 3.8 with $g(t) := p_{\nu_0+\delta}(t)$, we get $p_{\nu_0+\delta} \in \operatorname{Int} \mathcal{V}^+(\omega)$, i. e., the assertion of the proposition holds for $r := \nu_0+\delta$. $\square$

The last two statements of this section deal with the existence of functions α and β appearing in Lemma 3.17 which are usually called lower and upper functions of problem (1.1), respectively.

Proposition 3.19. *Let $p \in \mathcal{V}^-(\omega)$ and $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying hypothesis (H_1). Let moreover, there exist a number $r > 0$ such that*

$$p - q_0(\cdot, r) \in \operatorname{Int} \mathcal{V}^+(\omega). \tag{3.67}$$

Then for any $c \geq r$, there is a function $\alpha \in AC^1([0,\omega])$ satisfying inequality (2.2) *and*

$$\alpha(0) = \alpha(\omega), \qquad \alpha'(0) = \alpha'(\omega), \tag{3.68}$$

$$\alpha(t) \geq c \quad \text{for } t \in [0,\omega]. \tag{3.69}$$

Proof. Let $\nu > 0$ be the number appearing in the assertion of Lemma 3.9 with $g(t) := p(t) - q_0(t,r)$ and let $c \geq r$ be arbitrary. Then, in view of assumption (3.67), it follows from Lemma 3.9 that the problem

$$\alpha'' = \big(p(t) - q_0(t,r)\big)\alpha + \frac{c}{\nu\omega}\,; \quad \alpha(0) = \alpha(\omega),\ \alpha'(0) = \alpha'(\omega)$$

has a unique solution α and this solution satisfies inequality (3.69). Consequently, (3.68) holds and since $c \geq r$, hypothesis (H_1) guarantees that the function α satisfies relation (2.2), as well. $\square$

Proposition 3.20. *Let $p \in \mathcal{V}^-(\omega)$ and $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$q(t,0) = 0 \quad \text{for a. e. } t \in [0,\omega]. \tag{3.70}$$

Then for any $c > 0$, there exists a function $\beta \in AC^1([0,\omega])$ satisfying inequality (3.22) *and*

$$\beta(0) = \beta(\omega), \quad \beta'(0) = \beta'(\omega), \tag{3.71}$$

$$0 < \beta(t) \leq c \quad \text{for } t \in [0,\omega]. \tag{3.72}$$

Proof. Since q is a Carathéodory function with property (3.70), there exist a non-negative function $h \in L(0,\omega)$ and a non-negative, non-decreasing function $\varphi \in C([0,+\infty[)$ such that $\varphi(0) = 0$ and

$$|q(t,x)| \leq h(t)\varphi(|x|) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R},\ |x| \leq 1. \tag{3.73}$$

We first show that there is a number $\varepsilon_0 \in]0,1]$ such that

$$p - h\varphi(\varepsilon) \in \mathcal{V}^-(\omega) \quad \text{for every } \varepsilon \in [0,\varepsilon_0]. \tag{3.74}$$

Indeed, assume on the contrary that there exists a sequence $\{\varepsilon_n\}_{n=1}^{+\infty} \subset]0,1]$ such that

$$p - h\varphi(\varepsilon_n) \notin \mathcal{V}^-(\omega) \quad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to+\infty} \varepsilon_n = 0.$$

Since $\varphi(0) = 0$, it is clear that

$$\lim_{n\to+\infty} \int_0^\omega |h(s)\varphi(\varepsilon_n)|\mathrm{d}s = 0.$$

Consequently, we get $p \notin \mathcal{V}^-(\omega)$ because the set $\mathcal{V}^-(\omega)$ is open (see Lemma 3.2). However, it contradicts the assumption $p \in \mathcal{V}^-(\omega)$.

Now let $c > 0$ be arbitrary and $\delta := \min\{\varepsilon_0, c\}$. Let, moreover, $\nu, \Delta > 0$ be numbers appearing in the assertion of Lemma 3.10 with $g(t) := p(t) - h(t)\varphi(\delta)$. Then, in view of (3.74), it follows from Lemma 3.10 that the problem

$$\beta'' = \big(p(t) - h(t)\varphi(\delta)\big)\beta - \frac{\delta}{\Delta\omega}\,; \quad \beta(0) = \beta(\omega),\ \beta'(0) = \beta'(\omega)$$

has a unique solution β and

$$\frac{\nu}{\Delta}\,\delta \le \beta(t) \le \delta \quad \text{for } t \in [0,\omega].$$

Taking now into account that the function φ is non-decreasing and inequality (3.73) holds, we easily conclude that the function β satisfies (3.22), (3.71), and (3.72). □

4. Proofs of main results

Proof of Theorem 2.1. According to Proposition 3.20, there exists a function $\beta \in AC_u([0,\omega])$ satisfying inequalities (3.22), (3.71), and

$$0 < \beta(t) \le \alpha(t) \quad \text{for } t \in [0,\omega].$$

Consequently, all the assumptions of Lemma 3.17 are fulfilled and thus, problem (1.1) has at least one positive solution u satisfying relation (2.4). □

Proof of Corollary 2.2. By virtue of Theorem 2.1, to prove the corollary it is sufficient to show that, in both cases (a) and (b), there exists a function $\alpha \in AC_\ell([0,\omega])$ satisfying relations (2.1), (2.2), and (2.3).

If condition (a) is fulfilled then it is clear that the constant function $\alpha(t) := c$ satisfies (2.1), (2.2), and (2.3).

On the other hand, if condition (b) holds then the existence of a function α fulfilling (2.1), (2.2), and (2.3) follows from Proposition 3.19. □

Proof of Corollary 2.3. Observe that hypothesis (H_1) and assumption $q(\cdot,0) \equiv 0$ yield that

$$q_0(t,0) \le 0 \quad \text{for a. e. } t \in [0,\omega].$$

Therefore, according to Proposition 3.18 with $f(t,x) := q_0(t,x)$, there exists $r > 0$ such that $p - q_0(\cdot,r) \in \operatorname{Int}\mathcal{V}^+(\omega)$. Hence, the assertion of the corollary follows from Corollary 2.2(b). □

Proof of Proposition 2.5. Assume that problem (1.1) has a positive solution u. Then there are numbers $u^* > u_* > 0$ such that

$$u_* \le u(t) \le u^* \quad \text{for } t \in [0,\omega]$$

and, by virtue of hypothesis (H_2), we have

$$q(t,u(t)) \ge h_{u_*u^*}(t) \quad \text{for a. e. } t \in [0,\omega].$$

Therefore, we get

$$u''(t) \le p(t)u(t) - u_* h_{u_*u^*}(t) \le p(t)u(t) \quad \text{for a. e. } t \in [0,\omega]$$

and

$$\operatorname{meas}\{t \in [0,\omega] : u''(t) < p(t)u(t)\} > 0.$$

Hence, it follows from Lemma 3.7 with $g(t) := p(t)$ and $\gamma(t) := u(t)$ that $p \in \mathcal{V}^-(\omega)$. □

Proof of Proposition 2.6. According to hypothesis (H_3), one can show that

$$\text{the function } q(t,\cdot)\colon [0,+\infty[\to\mathbb{R} \text{ is non-decreasing for a.e. } t\in[0,\omega] \tag{4.1}$$

which together with assumption (2.9) yields that

$$q(t,x)\geq 0 \quad \text{for a.e. } t\in[0,\omega] \text{ and all } x\geq 0. \tag{4.2}$$

Suppose on the contrary that the assertion of the proposition is violated. Then we can assume without loss of generality that either

(a)

$$u(t)>v(t) \quad \text{for } t\in[0,\omega],$$

or

(b) there exists $t_0\in[0,\omega]$ such that

$$u(t)\geq v(t) \quad \text{for } t\in[0,\omega], \quad u(t_0)=v(t_0), \quad u'(t_0)=v'(t_0). \tag{4.3}$$

In the case (a), there exist positive numbers v_*, v^*, e_0 such that

$$u(t)\geq v(t)+e_0, \quad v_*\leq v(t)\leq v^* \quad \text{for } t\in[0,\omega]. \tag{4.4}$$

Therefore, in view of condition (4.1) and hypothesis (H_3), for a.e. $t\in[0,\omega]$ we get

$$q(t,u(t))-q(t,v(t))\geq q(t,v(t)+e_0)-q(t,v(t))\geq h_{v_*v^*e_0}(t). \tag{4.5}$$

On the other hand, it follows immediately from the equation in (1.1) that u and v are periodic solutions, respectively, to the equations

$$\begin{aligned} z''&=\big(p(t)-q(t,v(t))\big)z-\big[q(t,u(t))-q(t,v(t))\big]u(t),\\ z''&=\big(p(t)-q(t,v(t))\big)z \end{aligned}$$

and thus, by virtue of (4.4) and (4.5), Fredholm's third theorem yields the contradiction

$$0=\int_0^\omega \big[q(s,u(s))-q(s,v(s))\big]u(s)v(s)\mathrm{d}s\geq (v_*+e_0)v_*\int_0^\omega h_{v_*v^*e_0}(s)\mathrm{d}s>0.$$

In the case (b), by virtue of conditions (4.1), (4.2), and (4.3), we have

$$u(t)q(t,u(t))\geq v(t)q(t,v(t)) \quad \text{for a.e. } t\in[0,\omega]. \tag{4.6}$$

Put

$$w(t):=u(t)-v(t) \quad \text{for } t\in[0,\omega].$$

Clearly, the function w is a solution to the periodic problem

$$w''=p(t)w-\big[u(t)q(t,u(t))-v(t)q(t,v(t))\big]; \quad w(0)=w(\omega),\ w'(0)=w'(\omega).$$

If $u(\cdot)q(\cdot,u(\cdot))\not\equiv v(\cdot)q(\cdot,v(\cdot))$ then, in view of (2.5) and (4.6), it follows from Lemma 3.10 with $g(t):=p(t)$ that $w(t)>0$ for $t\in[0,\omega]$, which contradicts (4.3).

On the other hand, if $u(\cdot)q(\cdot,u(\cdot))\equiv v(\cdot)q(\cdot,v(\cdot))$ then, in view of (4.3), the function w is a solution to the initial value problem

$$w''=p(t)w; \quad w(t_0)=0,\ w'(t_0)=0.$$

Consequently, we get $w\equiv 0$ which is in a contradiction with the assumption that u, v are distinct solutions to problem (1.1). $\square$

Proof of Theorem 2.7. Put

$$q(t,x) := h(t)\varphi(x) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R}.$$

It is clear that q is a Carathéodory function satisfying $q(\cdot,0) \equiv 0$.

Assume that condition (i) holds. It follows from assumption (2.13) that there exists a number $\varphi_0 \leq 0$ such that $\varphi(x) \geq \varphi_0$ for $x \geq 0$. Therefore, in view of (2.12), the function q satisfies hypothesis (H_1) with $q_0(t,x) := h(t)\varphi_0$. Moreover, (2.14) implies the validity of (2.6) and thus, Corollary 2.2(a) yields that problem (2.11) has at least one positive solution.

Now assume that condition (ii) is satisfied. Let

$$\psi(x) := \inf\{\varphi(z) : z \in [x,+\infty[\} \quad \text{for } x \geq 0.$$

One can easily verify that the function ψ is well defined. Moreover, the function ψ is continuous, non-decreasing, and

$$\varphi(x) \geq \psi(x) \quad \text{for } x \geq 0, \qquad \lim_{x\to+\infty} \psi(x) = +\infty.$$

Consequently, in view of assumption (2.12), the function q satisfies hypothesis (H_1) with $q_0(t,x) := h(t)\psi(x)$ and condition (2.7) is fulfilled. Therefore, Corollary 2.3 yields that problem (2.11) has at least one positive solution.

Assume, in addition, that the function φ is increasing. Then, in view of (2.12), the function q satisfies hypothesis (H_3) and $q(\cdot,0) \equiv 0$. Consequently, if u, v are two distinct positive solutions to problem (2.11) then Proposition 2.6 yields that (2.10) holds with some $t_1, t_2 \in [0,\omega]$. □

Proof of Proposition 2.10. Assume on the contrary that problem (2.21) has a positive solution u such that $u \not\equiv 1$. Let

$$q(t,x) := p(t)|x|^{\lambda-1} \quad \text{for a. e. } t \in [0,\omega] \text{ and } x \in \mathbb{R}.$$

In view of assumption (2.18), the function q satisfies hypothesis (H_3) and $q(\cdot,0) \equiv 0$. Therefore, Proposition 2.6 yields that there exist $t_1, t_2 \in [0,\omega]$ such that

$$u(t_1) < 1, \qquad u(t_2) > 1. \tag{4.7}$$

It immediately follows from (2.21) that

$$\int_0^\omega p(s)u(s)\mathrm{d}s = \int_0^\omega p(s)u^\lambda(s)\mathrm{d}s.$$

Hence, by Hölder's inequality, we get

$$\begin{aligned}\int_0^\omega p(s)u^\lambda(s)\mathrm{d}s &= \int_0^\omega p^{\frac{\lambda-1}{\lambda}}(s)p^{\frac{1}{\lambda}}(s)u(s)\mathrm{d}s \\ &\leq \left(\int_0^\omega p(s)\mathrm{d}s\right)^{\frac{\lambda-1}{\lambda}} \left(\int_0^\omega p(s)u^\lambda(s)\mathrm{d}s\right)^{\frac{1}{\lambda}}\end{aligned}$$

and thus, the inequality

$$\int_0^\omega p(s)u^\lambda(s)\mathrm{d}s \leq \int_0^\omega p(s)\mathrm{d}s \tag{4.8}$$

holds. On the other hand, Hölder's inequality yileds that

$$\int_0^\omega p(s)u^\mu(s)\mathrm{d}s = \int_0^\omega p^{\frac{\lambda-\mu}{\lambda}}(s)p^{\frac{\mu}{\lambda}}(s)u(s)^\mu\mathrm{d}s$$

$$\leq \left(\int_0^\omega p(s)\mathrm{d}s\right)^{\frac{\lambda-\mu}{\lambda}} \left(\int_0^\omega p(s)u^\lambda(s)\mathrm{d}s\right)^{\frac{\mu}{\lambda}} \quad \text{for } \mu \in \,]0,\lambda[\,.$$

Consequently, by virtue of (4.8), we get

$$\int_0^\omega p(s)u^\mu(s)\mathrm{d}s \leq \int_0^\omega p(s)\mathrm{d}s \quad \text{for } \mu \in [0,\lambda]. \tag{4.9}$$

Put

$$\widetilde{p}(t) := p(t)\psi(u(t)) \quad \text{for a. e. } t \in [0,\omega], \tag{4.10}$$

where

$$\psi(x) := \begin{cases} \frac{x^\lambda - x}{x-1} & \text{for } x \in [0,1[\,\cup\,]1,+\infty[\,, \\ \lambda - 1 & \text{for } x = 1. \end{cases}$$

It is clear that the function ψ is continuous and non-negative on $[0,+\infty[$ and thus, we have $\widetilde{p} \in L([0,\omega])$ and

$$\widetilde{p}(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega]. \tag{4.11}$$

First suppose that $\lambda \in \,]1,2[$ and put

$$n := \Big\lfloor \frac{1}{\lambda - 1} \Big\rfloor. \tag{4.12}$$

Obviously, $n \in \mathbb{N}$ and

$$\lambda - 1 \leq \tfrac{1}{n}\,. \tag{4.13}$$

Therefore, we have

$$\frac{u^{\lambda-1}(t) - 1}{u(t) - 1} \leq \frac{u^{\frac{1}{n}}(t) - 1}{u(t) - 1} = \frac{1}{\sum_{k=0}^{n-1} u^{\frac{k}{n}}(t)} \quad \text{for } t \in [0,\omega],\ u(t) \neq 1$$

and thus, Lemma 3.11 yields that

$$\psi(u(t)) \leq \frac{u(t)}{\sum_{k=0}^{n-1} u^{\frac{k}{n}}(t)} \leq \frac{1}{n}\, u^{\frac{n+1}{2n}}(t) \quad \text{for } t \in [0,\omega],\ u(t) \neq 1.$$

Taking now into account (4.13), from the latter inequality we get

$$\psi(u(t)) \leq \frac{1}{n}\, u^{\frac{n+1}{2n}}(t) \quad \text{for } t \in [0,\omega]. \tag{4.14}$$

Hence, in view of (2.19), (2.20), (4.9), (4.12), and (4.14), notation (4.10) yields that

$$\int_0^\omega \widetilde{p}(s)\mathrm{d}s \leq \frac{1}{n}\int_0^\omega p(s)u^{\frac{n+1}{2n}}(s)\mathrm{d}s \leq \frac{1}{\left\lfloor \frac{1}{\lambda-1} \right\rfloor}\int_0^\omega p(s)\mathrm{d}s \leq \frac{16}{\omega}\,. \tag{4.15}$$

On the other hand, the function $u - 1$ is a nontrivial solution to the problem

$$v'' = -\widetilde{p}(t)v; \quad v(0) = v(\omega), \quad v'(0) = v'(\omega) \tag{4.16}$$

with at least two zeros in the interval $[0,\omega]$ (see (4.7)). Therefore, taking into account inequality (4.11), it follows from Lemma 3.12 that

$$\int_0^\omega \widetilde{p}(s)\mathrm{d}s > \frac{16}{\omega}\,, \tag{4.17}$$

which is in a contradiction with (4.15).

Now suppose that $\lambda \geq 2$ and put

$$n := \lceil \lambda - 1 \rceil. \tag{4.18}$$

Obviously, $n \in \mathbb{N}$ and
$$\lambda - 1 \leq n. \tag{4.19}$$
Therefore, we have
$$\frac{u^{\lambda-1}(t) - 1}{u(t) - 1} \leq \frac{u^n(t) - 1}{u(t) - 1} = \sum_{k=0}^{n-1} u^k(t) \quad \text{for } t \in [0, \omega],\ u(t) \neq 1$$
and thus, taking into account (4.19), from the latter inequality we get
$$\psi(u(t)) \leq \sum_{k=0}^{n-1} u^{k+1}(t) \quad \text{for } t \in [0, \omega]. \tag{4.20}$$
Hence, in view of (2.19), (2.20), (4.9), (4.18), and (4.20), notation (4.10) yields that
$$\int_0^\omega \widetilde{p}(s)\mathrm{d}s \leq \sum_{k=0}^{n-1} \int_0^\omega p(s)u^{k+1}(s)\mathrm{d}s \leq n \int_0^\omega p(s)\mathrm{d}s \leq \frac{16}{\omega}\,. \tag{4.21}$$

On the other hand, the function $u - 1$ is a nontrivial solution to problem (4.16) with at least two zeros in the interval $[0, \omega]$ (see (4.7)). Therefore, taking into account (4.11), it follows from Lemma 3.12 that inequality (4.17) holds, which contradicts (4.21). □

Proof of Corollary 2.11. Put
$$q(t, x) := h(t)|x|^{\lambda-1} \quad \text{for a. e. } t \in [0, \omega] \text{ and } x \in \mathbb{R}.$$

Assertion (1)*:* In view of assumption (2.12), the function q satisfies hypotheses (H_1) with $q_0(t, x) := h(t)x^{\lambda-1}$ and (H_2). Consequently, assertion (1) of the corollary follows from Corollary 2.3 and Proposition 2.5.

Assertion (2)*:* By virtue of assumption (2.12), hypothesis (H_3) is fulfilled and $q(\cdot, 0) \equiv 0$. Therefore, assertion (2) of the corollary follows from Proposition 2.6.

Assertion (3)*:* Assume that $p \in \mathcal{V}^-(\omega)$ and inequality (2.22) holds. Then the above-proved assertion (1) yields that problem (1.4) has a positive solution u_0. It remains to show that this problem has no other positive solution. Indeed, let
$$\gamma(t) := \int_0^t \frac{1}{u_0^2(s)}\,\mathrm{d}s \quad \text{for } t \in [0, \omega], \qquad \omega_0 := \int_0^\omega \frac{1}{u_0^2(s)}\,\mathrm{d}s,$$
and
$$g(z) := h(\gamma^{-1}(z))u_0^{\lambda+3}(\gamma^{-1}(z)) \quad \text{for a. e. } z \in [0, \omega_0].$$
Then the function g is well defined, $g \in L([0, \omega_0])$, and
$$g(z) \geq 0 \quad \text{for a. e. } z \in [0, \omega_0], \qquad g \not\equiv 0, \tag{4.22}$$
because the solution u_0 is positive and the function h satisfies assumption (2.12).

On the interval $[0, \omega_0]$, we consider the periodic problem
$$v'' = g(z)v\big(1 - |v|^{\lambda-1}\big); \quad v(0) = v(\omega_0), \quad v'(0) = v'(\omega_0). \tag{4.23}$$
One can easily verify that if u is a positive solution to problem (1.4) then the function
$$v(z) := \frac{u(\gamma^{-1}(z))}{u_0(\gamma^{-1}(z))} \quad \text{for } z \in [0, \omega_0]$$
is a positive solution to problem (4.23). Moreover, we have
$$\int_0^{\omega_0} g(\xi)\mathrm{d}\xi = \int_0^{\omega_0} h(\gamma^{-1}(\xi))u_0^{\lambda+3}(\gamma^{-1}(\xi))\mathrm{d}\xi = \int_0^\omega h(s)u_0^{\lambda+1}(s)\mathrm{d}s$$

and thus, in view of assumption (2.22), Lemma 3.16 with $\ell(t) := p(t)$ and $u(t) := u_0(t)$ yields that

$$\int_0^{\omega_0} g(\xi)\mathrm{d}\xi \leq \frac{2}{\omega_0}\, \mathrm{e}^{-1+\sqrt{1+\frac{\omega}{4}\int_0^\omega p(s)\mathrm{d}s}} \left(-1+\sqrt{1+\frac{\omega}{4}\int_0^\omega p(s)\mathrm{d}s}\right) \leq \frac{16\lambda^*}{\omega_0}\,,$$

where the number λ^* is defined by formula (2.20). Therefore, in view of (4.22), it follows from Proposition 2.10 with $p(t) := g(z)$ that problem (4.23) has the unique positive solution $v(z) := 1$ for $z \in [0, \omega_0]$. However, it guarantees that u_0 is a unique positive solution to problem (1.4). □

Proof of Theorem 2.13. Put

$$q(t, x) := h(t)|x|^{\lambda-1} - f(t)|x|^{\mu-1} \quad \text{for a. e. } t \in [0, \omega] \text{ and all } x \in \mathbb{R}. \tag{4.24}$$

By virtue of assumptions (2.12) and (2.24), we have

$$\begin{aligned} q(t, x) &\geq x^{\mu-1}\big(h(t)x^{\lambda-\mu} - [f(t)]_+\big) \\ &\geq h(t)x^{\mu-1}\big(x^{\lambda-\mu} - c\big) \\ &\geq h(t)\psi(x) \quad \text{for a. e. } t \in [0, \omega] \text{ and all } x \geq 0, \end{aligned}$$

where

$$\psi(x) := \begin{cases} -\frac{\lambda-\mu}{\lambda-1}\left[\frac{\mu-1}{\lambda-1}\right]^{\frac{\mu-1}{\lambda-\mu}} c^{\frac{\lambda-1}{\lambda-\mu}} & \text{for } 0 \leq x \leq \left[\frac{\mu-1}{\lambda-1}\, c\right]^{\frac{1}{\lambda-\mu}}, \\ x^{\mu-1}\big(x^{\lambda-\mu} - c\big) & \text{for } x > \left[\frac{\mu-1}{\lambda-1}\, c\right]^{\frac{1}{\lambda-\mu}}. \end{cases}$$

Consequently, the function q satisfies $q(\cdot, 0) \equiv 0$ and hypothesis (H_1) with $q_0(t, x) := h(t)\psi(x)$ and, in view of (2.12), condition (2.7) holds. Therefore, the assertion of the theorem follows from Corollary 2.3. □

Proof of Theorem 2.15. Let the function q be defined by formula (4.24). It is clear that $q(\cdot, 0) \equiv 0$ and, in view of assumption (2.27), we get

$$q(t, x) \geq h(t)x^{\mu-1}\left(x^{\lambda-\mu} - \frac{[f(t)]_+}{h(t)}\right) \quad \text{for a. e. } t \in [0, \omega] \text{ and all } x \geq 0.$$

Now for a. e. $t \in [0, \omega]$ and all $x \geq 0$, we put

$$q_0(t, x) := \begin{cases} -\frac{\lambda-\mu}{\lambda-1}\left[\frac{\mu-1}{\lambda-1}\right]^{\frac{\mu-1}{\lambda-\mu}} [f(t)]_+^{\frac{\lambda-1}{\lambda-\mu}} h^{-\frac{\mu-1}{\lambda-\mu}}(t) & \text{if } 0 \leq x \leq \left[\frac{\mu-1}{\lambda-1}\,\frac{[f(t)]_+}{h(t)}\right]^{\frac{1}{\lambda-\mu}}, \\ h(t)x^{\lambda-1} - [f(t)]_+ x^{\mu-1} & \text{if } x > \left[\frac{\mu-1}{\lambda-1}\,\frac{[f(t)]_+}{h(t)}\right]^{\frac{1}{\lambda-\mu}}. \end{cases}$$

Then, by virtue of assumption (2.28), one can verify that $q_0 \colon [0, \omega] \times [0, +\infty[\to \mathbb{R}$ is a Carathéodory function and hypothesis (H_1) holds. Moreover, the function q_0 satisfies

$$\lim_{x \to +\infty} q_0(t, x) = +\infty \quad \text{for a. e. } t \in [0, \omega]$$

which, in view of Remark 2.4, yields that (2.7) is fulfilled. Therefore, the assertion of the theorem follows from Corollary 2.3. □

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Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2, 616 69 Brno, Czech Republic

Institute of Mathematics, Czech Academy of Sciences, branch in Brno, Žižkova 22, 616 62 Brno, Czech Republic

E-mail address: `lomtatidze@fme.vutbr.cz`

Institute of Mathematics, Czech Academy of Sciences, branch in Brno, Žižkova 22, 616 62 Brno, Czech Republic

E-mail address: `sremr@ipm.cz`