PHASES OF LINEAR DIFFERENCE EQUATIONS AND SYMPLECTIC SYSTEMS

Zuzana Došlá, Denisa Škrabáková, Brno

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Abstract. The second order linear difference equation

$$\Delta(r_k \Delta x_k) + c_k x_{k+1} = 0, \tag{1}$$

where $r_k \neq 0$ and $k \in \mathbb{Z}$, is considered as a special type of symplectic systems. The concept of the phase for symplectic systems is introduced as the discrete analogy of the Borůvka concept of the phase for second order linear differential equations. Oscillation and nonoscillation of (1) and of symplectic systems are investigated in connection with phases and trigonometric systems. Some applications to summation of number series are given, too.

Keywords: second order linear difference equation, symplectic system, phase, oscillation, nonoscillation, trigonometric transformation

1. Introduction

In the fifties, O. Borůvka developed an original and fruitful theory of global transformation of linear differential equations of the second order in the real domain. To this purpose he introduced the phase theory of these equations and using it he solved some open problems concerning the qualitative theory of a global character. These results were surveyed in the monograph [5] and were extended in several directions—for linear differential equations of an arbitrary order by F. Neuman [8], for second order linear differential equation in the complex domain by S. Staněk [9], for linear differential systems by O. Došlý [6]. Concerning the extensive literature on this topic

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we refer to [5], [8], [10] and references therein. Here we show how it is possible to extend some results of the Borůvka theory of phases to the discrete case.

We consider a linear difference equation of the second order

$$\Delta(r_k \Delta x_k) + c_k x_{k+1} = 0, \qquad k \in \mathbb{Z} \tag{1}$$

where (r_k), (c_k) are sequences of real numbers such that $r_k \neq 0$, and a symplectic difference system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \qquad k \in \mathbb{Z} \tag{S}$$

where (a_k), (b_k), (c_k), (d_k) are sequences of real numbers such that $a_k d_k - b_k c_k = 1$. Under this assumption the matrix S_k in (S) is symplectic, i.e. it satisfies

$$S_k^T J S_k = J \qquad \text{with} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where ^{T} stands for the transpose of the matrix indicated.

Difference equation (1) can be written as a system of two equations of the first order for $(x_k, u_k) = (x_k, r_k \Delta x_k)$

$$\Delta x_k = \frac{1}{r_k} u_k, \qquad \Delta u_k = -c_k x_{k+1}$$

and, in turn, as a symplectic system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{r_k} \\ -c_k & 1 - \frac{c_k}{r_k} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}. \tag{G}$$

The aim of this paper is to introduce phases of difference equations and symplectic systems in a similar manner as for differential equations, to classify symplectic systems with respect to their oscillatory properties and to study oscillatory and nonoscillatory properties of these systems on $\mathbb{Z}$ in terms of phases.

The plan of the paper is the following. In §2 we explain why we consider equation (1) as a special case of symplectic system (S)—the reason is the discrepancy between the reciprocity of the linear differential and difference equations of the second order. The lack of a convenient class of self-reciprocal difference equations makes it necessary to consider the wider class of equations than equations of the form (1) and, as we will show, such a class is the class of symplectic difference systems (S). The basic notions for (S) and some properties of the self-reciprocal systems, which are called trigonometric systems, are also given in §2. §3 is devoted to the trigonometric transformation and further properties of trigonometric systems. The concept of the phase for symplectic difference systems is introduced in §4. In §5 oscillatory and nonoscillatory properties of these systems are described in terms of phases and an application of phases to the summation of number series is given.

2. Preliminaries

An important role in the Borůvka theory of phases for second order differential equations is played by the property of reciprocity and self-reciprocity. So, let us compare the reciprocity in the continuous and the discrete case.

Consider a second order linear differential equation

$$(r(t)x')' + c(t)x = 0, \qquad t \in (a,b), \tag{2}$$

where r, c are real-valued continuous functions on (a,b), $r(t) > 0$, and $-\infty \leqslant a < b \leqslant \infty$. If x is a solution of (2) and $c(t) \neq 0$, then $y = rx'$ is a solution of the *reciprocal* equation to (2)

$$\left(\frac{1}{c(t)}y'\right)' + \frac{1}{r(t)}y = 0.$$

An equation is said to be *self-reciprocal* if it coincides with its reciprocal equation.

The starting point of the Borůvka phase and transformation theory for differential equations (2) is based on the fact that in the class of all globally equivalent (transformable) differential equations (2) there exists a self-reciprocal equation which is chosen as the canonical form of this class. More precisely, if x_1, x_2 are two linearly independent solutions of (2) with the Wronskian $w \equiv r(x_1'x_2 - x_1x_2') \equiv 1$ and $h(t) = \sqrt{x_1^2(t) + x_2^2(t)}$, then the transformation $x(t) = h(t)y(t)$ transforms equation (2) into the self-reciprocal equation

$$\left(\frac{1}{q(t)}u'\right)' + q(t)u = 0, \quad q(t) = \frac{1}{r(t)h^2(t)}. \tag{q}$$

The transformation of the independent variable $s(t) = \int^t q(\tau)\,\mathrm{d}\tau$ transforms this equation into the equation $y''(s) + y(s) = 0$, $s \in (\alpha, \beta)$, therefore solutions of (2) are functions

$$x_1(t) = h(t)\sin\int^t q(\tau)\,\mathrm{d}\tau, \quad x_2(t) = h(t)\cos\int^t q(\tau)\,\mathrm{d}\tau. \tag{3}$$

In the discrete case, if $c_k \neq 0$, $r_k \neq 0$ and x_k is a solution of (1) then $y_k = r_k\Delta x_k$ is a solution of the *reciprocal* equation

$$\Delta\left(\frac{1}{c_k}\Delta y_k\right) + \frac{1}{r_{k+1}}y_{k+1} = 0.$$

The form of the reciprocal equation shows that no difference equation of the form (1) is self-reciprocal except the equation $\Delta^2 x_k + x_{k+1} = 0$ which is oscillatory on $\mathbb{Z}$ and so does not seem to be a good representative of nonoscillatory equations on $\mathbb{Z}$. This fact makes the main difference in establishing the phase theory for continuous and discrete equations.

In the sequel, we will need the following definitions (see e.g. [1]) and auxiliary results for 2×2 symplectic systems (S).

A pair of solutions $z^{[1]} = \binom{x_k}{u_k}^{[1]}$, $z^{[2]} = \binom{x_k}{u_k}^{[2]}$ of (S) with the Casoratian $w \equiv x_k^{[1]} u_k^{[2]} - x_k^{[2]} u_k^{[1]} = 1$ is said to be a *normalized basis* of system (S).

Definition 1. The *reciprocal system* to (S) is the symplectic system with the matrix $\overline{S}_k = J^{-1} S_k J$, i.e. the system

$$\text{(S}^r\text{)} \qquad \begin{pmatrix} \overline{x}_{k+1} \\ \overline{u}_{k+1} \end{pmatrix} = \begin{pmatrix} d_k & -c_k \\ -b_k & a_k \end{pmatrix} \begin{pmatrix} \overline{x}_k \\ \overline{u}_k \end{pmatrix}.$$

System (S) is said to be *self-reciprocal* if it coincides with its reciprocal system.

Remark 1. From Definition 1 it follows that if $\binom{x_k}{u_k}$ is a solution of (S) then $\binom{u_k}{-x_k}$ is a solution of its reciprocal system (S^r).

Any symplectic self-reciprocal system (S) takes the form

$$\text{(T)} \qquad \begin{pmatrix} s_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} p_k & q_k \\ -q_k & p_k \end{pmatrix} \begin{pmatrix} s_k \\ c_k \end{pmatrix}$$

where $p_k^2 + q_k^2 = 1$. Such a system is called the *trigonometric system* because any of its solutions can be expressed using the functions sin, cos as the following lemma shows.

Lemma 1. *Let $\varphi_k \in [0, 2\pi)$, $k \in \mathbb{Z}$, be defined by the relations*

$$(4) \qquad \sin \varphi_k = q_k, \qquad \cos \varphi_k = p_k.$$

Then the general solution of (T) *is of the form*

$$\begin{pmatrix} s_k \\ c_k \end{pmatrix} = \beta \begin{pmatrix} \sin(\xi_k + \alpha) \\ \cos(\xi_k + \alpha) \end{pmatrix}$$

where $k \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{R}$ and (ξ_k) is any sequence such that $\Delta \xi_k = \varphi_k$.

Proof. By direct computation and Remark 1 it follows that

$$\begin{pmatrix} s_k \\ c_k \end{pmatrix}^{[1]} = \begin{pmatrix} \sin \xi_k \\ \cos \xi_k \end{pmatrix}, \qquad \begin{pmatrix} s_k \\ c_k \end{pmatrix}^{[2]} = \begin{pmatrix} \cos \xi_k \\ -\sin \xi_k \end{pmatrix}$$

forms a basis of system (T). Hence any solution is a linear combination of these two solutions. $\square$

Definition 2. An interval $(m, m+1]$ is said to contain a *generalized zero* of a solution $\binom{x_k}{u_k}$ of system (S) if $x_m \neq 0$ and

$$x_{m+1} = 0 \qquad \text{or} \qquad b_m x_m x_{m+1} < 0.$$

System (S) is said to be *nonoscillatory at* $+\infty$ if there exists $k_0 \in \mathbb{N}$ such that every solution of (S) has at most one generalized zero on $[k_0, \infty)$, and it is said to be *oscillatory at* $+\infty$ in the opposite case. In a similar way, the oscillation [nonoscillation] of (S) at $-\infty$ is defined.

Remark 2. If (S) is nonoscillatory at $+\infty$, then for every solution $\binom{x_k}{u_k}$ of this system there exists $k_1 \in \mathbb{N}$ such that either $x_k = 0$ for $k \geqslant k_1$ or $x_k \neq 0$ and $b_k x_k x_{k+1} \geqslant 0$ for $k \geqslant k_1$.

Remark 3. If $b_m = 0$ and $x_m \neq 0$ for some $m \in \mathbb{Z}$ then $x_{m+1} \neq 0$, i.e. $(m, m+1]$ does not contain a generalized zero. Indeed, since $a_m d_m - b_m c_m = 1$, we have $a_m \neq 0$ and thus $x_{m+1} = a_m x_m \neq 0$. Hence, if $b_k \equiv 0$ for all $k \in \mathbb{Z}$ then any nontrivial solution of (S) has no generalized zero.

Remark 4. Observe that the Sturm type separation theorem holds for symplectic systems, see e.g. Theorem 1 and its proof in [2]. This means that the number of generalized zeros of any pair of solutions differs at most by 1.

Similarly to the continuous case, for nonoscillatory symplectic systems there exist the so called recessive solutions having a certain extremal property at $+\infty$ as the following lemma shows.

Lemma 2. *Let system* (S) *be nonoscillatory at* $+\infty$. *Then there exists a nontrivial solution* $\binom{x_k}{u_k}$ *of* (S) *with the property*

$$\lim_{k \to +\infty} \frac{x_k}{\tilde{x}_k} = 0 \tag{5}$$

for any other linearly independent solution $\binom{\tilde{x}_k}{\tilde{u}_k}$ *of* (S). *The solution* $\binom{x_k}{u_k}$ *is called the recessive solution at* $+\infty$ *and is given uniquely up to a nonzero constant multiple.*

Proof. Let $\binom{x_k}{u_k}$, $\binom{\tilde{x}_k}{\tilde{u}_k}$ be linearly independent solutions of (S) with a (constant) Casoratian $w \neq 0$. By direct computation we get

$$\begin{aligned} \Delta\Big(\frac{\tilde{x}_k}{x_k}\Big) &= \frac{\Delta \tilde{x}_k x_k - \tilde{x}_k \Delta x_k}{x_k x_{k+1}} \\ &= \frac{\big((a_k - 1)\tilde{x}_k + b_k \tilde{u}_k\big)x_k - \tilde{x}_k\big((a_k - 1)x_k + b_k u_k\big)}{x_k x_{k+1}} \\ &= \frac{b_k(x_k \tilde{u}_k - u_k \tilde{x}_k)}{x_k x_{k+1}} = \frac{w b_k}{x_k x_{k+1}}. \end{aligned}$$

Since (S) is nonoscillatory at $+\infty$, there exists $k_1 \in \mathbb{N}$ such that either $x_k = 0$ for $k \geqslant k_1$ or $x_k \neq 0$ and $b_k x_k x_{k+1} \geqslant 0$ for $k \geqslant k_1$. In the former case $u_k \neq 0$ and, because $w = x_k \tilde{u}_k - \tilde{x}_k u_k \neq 0$, $\tilde{x}_k \neq 0$ for $k \geqslant k_1$ and (5) holds. In the latter case $b_k/(x_k x_{k+1}) \geqslant 0$, hence the sequence $\tilde{x}_k/x_k$ is monotone for $k \geqslant k_1$ and there exists a limit $L = \lim\limits_{k\to+\infty} \tilde{x}_k/x_k$. If $L = \infty$ or $L = 0$, then $\binom{\tilde{x}_k}{\tilde{u}_k}$ or $\binom{x_k}{u_k}$ is the recessive solution, respectively. If $0 < L < \infty$ then the solution $\binom{\tilde{x}_k}{\tilde{u}_k} - L\binom{x_k}{u_k}$ is the recessive one.

The uniqueness of the recessive solution follows from (5)—if there existed two linearly independent recessive solutions then $\lim\limits_{k\to+\infty} x_k^{[1]}/x_k^{[2]} = 0$ and $\lim\limits_{k\to+\infty} x_k^{[2]}/x_k^{[1]} = 0$, a contradiction. □

Analogously, if (S) is nonoscillatory at $-\infty$ then there exists a solution $\binom{x_k}{u_k}$ with the property $\lim\limits_{k\to-\infty} x_k/\tilde{x}_k = 0$ and it is called the recessive solution at $-\infty$.

Lemma 3. *Let the trigonometric system* (T) *be nonoscillatory at $+\infty$ and let $z^{[+\infty]}$ be a recessive solution of* (T) *at $+\infty$. Then $\lim\limits_{k\to+\infty} s_k = 0$.*

Proof. By the self-reciprocity of (T) and by Remark 1, solutions $\binom{s_k}{c_k}$, $\binom{c_k}{-s_k}$ form the basis of system (T) . Since c_k is bounded, we have $\lim\limits_{k\to+\infty} s_k/c_k = 0$ if and only if $\lim\limits_{k\to+\infty} s_k = 0$. □

Lemma 4. *Let* (S) *be nonoscillatory at $+\infty$, let $z^{[+\infty]}$ be its recessive solution at $+\infty$ and let the interval $(m, m+1]$ contain the largest generalized zero of $z^{[+\infty]}$. Then any solution of* (S) *which is linearly independent of $z^{[+\infty]}$ has a generalized zero in $[m, \infty)$.*

Proof. If $b_k \equiv 0$ for large k then the statement is obvious. In the opposite case the proof is the same as for the linear Hamiltonian system, see [4, Theorem 1]. □

Lemma 5. *Let* (S) *be nonoscillatory at $+\infty$ and at $-\infty$. Then recessive solutions of* (S) *at $-\infty$ and at $+\infty$ have the same number of generalized zeros.*

Proof. Let $z^{[+\infty]}$ be the recessive solution of (S) at $+\infty$ and let $z^{[-\infty]}$ be the recessive solution of (S) at $-\infty$. If $z^{[-\infty]}$ and $z^{[+\infty]}$ are linearly dependent, then the statement holds. Assume that $z^{[-\infty]}$, $z^{[+\infty]}$ are linearly independent. By Remark 4, the number of their generalized zeros differs at most by 1. Suppose by contradiction that $z^{[-\infty]}$ has $k-1$ generalized zeros and $z^{[+\infty]}$ has k generalized zeros on $\mathbb{Z}$. Let the interval $(m, m+1]$ contain the largest generalized zero of $z^{[+\infty]}$. Then by the Sturm separation theorem (see Remark 4) the interval $[m, \infty)$ contains no generalized zero of $z^{[-\infty]}$, which contradicts Lemma 4. The same arguments hold in the opposite case. □

3. Trigonometric transformations

In this section we study the problem of transformation of any system (S) into a trigonometric system (T) and show that the trigonometric system (T) can be viewed as a discrete analogue of the self-reciprocal differential equation (q).

We start with some general facts concerning transformations of 2×2 symplectic systems. Let

$$R_k = \begin{pmatrix} h_k & l_k \\ g_k & m_k \end{pmatrix}$$

be a 2×2 symplectic matrix, i.e. $h_k m_k - g_k l_k = 1$. The transformation

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = R_k \begin{pmatrix} y_k \\ v_k \end{pmatrix} \tag{6}$$

transforms system (S) into the system

$$\begin{pmatrix} y_{k+1} \\ v_{k+1} \end{pmatrix} = \overline{S}_k \begin{pmatrix} y_k \\ v_k \end{pmatrix}, \quad \overline{S}_k = R_{k+1}^{-1} S_k R_k \tag{7}$$

and this system is again a symplectic system because symplectic matrices form a group with respect to multiplication. Moreover, if $l_k \equiv 0$ then $h_k \neq 0$, $m_k = 1/h_k$ and the transformation (6) preserves generalized zeros, i.e. the interval $(m, m+1]$ contains a generalized zero of a solution $\binom{x_k}{u_k}$ of system (S) if and only if it contains a generalized zero of the solution $\binom{y_k}{v_k}$ of system (7). Indeed, if $x_m \neq 0$ and $x_{m+1} = 0$ or $b_m x_m x_{m+1} < 0$, then $y_m = h_m^{-1} x_m \neq 0$ and

$$y_{m+1} = h_{m+1}^{-1} x_{m+1} = 0 \quad \text{or} \quad \overline{b}_m y_m y_{m+1} = \frac{1}{(h_m h_{m+1})^2} b_m x_m x_{m+1} < 0.$$

In [3, Th. 3.1] the following trigonometric transformation was stated for $2n \times 2n$ symplectic systems.

Theorem A. *Let* $z^{[1]} = \binom{x_k}{u_k}^{[1]}$, $z^{[2]} = \binom{x_k}{u_k}^{[2]}$ *form a normalized basis of* (S) *and let*

$$h_k^2 = \big(x_k^{[1]}\big)^2 + \big(x_k^{[2]}\big)^2, \quad g_k = \frac{x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]}}{h_k}. \tag{8}$$

Then the transformation

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & 1/h_k \end{pmatrix} \begin{pmatrix} s_k \\ c_k \end{pmatrix} \tag{9}$$

transforms system (S) *into the trigonometric system* (T) *with*

$$p_k = \frac{1}{h_{k+1}}(a_k h_k + b_k g_k), \quad q_k = \frac{1}{h_k h_{k+1}} b_k \tag{10}$$

without changing the oscillatory behavior. The sequence (h_k) *satisfying* (8) *can be chosen in such a way that* $q_k \geqslant 0$ *and if in addition* $b_k \neq 0$, *then it can be chosen in such a way that* $q_k > 0$ *for* $k \in \mathbb{Z}$.

Definition 3. Let $(z^{[1]}, z^{[2]})$ be a normalized basis of system (S). Transformation (9) with h, g given by (8) is said to be a *trigonometric transformation of the basis* $(z^{[1]}, z^{[2]})$, and system (T) with p, q given by (10) and $q_k \geqslant 0$ is said to be a *trigonometric system of the basis* $(z^{[1]}, z^{[2]})$. The trigonometric system of any basis of system (S) is said to be an *associated trigonometric system* to system (S).

The following lemma plays a crucial role in our later consideration. It shows that solutions of 2×2 symplectic systems can be expressed by means of a certain solution of (T) and can be regarded as a discrete version of (3).

Lemma 6. *Let* $z^{[1]} = \binom{x_k}{u_k}^{[1]}$, $z^{[2]} = \binom{x_k}{u_k}^{[2]}$ *be a normalized basis of* (S) *and let* (T) *be a trigonometric system of this basis. Then there exists a solution* $\binom{s_k}{c_k}$ *of* (T) *such that*

$$\begin{pmatrix} x_k^{[1]} \\ u_k^{[1]} \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & \frac{1}{h_k} \end{pmatrix} \begin{pmatrix} s_k \\ c_k \end{pmatrix}, \quad \begin{pmatrix} x_k^{[2]} \\ u_k^{[2]} \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & \frac{1}{h_k} \end{pmatrix} \begin{pmatrix} c_k \\ -s_k \end{pmatrix}, \tag{11}$$

where $k \in \mathbb{Z}$, (h_k), (g_k) *are given by* (8),

$$\begin{pmatrix} s_k \\ c_k \end{pmatrix} = \begin{pmatrix} \sin \xi_k \\ \cos \xi_k \end{pmatrix}, \tag{12}$$

and (ξ_k) *is an arbitrary sequence such that* $\Delta \xi_k = \varphi_k$ *and* $\varphi_k \in [0, \pi)$ *satisfy* (4).

P r o o f. By Theorem A there exist solutions $\binom{s_k}{c_k}^{[1]}$, $\binom{s_k}{c_k}^{[2]}$ of (T) such that

$$\begin{pmatrix} x_k^{[1]} \\ u_k^{[1]} \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & \frac{1}{h_k} \end{pmatrix} \begin{pmatrix} s_k^{[1]} \\ c_k^{[1]} \end{pmatrix}, \quad \begin{pmatrix} x_k^{[2]} \\ u_k^{[2]} \end{pmatrix} = \begin{pmatrix} h_k & 0 \\ g_k & \frac{1}{h_k} \end{pmatrix} \begin{pmatrix} s_k^{[2]} \\ c_k^{[2]} \end{pmatrix}, \tag{13}$$

that is,

$$s^{[1]} = h^{-1} x^{[1]}, \quad c^{[1]} = -g x^{[1]} + h u^{[1]}, \quad s^{[2]} = h^{-1} x^{[2]}, \quad c^{[2]} = -g x^{[2]} + h u^{[2]}.$$

By direct computation we have

$$s^{[1]} c^{[2]} - c^{[1]} s^{[2]} = 1 \tag{14}$$

and

$$
\begin{aligned}
(s^{[1]})^2 + (c^{[1]})^2 &= \frac{(x^{[1]})^2}{h^2} + g^2 (x^{[1]})^2 - 2ghx^{[1]}u^{[1]} + (u^{[1]})^2 h^2 \\
&= (x^{[1]})^2 \biggl(\frac{1}{h^2} + \frac{(x^{[1]}u^{[1]} + x^{[2]}u^{[2]})^2}{h^2} - (u^{[1]})^2 \biggr) \\
&\quad + (x^{[2]}u^{[1]})^2 - 2x^{[1]}x^{[2]}u^{[1]}u^{[2]} \\
&= \Bigl(\frac{x^{[1]}}{h} \Bigr)^2 ((x^{[1]}u^{[2]} - x^{[2]}u^{[1]})^2 + (x^{[1]}u^{[1]} + x^{[2]}u^{[2]})^2 \\
&\quad - (x^{[1]})^2 (u^{[1]})^2 - (x^{[2]})^2 (u^{[1]})^2) \\
&\quad + (x^{[2]}u^{[1]})^2 - 2x^{[1]}x^{[2]}u^{[1]}u^{[2]} \\
&= (x^{[1]}u^{[2]})^2 + (x^{[2]}u^{[1]})^2 - 2x^{[1]}x^{[2]}u^{[1]}u^{[2]} \\
&= (x^{[1]}u^{[2]} - x^{[2]}u^{[1]})^2 = 1.
\end{aligned}
$$

Similarly, we have $(s^{[2]})^2 + (c^{[2]})^2 = 1$. By Lemma 1 there exist real constants $\alpha^{[i]}$, $\beta^{[i]}$ for $i = 1, 2$ such that

$$
\begin{pmatrix} s_k^{[1]} \\ c_k^{[1]} \end{pmatrix} = \beta^{[1]} \begin{pmatrix} \sin(\xi_k + \alpha^{[1]}) \\ \cos(\xi_k + \alpha^{[1]}) \end{pmatrix}, \qquad \begin{pmatrix} s_k^{[2]} \\ c_k^{[2]} \end{pmatrix} = \beta^{[2]} \begin{pmatrix} \sin(\xi_k + \alpha^{[2]}) \\ \cos(\xi_k + \alpha^{[2]}) \end{pmatrix},
$$

where (ξ_k) is an arbitrary sequence such that $\Delta\xi_k = \varphi_k$ and (φ_k) is given by (4). Since $(s^{[i]})^2 + (c^{[i]})^2 = 1$, we have $\beta^{[i]} = 1$ for $i = 1, 2$. In addition, by (14) we obtain

$$
\begin{aligned}
s^{[1]}c^{[2]} - c^{[1]}s^{[2]} &= \sin(\xi_k + \alpha^{[1]}) \cos(\xi_k + \alpha^{[2]}) - \sin(\xi_k + \alpha^{[2]}) \cos(\xi_k + \alpha^{[1]}) \\
&= \sin(\alpha^{[1]} - \alpha^{[2]}) = 1,
\end{aligned}
$$

that is $\alpha^{[2]} - \alpha^{[1]} = \frac{1}{2}\pi \pmod{2\pi}$. Hence $s^{[2]} = c^{[1]}$, $c^{[2]} = -s^{[1]}$ and (11) holds. Since (ξ_k) was arbitrary such that $\Delta\xi_k = \varphi_k$, changing ξ_k to $\xi_k - \alpha^{[1]}$ we get (12). $\square$

Lemma 7. *Trigonometric transformation* (9) *of any basis of system* (S) *transforms the recessive solution of system* (S) *at* $+\infty$ *[at* $-\infty$*] into the recessive solution at* $+\infty$ *[at* $-\infty$*] of any associated trigonometric system* (T) *to* (S).

Proof. Let $\binom{x_k}{u_k}$ be a recessive solution at $+\infty$ of system (S). By (5) and (9), we conclude

$$
\lim_{k\to+\infty} \frac{x_k}{\tilde{x}_k} = \lim_{k\to+\infty} \frac{h_k s_k}{\tilde{h}_k \tilde{s}_k} = 0,
$$

i.e. $\binom{s_k}{c_k}$ is a recessive solution at $+\infty$ of system (T). $\square$

4. The phase of symplectic difference systems

The trigonometric transformation and trigonometric systems enable us to introduce phases for 2×2 symplectic systems and for second order linear difference equations.

We start with the continuous case where the phase of equation (2) is defined in the following way. Let u, v be linearly independent solutions of (2) with the Wronskian $w \equiv 1$. A function $\alpha \in C^3$ is called a *phase of the basis* (u, v), if

$$\tan \alpha(t) = \frac{u(t)}{v(t)} \qquad \text{for} \quad t \in (a, b)$$

except at the zeros of v. Observe that α is uniquely determined by the continuity at the zeros of v. The phase of equation (2) is any phase of a basis of this equation.

In the discrete case, based on Theorem A and Lemma 1, we proceed in a similar way and introduce the following definition.

Notation. Arctan and Arccot denote the particular branch of the multivalued function arctan with the image $(-\pi/2, \pi/2)$ and of the function arccot with the image $(0, \pi)$, respectively.

Definition 4. Let $z^{[1]} = \binom{x_k}{u_k}^{[1]}$, $z^{[2]} = \binom{x_k}{u_k}^{[2]}$ be a normalized basis of system (S). By a *phase of the basis* $(z^{[1]}, z^{[2]})$ we understand any sequence $\psi = (\psi_k)$, $k \in \mathbb{Z}$, such that $\Delta\psi_k \in [0, \pi)$ and

$$\psi_k = \begin{cases} \arctan \frac{x_k^{[1]}}{x_k^{[2]}} & \text{if } x_k^{[2]} \neq 0, \\ \text{odd multiple of } \frac{\pi}{2} & \text{if } x_k^{[2]} = 0. \end{cases}$$

The *phase of system* (S) is any phase of a basis of this system.

Obviously, if ψ is a phase of the basis $(z^{[1]}, z^{[2]})$ then $\psi + k\pi$, $k \in \mathbb{Z}$ is a phase of this basis as well. Conversely, if $\psi^{[1]}$, $\psi^{[2]}$ are two phases of the basis $(z^{[1]}, z^{[2]})$, then $\psi^{[1]} - \psi^{[2]} = 0 \pmod \pi$.

The next theorem shows the fundamental relation between the phase of a given basis of (S) and the trigonometric system of this basis.

Theorem 1. *Let* $(z^{[1]}, z^{[2]})$ *be a normalized basis of system* (S), ψ *a phase of this basis and let* (T) *be the trigonometric system of this basis, i.e. the system* (T) *with* p, q *satisfying* (10) *and* $q_k \geqslant 0$. *Then* $\sin \Delta\psi_k = q_k$, $\cos \Delta\psi_k = p_k$, *that is,*

$$\Delta\psi_k = \operatorname{Arccot} \frac{p_k}{q_k} \tag{15}$$

if $q_k > 0$ *and* $\Delta\psi_k = 0$ *if* $q_k = 0$.

P r o o f. Let (T) be the trigonometric system and ψ a phase of the basis $(z^{[1]}, z^{[2]})$. By Lemma 6 there exists a solution $\binom{s_k}{c_k}$ of (T) such that $s_k = \sin \xi_k$, $c_k = \cos \xi_k$ and $z^{[i]} = \binom{x_k}{u_k}^{[i]}$, $i = 1, 2$ satisfy

$$x_k^{[1]} = h_k \sin \xi_k, \qquad x_k^{[2]} = h_k \cos \xi_k, \tag{16}$$

where h is given by (8) and $\Delta \xi_k = \varphi_k$. Since $q_k \geqslant 0$ we have $\varphi_k \in [0, \pi)$ and (4) holds. Hence for $x_k^{[2]} \neq 0$,

$$\tan \xi_k = \frac{x_k^{[1]}}{x_k^{[2]}}.$$

On the other hand, by Definition 4,

$$\tan \psi_k = \frac{x_k^{[1]}}{x_k^{[2]}} \tag{17}$$

for all $k \in \mathbb{Z}$ for which $x_k^{[2]} \neq 0$, and $\psi_k = \frac{\pi}{2} \pmod{\pi}$ when $x_k^{[2]} = 0$. Consequently, $\psi_k = \xi_k \pmod{\pi}$. This together with the fact that $\Delta \psi_k = \Delta \xi_k = \varphi_k \in [0, \pi)$ gives the conclusion. □

The relationship between the phases of two different bases of system (S) is similar to the continuous case (see [5], pp. 43 and 46) and is given by the following theorem.

Theorem 2. *If ψ_k, $\overline{\psi}_k$ are two phases of the same symplectic system* (S) *then there exist a, b, c, d such that $ad - bc = 1$ and*

$$\tan \overline{\psi}_k = \frac{a \tan \psi_k + b}{c \tan \psi_k + d}$$

for all k for which this expression has a sense.

P r o o f. Let $(z^{[1]}, z^{[2]})$ and $(\bar{z}^{[1]}, \bar{z}^{[2]})$ be normalized bases of system (S) which determine the phase ψ and $\overline{\psi}$, respectively. There exist constants $a, b, c, d \in \mathbb{R}$ such that

$$\bar{z}^{[1]} = a z^{[1]} + b z^{[2]}, \quad \bar{z}^{[2]} = c z^{[1]} + d z^{[2]}.$$

Moreover, since $(z^{[1]})^T J z^{[1]} = 1$, we have $ad - bc = 1$. Taking into account (17), we get

$$\tan \overline{\psi}_k = \frac{\bar{x}^{[1]}}{\bar{x}^{[2]}} = \frac{a x^{[1]} + b x^{[2]}}{c x^{[1]} + d x^{[2]}} = \frac{a \tan \psi_k + b}{c \tan \psi_k + d}.$$

□

The basic geometric interpretation of the phase of system (S) is the following.

Theorem 3. *System* (S) *is oscillatory at* $+\infty$ *if and only if any phase* ψ_k *of this system satisfies* $\lim\limits_{k\to+\infty} \psi_k = \infty$, *i.e. there exists* $k_0 \in \mathbb{N}$ *such that*

$$\sum_{k\geqslant k_0,\, q_k>0}^{\infty} \operatorname{Arccot} \frac{p_k}{q_k} = \infty. \tag{18}$$

P r o o f. Let $(z^{[1]}, z^{[2]})$ be a normalized basis of (S), ψ_k the phase and (T) the trigonometric system of this basis. By Lemma 6 there exists a solution $\binom{s_k}{c_k}$ of (T) such that (11) and (12) hold. From this and Theorem 1 we have

$$s_k = \sin \psi_k, \quad \Delta\psi_k \in [0, \pi).$$

Hence the sequence $\{\psi_k\}$ is nondecreasing and $\lim\limits_{k\to+\infty} \psi_k$ exists. Two cases are possible:

a) Assume $\lim\limits_{k\to+\infty} \psi_k = +\infty$. Since $\psi_{k+1} = \psi_k + \Delta\psi_k < \psi_k + \pi$, we have $\operatorname{sgn} \sin \psi_k = -\operatorname{sgn} \sin \psi_{k+1}$ for infinitely many k and for these indices

$$q_k s_k s_{k+1} = q_k \sin \psi_k \sin \psi_{k+1} \leqslant 0.$$

We will show that the fact $\lim\limits_{k\to+\infty} \psi_k = +\infty$ implies $b_k \not\equiv 0$, i.e. $q_k \not\equiv 0$ for large k. If $b_k \equiv 0$ for large k then $q_k \equiv 0$ and $p_k^2 \equiv 1$ for such k. The case $p_k = -1$ gives $\Delta\psi_k = \pi$, which is impossible in view of Definition 4. If $p_k \equiv 1$ then $\Delta\psi_k \equiv 0$ and $\{\psi_k\}$ is an eventually constant sequence, a contradiction. Therefore there exist infinitely many k for which $q_k > 0$ and any nontrivial solution $\binom{s_k}{c_k}$ of (T) satisfies either $s_k s_{k+1} < 0$ or $s_k \neq 0$ and $s_{k+1} = 0$ for these k's. Hence, system (T) is oscillatory and, by Theorem A, system (S) is oscillatory as well.

b) Assume $\lim\limits_{k\to+\infty} \psi_k < \infty$. Since $\{\psi_k\}$ is nondecreasing, there exists k_0 such that $\sin \psi_k \sin \psi_{k+1} \geqslant 0$ for $k \geqslant k_0$, which means $\sin \psi_k \sin \psi_{k+1} > 0$ or $\psi_k = \ell\pi$ ($\ell \in \mathbb{Z}$) for $k \geqslant k_0$. If $q_k > 0$ for large k then $q_k s_k s_{k+1} = q_k \sin \psi_k \sin \psi_{k+1} > 0$ and systems (T) and (S) are nonoscillatory. If $q_k \equiv 0$ for large k (which includes the case $\psi_k = \ell\pi$) then $p_k \equiv 1$ and so $s_k = s_{k+1}$ for large k. Thus no solution of (T) has a generalized zero in the neighbourhood of infinity and both systems (T) and (S) are nonoscillatory.

Finally, taking into account (15) we get (18). $\square$

All solutions of the symplectic system (S) have the same oscillatory character, that is to say they all have either a finite or an infinite number of generalized zeros on $\mathbb{Z}$. In the first case system (S) is said to be *of finite type* or *nonoscillatory*, in the second case *of infinite type* or *oscillatory*. More precisely, in accordance with O. Borůvka, we introduce the following classification of nonoscillatory symplectic systems.

Definition 5. Symplectic system (S) is said to be *of finite type* m, $m \in \mathbb{N}$, if this system possesses solutions with m generalized zeros in $\mathbb{Z}$ but none with $m+1$ generalized zeros.

If (S) is of finite type m, then it is called *of general kind* if it admits two linearly independent solutions with $m-1$ generalized zeros on $\mathbb{Z}$. Otherwise, system (S), being of the finite type m, is *of special kind*.

Theorem 3 yields a criterion of boundedness of a phase as in the continuous case, cf. [5, §5. 4].

Corollary 1. *The phase ψ is bounded on $\mathbb{Z}$ if and only if system* (S) *is of finite type.*

The type and kind of system (S) is the same as those of any associated trigonometric system and is uniquely determined by the boundary values of any phase ψ as the following theorem states.

Theorem 4. *The following statements are equivalent:*

(a) *Symplectic system* (S) *is of type m and of special kind on* $\mathbb{Z}$.

(b) *Any trigonometric system* (T) *associated to system* (S) *is of type m and of special kind on* $\mathbb{Z}$.

(c) *Recessive solutions of* (S) *at* $\pm\infty$ *are linearly dependent and possess* $m-1$ *generalized zeros.*

(d) *Any phase ψ of system* (S) *satisfies*

$$\sum_{k=-\infty}^{+\infty} \Delta\psi_k = m\pi. \tag{19}$$

(e) *Any trigonometric system* (T) *associated to system* (S) *satisfies*

$$\sum_{k=-\infty,\, q_k>0}^{+\infty} \operatorname{Arccot} \frac{p_k}{q_k} = m\pi. \tag{20}$$

P r o o f. "(a) $\Leftrightarrow$ (b)" follows from the fact that the trigonometric transformation is a special case of the transformation (6) and, as has been shown in §3, such transformation preserves generalized zeros.

"(a) $\Rightarrow$ (c)". Let (S) be of type m and of special kind. By Lemma 5 the recessive solution $z^{[-\infty]}$ at $-\infty$ and the recessive solution $z^{[+\infty]}$ at $+\infty$ of (S) have the same number of generalized zeros on $\mathbb{Z}$ which is either $m-1$ or m. Since (S) is of special kind, only one solution has $m-1$ generalized zeros. Thus if $z^{[-\infty]}$ and $z^{[+\infty]}$ are linearly independent, they must have m generalized zeros and by Lemma 4 there exists a solution with $m+1$ generalized zeros, which is a contradiction with the fact that (S) is of type m. Consequently, $z^{[-\infty]}$ and $z^{[+\infty]}$ are linearly dependent and have $m-1$ generalized zeros.

"(c) $\Rightarrow$ (d)". Let ψ be a phase of (S) and let the recessive solutions of (S) at $\pm\infty$ be linearly dependent with $m-1$ zeros. By Lemma 4, recessive solutions of (T) at $\pm\infty$ have the same property. Denote by $\binom{s_k}{c_k}^{[+\infty]}$ the recessive solution of (T) at $+\infty$ and by $\binom{s_k}{c_k}^{[-\infty]}$ the recessive solution at $-\infty$. By Lemma 3,

$$\lim_{k\to+\infty} s_k^{[+\infty]} = \lim_{k\to-\infty} s_k^{[-\infty]} = 0.$$

Recessive solutions at $+\infty$ and at $-\infty$ are determined uniquely up to a nonzero constant multiple and by a direct computation one can check that they are of the form

$$s_k^{[+\infty]} = \sin\sum_{j=k}^{\infty}\varphi_j \quad \text{and} \quad s_k^{[-\infty]} = \sin\sum_{j=-\infty}^{k-1}\varphi_j, \tag{21}$$

where $\varphi_k = \Delta\psi_k$. Since both these solutions are linearly dependent, we have

$$\sum_{j=k}^{\infty}\varphi_j \pm \sum_{j=-\infty}^{k-1}\varphi_j = \ell\pi, \qquad \ell\in\mathbb{Z}.$$

Passing $k\to-\infty$ and taking into account that $s_k^{[+\infty]}$ has $m-1$ zeros, we have $\ell = m$, $m\in\mathbb{N}$. Using the fact that $\varphi_k = \Delta\psi_k$ we get (19).

"(d) $\Leftrightarrow$ (e)" follows from (15).

"(e) $\Rightarrow$ (b)". Let (T) be a trigonometric system associated to (S) satisfying (20). Let $\binom{s_k}{c_k}^{[+\infty]}$ and $\binom{s_k}{c_k}^{[-\infty]}$ be recessive solutions of (T) at $+\infty$ and $-\infty$, respectively.

It follows from (21) that

$$
\begin{aligned}
s_k^{[-\infty]} &= \sin \sum_{j=-\infty}^{k-1} \varphi_j = (-1)^m \sin\Big(\sum_{j=-\infty}^{k-1} \varphi_j - m\pi \Big) \\
&= (-1)^m \sin\Big(\sum_{j=-\infty}^{k-1} \varphi_j - \sum_{j=-\infty}^{\infty} \varphi_j \Big) = (-1)^m \sin\Big(-\sum_{j=k}^{\infty} \varphi_j \Big) \\
&= (-1)^{m+1} \sin \sum_{j=k}^{\infty} \varphi_j = (-1)^{m+1} s_k^{[+\infty]},
\end{aligned}
$$

that is the recessive solutions at $+\infty$ and $-\infty$ are linearly dependent. In addition, in view of (15),

$$
\sum_{j=k}^{\infty} \varphi_j < m\pi \quad \text{and} \quad \sum_{j=-\infty}^{k-1} \varphi_j < m\pi,
$$

so the recessive solutions have $m-1$ generalized zeros. From here and Lemma 4 it follows that (T) is of type m and of special kind. $\square$

R e m a r k 5. Claims (a), (b) and (d) can be regarded as a discrete version of the Borůvka theory of phases, cf. [5, §7. 2]. In accordance with this theory, denote the number $|c-d| = \sum\limits_{k=-\infty}^{+\infty} \Delta\psi_k$ by $O(\psi)$, the so called *oscillation* of the phase ψ. From Definition 5 and Theorem 4 it follows that (S) of finite type m is general or special according as, for the oscillation of each of its phases ψ, we have $(m-1)\pi < O(\psi) < m\pi$ or $O(\psi) = m\pi$.

R e m a r k 6. Coming back to difference equations, by a *phase* ψ of equation (1) we mean any phase of the corresponding symplectic system (G). All statements of §3, 4 can be formulated for difference equations using the corresponding system (G).

We conclude the paper with an application of Theorem 4 to difference equations showing how the formula (20) can be used for the summation of certain number series.

E x a m p l e. Consider the second order linear equation

$$
\Delta^2 x_k = 0, \qquad k \in \mathbb{Z}. \tag{22}
$$

This equation has linearly independent solutions $x_k^{[1]} = 1$, $x_k^{[2]} = k$ such that $x_k^{[1]} \Delta x_k^{[2]} - \Delta x_k^{[1]} x_k^{[2]} = 1$. The recessive solution at $\pm\infty$ is $x_k^{[1]} = 1$ and it satisfies the relation $x_k^{[1]} x_{k+1}^{[1]} > 0$ for $k \in \mathbb{Z}$, i.e. has no generalized zero. Therefore the equation is of finite type 1 and of special kind.

Equation (22) can be written as a symplectic system (S) with $a_k = b_k = d_k = 1$ and $c_k = 0$. By Theorem A we have

$$\frac{p_k}{q_k} = h_k^2 + x_k^{[1]} \Delta x_k^{[1]} + x_k^{[2]} \Delta x_k^{[2]} = x_k^{[1]} x_{k+1}^{[1]} + x_k^{[2]} x_{k+1}^{[2]} = 1 + k(k+1) = k^2 + k + 1.$$

Using Theorem 4 we get

$$\sum_{k=-\infty}^{+\infty} \operatorname{Arccot}(k^2 + k + 1) = \pi.$$

Concluding remarks.

(1) One of the important applications of phases is the construction of difference equations with prescribed properties. This will be given elsewhere.

(2) In the Borůvka theory the phase α of the differential equation (2) is called the *first phase* while the *second phase* β is defined as the first phase of the reciprocal equation. The relation between both the phases α, β and among other types of phases (hyperbolic phase) have been investigated. It is the subject of the present investigation whether similar problems can be solved also in the discrete case.

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Authors' addresses: *Zuzana Došlá*, *Denisa Škrabáková*, Department of Mathematics, Masaryk University, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic, e-mails: dosla@math.muni.cz, denisa@math.muni.cz.