CONSTRUCTIONS OF CELL ALGEBRAS

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Abstract. A construction of cell algebras is introduced and some of their properties are investigated. A particular case of this construction for lattices of nets is considered.

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1. Introduction

There are many ways how to construct a "new" algebra from algebras of the same type. The relationship between the resulting algebra and the original ones depends on the construction. For instance, the direct product $\prod\limits_{i\in I}\mathscr{A}_i$ of algebras of the same type is an algebra satisfying the identities which hold in all algebras $\mathscr{A}_i$, $i\in I$. On the other hand, the Plonka sum $\sum\limits_{i\in I}\mathscr{A}_i$ [9] satisfies only the regular identities which hold in all algebras $\mathscr{A}_i$, $i\in I$. A less known construction was introduced by Hecht in [7]. The algebra he constructed preserves only identities of the type

$$\begin{aligned} f(r(x_1,\dots,x_n),p_2(x_1,\dots,x_n),\dots,p_k(x_1,\dots,x_n)) \\ = f(r(x_1,\dots,x_n),q_2(x_1,\dots,x_n),\dots,q_k(x_1,\dots,x_n)) \end{aligned} \tag{1.1}$$

and all their consequences, where f is a k-ary operational symbol and r, p_i, q_i, $i=2,\dots,k$, are polynomials of variables $x_1,\dots,x_n$.

We introduce a construction of algebras which is similar both to Plonka sums and Hecht's construction.

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Throughout the paper we assume that all algebras considered are of a given type τ. By F we denote the set of all operational symbols of the type τ, i.e. $F = \{f_t;\ t \in \tau\}$. We write $f_t^{(A)}$ for the realization of f_t on a set A. We often denote briefly by f an operational symbol and also its realization (when no confusion can arise).

Let $\mathscr{A} = (A, F)$ be an algebra of a type τ. For each element $a \in A$ let an algebra $\mathscr{B}_a = (B_a, F)$ of the type τ be given and let $B_a \cap B_b = \emptyset$ if $a \neq b$. Moreover, for each k-ary ($k \geqslant 1$) operation $f \in F$ let $\mathscr{S}^{(f)}$ be a system of mappings with the following property:

$$\text{(2.1)} \qquad \begin{aligned} &\text{if } f(a_1, \ldots, a_k) = a \quad \text{for } f \in F \text{ and } a_1, \ldots, a_k \in A,\\ &\text{then there exists a mapping}\\ &\varphi^{(f)}_{a_i,a}\colon\ B_{a_i} \to B_a \quad \text{from } \mathscr{S}^{(f)} \text{ for each } i \in \{1, \ldots, k\}. \end{aligned}$$

Let us denote $S^{(F)} = \{S^{(f)}; f \in F\}$.

Definition 1. Let $\mathscr{A} = (A, F)$ be an algebra of the type τ, let $\mathscr{B}_a = (B_a, F)$, $a \in A$, be a system of algebras of the same type τ and $S^{(F)}$ a system of mappings satisfying (2.1). By the cell algebra with the basic algebra $\mathscr{A}$, the cells $\mathscr{B}_a$, $a \in A$ and with the system $S^{(F)}$ we mean the algebra of the type τ with the carrier $M = \bigcup\limits_{a \in A} B_a$ and the operations $f^{(M)}$ defined on M as follows:

1. if $f \in F$ is a k-ary operational symbol, $k \geqslant 1$, $x_1 \in B_{a_1}, \ldots, x_k \in B_{a_k}$ and $f(a_1, \ldots, a_k) = a$ then

$$f^{(M)}(x_1, \ldots, x_k) = f^{(B_a)}(\varphi^{(f)}_{a_1,a}(x_1), \ldots, \varphi^{(f)}_{a_k,a}(x_k)); \tag{2.2}$$

2. if f is a nullary operational symbol and $f^{(A)} = c$ then $f^{(M)} = f^{(B_c)}$.

We denote it by $\mathscr{A}(\mathscr{B}_a; a \in A)$ or briefly by $\mathscr{A}(\mathscr{B})$.

The next construction is described in [7]. Let $\mathscr{A} = (A, F)$ be an algebra of the type τ, $\{S_a; a \in A\}$ a family of pairwise disjoint nonvoid sets and $\varphi^{(f)}_{a,\bar{a}}\colon\ S_a \to S_{\bar{a}}$ a family of mappings for all $a \in A$, $f \in F$, $\bar{a} \in \{b \in A; b = f(a, a_1 \ldots, a_{k-1})$ for some $a_1, \ldots, a_{k-1} \in A\}$. For a k-ary operational symbol f, $k \geqslant 1$, the operation $f^{(M)}$ on $M = \bigcup\limits_{a \in A} S_a$ is defined by

$$f^{(M)}(x_1, \ldots, x_k) = \varphi^{(f)}_{a_1,a}(x_1), \tag{2.2a}$$

where $x_1 \in S_{a_1}, \ldots, x_k \in S_{a_k}$, $f(a_1, \ldots, a_k) = a$. If for each $a \in A$, $f \in F$ we define an operation $f^{(S_a)}$ on S_a by $f^{(S_a)}(x_1, \ldots, x_k) = x_1$, we get an algebra $\mathscr{B}_a = (S_a, F)$

of the same type τ and the identity (2.2) is of the form (2.2a). So, the algebra constructed in [7] is a special case of a cell algebra.

If we do not require any additional conditions for the system of mappings $\mathscr{S}^{(F)}$ (analogously to [9]) then the algebra $\mathscr{A}(\mathscr{B})$ has no close relationship to algebras $\mathscr{A}$ and $\mathscr{B}_a$. However, there are some identities preserved by the construction of cell algebras.

Theorem 2. *Let $\mathscr{A}(\mathscr{B}) = (M, F)$ be a cell algebra with a basic algebra $\mathscr{A} = (A, F)$, cells $\mathscr{B}_a = (B_a, F)$, $a \in A$, and let the system $\mathscr{S}^{(F)}$ satisfy* (2.1) *and moreover $S^{(f)} = S^{(g)}$ for every $f, g \in F$. If the basic algebra $\mathscr{A}$ and also every cell $\mathscr{B}_a$ satisfies an identity*

$$f(x_1, \ldots, x_m) = g(y_1, \ldots, y_n) \tag{2.3}$$

where f, g are m-ary and n-ary operational symbols, $m \geqslant 1$, $n \geqslant 1$, then the identity (2.3) *holds in the cell algebra $\mathscr{A}(\mathscr{B})$, too.*

P r o o f. Let $x_1 \in B_{a_1}, \ldots, x_m \in B_{a_m}$, $y_1 \in B_{b_1}, \ldots, y_n \in B_{b_n}$ and $f(a_1, \ldots, a_m) = a$, $g(b_1, \ldots, b_n) = b$. By assumption we have $a = b$, and moreover

$$\begin{aligned} f^{(M)}(x_1, \ldots, x_m) &= f^{(B_a)}(\varphi^{(f)}_{a_1,a}(x_1), \ldots, \varphi^{(f)}_{a_m,a}(x_m)) \\ &= g^{(B_a)}(\varphi^{(g)}_{b_1,a}(y_1), \ldots, \varphi^{(g)}_{b_n,a}(y_n)) = g^{(M)}(y_1, \ldots, y_n). \end{aligned}$$

□

Corollary 3. *If a basic algebra $\mathscr{A}$ and each cell $\mathscr{B}_a$ $(a \in A)$ is an abelian groupoid, then the cell algebra $\mathscr{A}(\mathscr{B})$ is also an abelian groupoid.*

Common identities of more complicated type than (2.3) are not preserved by the cell algebra construction. For example, let us consider an identity of the type

$$f(p(x_1, \ldots, x_m), x_2, \ldots, x_m) = g(y_1, \ldots, y_n), \tag{2.4}$$

where f, g are operational symbols of the type τ and p is a term of the type τ which is not a projection. There exist a basic algebra $\mathscr{A} = (A, F)$, cells $\mathscr{B}_a = (B_a, F)$, $a \in A$ and a system of mappings $\mathscr{S}^{(F)}$ such that the identity (2.4) holds in $\mathscr{A}$ and in each cell $\mathscr{B}_a$, $a \in A$, but (2.4) does not hold in the cell algebra $\mathscr{A}(\mathscr{B}) = (M, F)$. Assume $x_1 \in B_{a_1}, \ldots, x_m \in B_{a_m}, y_1 \in B_{b_1}, \ldots, y_n \in B_{b_n}$ and $p^{(A)}(a_1, \ldots, a_m) = a_0$, $f^{(A)}(a_0, a_2, \ldots, a_m) = a = g^{(A)}(b_1, \ldots, b_n)$. We get

$$\begin{aligned} & f^{(M)}(p^{(M)}(x_1, \ldots, x_m), x_2, \ldots, x_m) \\ & \quad = f^{(B_a)}(\varphi_1(p^{(B_{a_0})}(x'_1, \ldots, x'_m)), \varphi^{(f)}_{a_2,a}(x_2), \ldots, \varphi^{(f)}_{a_m,a}(x_m)) \end{aligned}$$

where $x'_1, \dots, x'_m$ are some elements and φ_1 is a mapping depending not only on $x_1, \dots, x_m$ but also on the term p. The result depends on the system of the maps $\mathscr{S}^{(f)}$, too. A special case of (2.4) is, for example, the identity

$$f(h(x,y),y) = g(x,y), \tag{2.4a}$$

where f, g, h are binary operational symbols. Let the realizations of these operational symbols in the basic algebra $\mathscr{A}$ and also in each cell $\mathscr{B}_a$, $a \in A$, satisfy the identity

$$h(x,y) = g(x,y) = y, \quad f(t,y) = t.$$

Then the identity (2.4a) is satisfied in the basic algebra and also in every cell. If $x \in B_a$, $y \in B_b$ in the cell algebra $\mathscr{A}(\mathscr{B})$ we get

$$f^{(M)}(h^{(M)}(x,y),y) = \varphi^{(f)}_{b,b}(\varphi^{(h)}_{b,b}(y)), \quad g^{(M)}(x,y) = \varphi^{(g)}_{b,b}(y) \tag{2.5}$$

and so the identity (2.4a) need not be satisfied in $\mathscr{A}(\mathscr{B})$.

A class of identities preserved by the cell algebra construction can be increased by assuming some suitable conditions for mappings $\varphi_{a,b}$ (analogous to the conditions for Plonka sums). First, let us consider algebras with one binary operation f. Which conditions are necessary for $S^{(f)}$ in order that f satisfies the associative law or the idempotency?

Let a basic algebra $\mathscr{A} = (A, f)$ and every cell $\mathscr{B}_a = (B_a, f)$ be semigroups, i.e. let

$$f(f(x,y),z) = f(x, f(y,z)) \tag{2.6}$$

hold in $\mathscr{A}$ and in every cell $\mathscr{B}_a$, $a \in A$. Let us assume that the realizations $f^{(A)}$ and $f^{(B_a)}$, $a \in A$, satisfy the identity

$$f(x,y) = x$$

(i.e. $\mathscr{A}$ and $\mathscr{B}_a$, $a \in A$, are left-zero semigroups). Let mappings $\varphi^{(f)}_{a,b}$ be given for every $a, b \in A$. Take elements $x \in B_{a_1}$, $y \in B_{a_2}$, $z \in B_{a_3}$ and let $f(a_1, a_2) = a_0$, $f(a_0, a_3) = a$, $f(a_2, a_3) = a_4$ (by assumption $f(a_1, a_4) = a$). Putting the elements considered to the left-hand side of the identity (2.6) we get (for the realization $f^{(M)}$ of the cell algebra)

$$\begin{aligned} f^{(M)}(f^{(M)}(x,y),z) &= f^{(B_a)}(\varphi^{(f)}_{a_0,a}(f^{(B_{a_0})}(\varphi^{(f)}_{a_1,a_0}(x), \varphi^{(f)}_{a_2,a_0}(y))), \varphi^{(f)}_{a_3,a}(z)) \\ &= \varphi^{(f)}_{a_0,a}(\varphi^{(f)}_{a_1,a_0}(x)). \end{aligned}$$

Analogously, putting the elements to the right-hand side of (2.6) we get

$$f^{(M)}(x, f^{(M)}(y,z)) = \varphi^{(f)}_{a_1,a}(x).$$

Thus (2.6) holds in the cell algebra $\mathscr{A}$ if

$$\varphi^{(f)}_{a_0,a}(\varphi^{(f)}_{a_1,a_0}(x)) = \varphi^{(f)}_{a_1,a}(x).$$

Hence

$$\varphi^{(f)}_{b,c} \circ \varphi^{(f)}_{a,b} = \varphi^{(f)}_{a,c} \tag{2.6a}$$

is a necessary condition for the associative law to hold in this case. The use of (2.6a) requires that $\varphi^{(f)}_{a,b}$ be homomorphisms (analogously to [9]).

Theorem 4. *Let a basic algebra $\mathscr{A} = (A, f)$ and every cell $\mathscr{B}_a = (B_a, f)$, $a \in A$, be semigroups. If $\mathscr{S}^{(f)}$ is a family of homomorphisms satisfying* (2.1) *and* (2.6a) *then the cell algebra $\mathscr{A}(\mathscr{B})$ is also a semigroup.*

P r o o f. Consider as above $x \in B_{a_1}$, $y \in B_{a_2}$, $z \in B_{a_3}$. If $f(a_1,a_2) = a_0$, $f(a_0,a_3) = a$, $f(a_2,a_3) = a_4$ we get

$$\begin{aligned} f^{(M)}(f^{(M)}(x,y),z) &= f^{(B_a)}(\varphi^{(f)}_{a_0,a}(f^{(B_{a_0})}(\varphi^{(f)}_{a_1,a_0}(x), (\varphi^{(f)}_{a_2,a_0}(y))), \varphi^{(f)}_{a_3,a}(z)) \\ &= f^{(B_a)}(f^{(B_a)}(\varphi^{(f)}_{a_0,a}(\varphi^{(f)}_{a_1,a_0}(x)), \varphi^{(f)}_{a_0,a}(\varphi^{(f)}_{a_2,a_0}(y))), \varphi^{(f)}_{a_3,a}(z)) \\ &= f^{(B_a)}(f^{(B_a)}(\varphi^{(f)}_{a_1,a}(x), \varphi^{(f)}_{a_2,a}(y)), \varphi^{(f)}_{a_3,a}(z)), \end{aligned}$$

and similarly

$$f^{(M)}(x, f^{(M)}(y,z)) = f^{(B_a)}(\varphi^{(f)}_{a_1,a}(x), f^{(B_a)}(\varphi^{(f)}_{a_2,a}(y), \varphi^{(f)}_{a_3,a}(z))).$$

Since (B_a, f) is a semigroup, it follows that

$$f^{(M)}(f^{(M)}(x,y),z) = f^{(M)}(x, f^{(M)}(y,z)).$$

□

Let a basic algebra (A, f) and every cell (B_a, f), $a \in A$, be idempotent groupoids, i.e. let

$$f(x,x) = x. \tag{2.7}$$

By taking an element $x \in B_a$ we get (in the cell algebra $\mathscr{A}(\mathscr{B})$)

$$f^{(M)}(x, x) = f^{(B_a)}(\varphi^{(f)}_{a,a}(x), \varphi^{(f)}_{a,a}(x)) = \varphi^{(f)}_{a,a}(x).$$

Hence the identity 2.7 holds if

$$\varphi^{(f)}_{a,a} = \mathrm{id} = \Delta_{B_a} \tag{2.7a}$$

for each element a from the set A.

Theorem 5. *Let a basic algebra $\mathscr{A} = (A, f)$ and every cell $\mathscr{B}_a = (B_a, f)$ be bands (idempotent semigroups) or monoids. If $\mathscr{S}^{(f)}$ is a family of homomorphisms satisfying* (2.1), (2.6a) *and* (2.7a) *then the cell algebra is also a band or a monoid, respectively.*

Proof. If $\mathscr{A}$ and every cell are bands and the conditions concerning $\mathscr{S}^{(f)}$ are fulfilled then $\mathscr{A}(\mathscr{B})$ is also a band by Theorem 4 and the above considerations.

Let $\mathscr{A}$ and each cell $\mathscr{B}_a$, $a \in A$ be monoids. We denote by 1 the neutral element in $\mathscr{A}$ and by 1_a the neutral element in $\mathscr{B}_a$. We are going to show that the element 1_1 is the neutral element in the cell algebra $\mathscr{A}(\mathscr{B}) = (M, f)$. For $x \in B_a$ we get

$$f^{(M)}(x, 1_1) = f^{(B_a)}(\varphi^{(f)}_{a,a}(x), \varphi^{(f)}_{1,a}(1_1)) = f^{(B_a)}(\varphi^{(f)}_{a,a}(x), 1_a) = \varphi^{(f)}_{a,a}(x) = x$$

(a homomorphic image of a neutral element is a neutral element and $a.1 = a$). Analogously, $f^{(M)}(1_1, x) = x$. □

When a basic algebra $\mathscr{A} = (A, f)$ is a group, for each $a, b \in A$ there exist elements $x, y \in A$ for which $f(x, a) = b$ and $f(a, y) = b$. It follows that for each homomorphism $\varphi^{(f)}_{a,b} \in S^{(f)}$ there exists a homomorphism $\varphi^{(f)}_{b,a} \in S^{(f)}$. Moreover, if (2.6a) and (2.7a) hold, we have

$$\varphi^{(f)}_{a,b} \circ \varphi^{(f)}_{b,a} = \varphi^{(f)}_{a,a} = \mathrm{id},$$

therefore $\varphi^{(f)}_{a,b}$ and $\varphi^{(f)}_{b,a}$ are bijections of B_a onto B_b and conversely. So, $\varphi^{(f)}_{a,b}$ and $\varphi^{(f)}_{b,a}$ are inverse isomorphisms. The next theorem shows that if a basic algebra and every cell are groups then one can obtain as cell algebras only direct products of groups.

Theorem 6. *Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{B}_a$, $a \in A$, be algebras of the type τ and for every $a \in A$ let there exist an isomorhism $\varphi_a\colon \mathscr{B}_a \to \mathscr{B}$. If $\mathscr{S}^{(F)}$ is a family of isomorphisms $\varphi_{a,b}\colon \mathscr{B}_a \to \mathscr{B}_b$ for every $a, b \in A$ (i.e. $\mathscr{S}^{(f)} = \mathscr{S}^{(g)}$ for any $f, g \in F$), then the cell algebra $\mathscr{A}(\mathscr{B})$ is isomorphic to the direct product $\mathscr{A} \times \mathscr{B}$.*

P r o o f. Without loss of generality we can assume that for each $a, b \in A$ we have $\varphi_b \circ \varphi_{a,b} = \varphi_a$ where φ_b, φ_a are isomorphisms of the cells $\mathscr{B}_b$, $\mathscr{B}_a$ onto algebra $\mathscr{B}$. We are going to show that the mapping

$$\varphi\colon\; M \to A \times B$$

defined by

$$\varphi(x) = [a, \varphi_a(x)] \text{ if } x \in B_a$$

is an isomorphism of the cell algebra $\mathscr{A}(\mathscr{B})$ onto the direct product $\mathscr{A} \times \mathscr{B}$. Evidently φ is a bijection. If f is a k-ary operational symbol, $x_1 \in B_{a_1}, \dots, x_k \in B_{a_k}$, $f^{(A)}(a_1, \dots, a_k) = a$ then

$$\begin{aligned}
\varphi(f^{(M)}(x_1, \dots, x_k)) &= [a, \varphi_a(f^{(M)}(x_1, \dots, x_k))] \\
&= [f^{(A)}(a_1, \dots, a_k), \varphi_a(f^{(B_a)}(\varphi_{a_1,a}(x_1), \dots, \varphi_{a_k,a}(x_k)))] \\
&= [f^{(A)}(a_1, \dots, a_k), f^{(B)}(\varphi_a(\varphi_{a_1,a}(x_1)), \dots, \varphi_a(\varphi_{a_k,a}(x_k)))] \\
&= [f^{(A)}(a_1, \dots, a_k), f^{(B)}(\varphi_{a_1}(x_1), \dots, \varphi_{a_k}(x_k))] \\
&= f^{(A\times B)}([a_1, \varphi_{a_1}(x_1)], \dots, [a_k, \varphi_{a_k}(x_k)]) \\
&= f^{(A\times B)}(\varphi(x_1), \dots, \varphi(x_k)).
\end{aligned}$$

□

Theorem 7. *Let a basic algebra $\mathscr{A}$ and every cell $\mathscr{B}_a$, $a \in A$, be algebras of the type τ. Let $\mathscr{S}^{(F)}$ be a family of homomorphisms $\varphi_{a,b}\colon B_a \to B_b$ such that* (2.1), (2.6a) *and* (2.7a) *hold and moreover $\mathscr{S}^{(f)} = \mathscr{S}^{(g)}$ for all operations $f, g \in F$ (i.e. the family $\mathscr{S}^{(F)}$ does not depend on operations). If an identity*

$$p(x_1, \dots, x_n) = q(x_1, \dots, x_n)$$

holds in $\mathscr{A}$ and also in each $\mathscr{B}_a$ then it holds in the cell algebra $\mathscr{A}(\mathscr{B})$, too.

P r o o f. First, we will show that

$$p^{(M)}(x_1, \dots, x_n) = p^{(B_a)}(\varphi_{a_1,a}(x_1), \dots, \varphi_{a_n,a}(x_n))$$

if $p(x_1,\dots,x_n)$ is an arbitrary term of the type τ, $x_1 \in B_{a_1},\dots,x_n \in B_{a_n}$ and $p^{(A)}(a_1,\dots,a_n) = a$. We prove it by induction with respect to the number of operational symbols in the term $p(x_1,\dots,x_n)$.

Let

$$p(x_1,\dots,x_n) = f(p_1(x_1,\dots,x_n),\dots,p_k(x_1,\dots,x_n))$$

where f is a k-ary operational symbol. Let $x_1 \in B_{a_1},\dots,x_n \in B_{a_n}$, $p_i^{(A)}(a_1,\dots,a_n) = b_i$ for $i = 1,2,\dots,k$. By induction hypothesis we have

$$p_i^{(M)}(x_1,\dots,x_n) = p_i^{(B_{b_i})}(\varphi_{a_1,b_i}(x_1),\dots,\varphi_{a_n,b_i}(x_n)) \tag{2.8}$$

for $i = 1,2,\dots,k$. Let $p^{(A)}(a_1,\dots,a_n) = a$, i.e. $f^{(A)}(b_1,\dots,b_k) = a$. We get

$$\begin{aligned}
p^{(M)}(x_1,\dots,x_n) &= f^{(B_a)}(\varphi_{b_1,a}(p_1^{(B_{b_1})}(x_1,\dots,x_n)),\dots,\varphi_{b_k,a}(p_k^{(B_{b_k})}(x_1,\dots,x_n)) \\
&= f^{(B_a)}(\varphi_{b_1,a}(p_1^{(B_{b_1})}(\varphi_{a_1,b_1}(x_1),\dots,\varphi_{a_n,b_1}(x_n))),\dots, \\
&\qquad \varphi_{b_k,a}(p_k^{(B_{b_k})}(\varphi_{a_1,b_k}(x_1),\dots,\varphi_{a_n,b_k}(x_n)))) \\
&= f^{(B_a)}(p_1^{(B_a)}(\varphi_{b_1,a}(\varphi_{a_1,b_1}(x_1)),\dots,\varphi_{b_1,a}(\varphi_{a_n,b_1}(x_n))),\dots, \\
&\qquad p_k^{(B_a)}(\varphi_{b_k,a}(\varphi_{a_1,b_k}(x_1)),\dots,\varphi_{b_k,a}(\varphi_{a_n,b_k}(x_n)))) \\
&= f^{(B_a)}(p_1^{(B_a)}(\varphi_{a_1,a}(x_1),\dots,\varphi_{a_n,a}(x_n)),\dots, \\
&\qquad p_k^{(B_a)}(\varphi_{a_1,a}(x_1),\dots,\varphi_{a_n,a}(x_n))) \\
&= p^{(B_a)}(\varphi_{a_1,a}(x_1),\dots,\varphi_{a_n,a}(x_n)).
\end{aligned}$$

Therefore under the above mentioned assumptions we obtain

$$\begin{aligned}
p^{(M)}(x_1,\dots,x_n) &= p^{(B_a)}(\varphi_{a_1,a}(x_1),\dots,\varphi_{a_n,a}(x_n)) \\
&= q^{(B_a)}(\varphi_{a_1,a}(x_1),\dots,\varphi_{a_n,a}(x_n)) = q^{(M)}(x_1,\dots,x_n).
\end{aligned}$$

□

Corollary 8. *Let a basic algebra $\mathscr{A} = (A,f)$ and every cell $\mathscr{B}_a = (B_a,f)$, $a \in A$, be groupoids. Let $\mathscr{S}^{(f)}$ be a family of homomorphisms $\varphi_{a,b}\colon B_a \to B_b$ such that* (2.1), (2.6a) *and* (2.7a) *hold. If an identity*

$$p(x_1,\dots,x_n) = q(x_1,\dots,x_n)$$

holds in $\mathscr{A}$ and also in each $\mathscr{B}_a$ then it holds in the groupoid $\mathscr{A}(\mathscr{B})$, too.

3. N-skew lattices

In this section we will give a characterization of N-skew lattices using the construction of cell algebras.

An algebra $(L,\wedge,\vee)$ of the type $(2,2)$ is called a noncommutative lattice if the binary operations $\wedge$ and $\vee$ are associative, idempotent and satisfy some absorption identities.

M. D. Gerhards has investigated noncommutative lattices satisfying the identities

$$x\wedge(x\vee y)=x \qquad \& \qquad (y\wedge x)\vee x=x \tag{3.1}$$

and

$$(z\vee y\vee x)\wedge(x\vee y)=y\vee x \qquad \& \qquad (y\wedge x)\vee(x\wedge y\wedge z)=x\wedge y \tag{3.2}$$

which are called prelattices (fastverbands). In [3] it is shown that every prelattice is the direct product of a lattice and a nest. In [2] M. D. Gerhards characterized prelattices as relational structures. Recall that a nest is an algebra $(L,\wedge,\vee)$ of the type $(2,2)$ satisfying the identities

$$x\wedge y=x \qquad \& \qquad y\vee x=x. \tag{3.3}$$

V. Slavík investigated prelattices in [11] and varieties of prelattices in [12].

M. Yamada and N. Kimura in [13] investigated idempotent semigroups (bands) satisfying the identity $xyz=xzy$ and showed that they are semilattices of trivial algebras (seminests). In [6] A. Haviar introduced a larger class of noncommutative lattices, so-called N-skew lattices, which can be characterized as relational systems, too. N-skew lattices are noncommutative lattices satisfying the identity (3.1) and the identities

$$x\wedge(y\wedge z)=x\wedge(z\wedge y) \qquad \& \qquad (z\vee y)\vee x=(y\vee z)\vee x. \tag{3.4}$$

Theorem 9. *An algebra $(L,\wedge,\vee)$ of the type $(2,2)$ is an N-skew lattice if and only if $(L,\wedge,\vee)$ is isomorphic to a cell algebra $\mathscr{A}(\mathscr{B})$ in which the basic algebra $\mathscr{A}$ is a lattice, every cell $\mathscr{B}_a$, $a\in A$, is a nest and the system of mappings $\varphi^{(\wedge)}_{b,a}\colon B_b\to B_a$ and $\varphi^{(\vee)}_{a,b}\colon B_a\to B_b$ for each $a\leqslant b$, $a,b\in A$, satisfies the conditions* (2.6a) *and* (2.7a).

P r o o f. a) Let $\mathscr{L}=(L,\wedge,\vee)$ be an N-skew lattice. We define a relation Θ on L as follows:

$a\,\Theta\,b \Longleftrightarrow a\wedge b=a \quad \& \quad b\wedge a=b.$

The relation Θ is a congruence relation of $\mathscr{L}$, the algebra $\mathscr{L}/\Theta$ is a lattice (a modification of $\mathscr{L}$ in the variety of lattices) and every block $a\Theta = B_a$ is a nest (see [11]).

For $a\Theta \leqslant b\Theta$ we define mappings

$$\varphi^{(\wedge)}_{b\Theta,a\Theta}\colon\ b\Theta \to a\Theta \qquad \text{and} \qquad \varphi^{(\vee)}_{a\Theta,b\Theta}\colon\ a\Theta \to b\Theta$$

by

$$\text{(i)} \qquad \forall x \in b\Theta \qquad \varphi^{(\wedge)}_{b\Theta,a\Theta}(x) = x \wedge a,$$

$$\text{(ii)} \qquad \forall x \in a\Theta \qquad \varphi^{(\vee)}_{a\Theta,b\Theta}(x) = b \vee x.$$

Let $a_1 \in a\Theta$ and $b_1 \in b\Theta$. Since $x \wedge a_1 = x \wedge a_1 \wedge a = x \wedge a \wedge a_1 = x \wedge a$ (by (3.4)) and similarly $b_1 \vee x = b \vee x$, the mappings $\varphi^{(\wedge)}_{b\Theta,a\Theta}$ and $\varphi^{(\vee)}_{a\Theta,b\Theta}$ are defined correctly. (Moreover, the mappings $\varphi^{(\wedge)}_{b\Theta,a\Theta}$ and $\varphi^{(\vee)}_{a\Theta,b\Theta}$ are homomorphisms because $a\Theta$ and $b\Theta$ are nests.)

If $a\Theta \leqslant b\Theta \leqslant c\Theta$ then

$$\varphi^{(\wedge)}_{b\Theta,a\Theta}(\varphi^{(\wedge)}_{c\Theta,b\Theta}(x)) = \varphi^{(\wedge)}_{b\Theta,a\Theta}(x \wedge b) = (x \wedge b) \wedge a = x \wedge (a \wedge b) = x \wedge a = \varphi^{(\wedge)}_{c\Theta,a\Theta}(x)$$

and

$$\varphi^{(\wedge)}_{a\Theta,a\Theta}(x) = x \wedge a = x$$

and dually for $\varphi^{(\vee)}_{a\Theta,b\Theta}$, hence the mappings $\varphi^{(\wedge)}_{b\Theta,a\Theta}$ and $\varphi^{(\vee)}_{a\Theta,b\Theta}$ satisfy the conditions (2.6a) and (2.7a).

Let $\mathscr{S}^{(\wedge)}$ and $\mathscr{S}^{(\vee)}$ be systems of mappings

$$\mathscr{S}^{(\wedge)} = \{\varphi^{(\wedge)}_{b\Theta,a\Theta};\, a\Theta \leqslant b\Theta\}, \quad \mathscr{S}^{(\vee)} = \{\varphi^{(\vee)}_{a\Theta,b\Theta};\, a\Theta \leqslant b\Theta\}.$$

Denote by $\sqcap$ and $\sqcup$ the operations of a cell algebra with the basic algebra $\mathscr{L}/\Theta$, cells $B_a = a\Theta$, $a\Theta \in L/\Theta$ and systems of mappings $\mathscr{S}^{(\wedge)}$, $\mathscr{S}^{(\vee)}$. For any elements $x, y \in \bigcup\limits_{a \in L} B_a = M$ we get

$$x \sqcap y = \varphi^{(\wedge)}_{x\Theta,x\wedge y\Theta}(x) \wedge \varphi^{(\wedge)}_{y\Theta,x\wedge y\Theta}(y) = (x \wedge (x \wedge y)) \wedge (y \wedge (x \wedge y)) = x \wedge y$$

and dually $x \sqcup y = x \vee y$.

b) Conversely, let $\mathscr{A}(\mathscr{B})$ be a cell algebra for which the basic algebra $\mathscr{A}$ is a lattice $(A, \wedge, \vee)$, let each cell B_a, $a \in A$, be a nest and for every $a \leqslant b$ let the mappings

$$\varphi^{(\wedge)}_{b,a}\colon\ B_b \to B_a, \quad \varphi^{(\vee)}_{a,b}\colon\ B_a \to B_b$$

satisfy the conditions (2.6a) and (2.7a).

The operations of the basic algebra as well as those of every cell are associative, idempotent and the mappings $\varphi_{b,a}^{(\wedge)}$, $\varphi_{a,b}^{(\vee)}$ are homomorphisms, hence by Theorem 5 the operations of the cell algebra $\mathscr{A}(\mathscr{B})$ are also associative and idempotent. By Corollary 8 the operations of the cell algebra $\mathscr{A}(\mathscr{B})$ satisfy the identity (3.4), too.

For any elements $x \in B_a$, $y \in B_b$ we get

$$\begin{aligned} x \sqcap (x \sqcup y) &= x \sqcap (\varphi_{a,a\vee b}^{(\vee)}(x) \vee \varphi_{b,a\vee b}^{(\vee)}(y)) = x \sqcap \varphi_{b,a\vee b}^{(\vee)}(y) \\ &= \varphi_{a,a\wedge(a\vee b)}^{(\wedge)}(x) \wedge \varphi_{a\vee b,a\wedge(a\vee b)}^{(\wedge)}(\varphi_{b,a\vee b}^{(\vee)}(y)) = \varphi_{a,a}^{(\wedge)}(x) = x \end{aligned}$$

and dually $(y \sqcap x) \sqcup x = x$. □

Now let us assume that the basic algebra of a cell algebra is a distributive lattice. A lattice is distributive if it satisfies the identity $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ which is satisfied in every nest, too.

A slight change in the proof of Theorem 9 enables us to show the next statement.

Theorem 10. *An algebra $(L, \wedge, \vee)$ of the type $(2,2)$ is a distributive N-skew lattice if and only if $(L, \wedge, \vee)$ is isomorphic to a cell algebra $\mathscr{A}(\mathscr{B})$ in which the basic algebra $\mathscr{A}$ is a distributive lattice, each cell $\mathscr{B}_a$, $a \in A$, is a nest and the system of mappings $\varphi_{b,a}^{(\wedge)}\colon B_b \to B_a$, $\varphi_{a,b}^{(\vee)}\colon B_a \to B_b$ for every $a \leqslant b$, $a, b \in A$, satisfies the conditions* (2.6a) *and* (2.7a).

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