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A p-form $\boldsymbol{F}$ is VSI (i.e., all its scalar invariants of arbitrary order vanish) in a n-dimensional spacetime *if and only if* it is of type N, its multiple null direction $\boldsymbol{\ell}$ is "degenerate Kundt", and $\pounds_{\boldsymbol{\ell}}\boldsymbol{F} = 0$. This recent result is reviewed in the present contribution and its main consequences are summarized. In particular, a subset of VSI Maxwell fields possesses a *universal* property, i.e., they also solve (virtually) any generalized (non-linear and with higher derivatives) electrodynamics, possibly also coupled to Einstein's gravity.

1. Introduction

The present contribution is summary of the main results of our recent work[1]. It is useful to start by defining the VSI property for a general tensor, i.e.,

Definition 1.1 (VSI tensors). *A tensor in an n-dimensional spacetime with metric g_{ab} is VSI_I if all the scalar polynomial invariants constructed from the tensor itself and its covariant derivatives up to order I ($I = 0, 1, 2, 3, \dots$) vanish. It is VSI if all its scalar polynomial invariants of* arbitrary order *vanish.*

As a generalization of the notion of null fields[2] to arbitrary p-forms, it is natural to introduce the following

Definition 1.2 (p-forms of type N). *At a spacetime point, a p-form $\boldsymbol{F}$ is of type N if it satisfies*

$$\ell^a F_{ab_1\dots b_{p-1}} = 0, \qquad \ell_{[a}F_{b_1\dots b_p]} = 0, \tag{1}$$

where $\boldsymbol{\ell}$ is a null vector (this follows from (1) and need not be assumed). The second condition can be equivalently replaced by $\ell^{a\,*}F_{ab_1\dots b_{n-p-1}} = 0$.

This is equivalent to the type N condition in the set-up of[3].

First, with the results of[4], it is easy to see that *a p-form $\boldsymbol{F}$ is VSI_0 iff it is is of type N*. Next, our main result is the VSI condition, given in the next section.

2. Main result: VSI p-forms

The main result of[1] is the following

Theorem 2.1 (VSI p-forms[1]). *The following two conditions are equivalent:*

(1) a non-zero p-form field $\boldsymbol{F}$ is VSI in a spacetime with metric g_{ab}
(2)(a) $\boldsymbol{F}$ possesses a multiple null direction $\boldsymbol{\ell}$, i.e., it is of type N
  (b) $\pounds_{\boldsymbol{\ell}}\boldsymbol{F} = 0$

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> *(c)* g_{ab} *is a* degenerate Kundt *metric, and* $\boldsymbol{\ell}$ *is the corresponding Kundt null direction.*

We observe that, in Theorem 2.1, the p-form $\boldsymbol{F}$ is not assumed to satisfy any particular field equations and the result is thus purely geometric (on the other hand, *if* $\boldsymbol{F}$ is taken to be *closed*, i.e., $\mathrm{d}\boldsymbol{F} = 0$, then condition 2b automatically follows from the type N condition 2a, and need not be assumed).

2.1. *Adapted coordinates*

From Theorem 2.1 with the results of[5,6], it follows that coordinates (u, r, x^α), adapted to $\boldsymbol{\ell} = \partial_r$, exist such that any VSI p-form can be written as

$$\boldsymbol{F} = \frac{1}{(p-1)!} f_{\alpha_1 \ldots \alpha_{p-1}}(u, x) \mathrm{d}u \wedge \mathrm{d}x^{\alpha_1} \wedge \ldots \wedge \mathrm{d}x^{\alpha_{p-1}}, \tag{2}$$

and the corresponding background metric as

$$\mathrm{d}s^2 = 2\mathrm{d}u \left[\mathrm{d}r + H(u, r, x)\mathrm{d}u + W_\alpha(u, r, x)\mathrm{d}x^\alpha \right] + g_{\alpha\beta}(u, x)\mathrm{d}x^\alpha \mathrm{d}x^\beta, \tag{3}$$

$$W_\alpha(u, r, x) = r W_\alpha^{(1)}(u, x) + W_\alpha^{(0)}(u, x), \tag{4}$$

$$H(u, r, x) = r^2 H^{(2)}(u, x) + r H^{(1)}(u, x) + H^{(0)}(u, x), \tag{5}$$

where $\alpha, \beta = 2 \ldots n-1$, and x denotes collectively the set of coordinates x^α.

The degenerate Kundt metric (3)–(5) includes, e.g., all VSI spacetimes[7,8], all pp-waves, and all Kundt Einstein (or aligned pure radiation) spacetimes – in particular, spacetimes of constant curvature (Minkowski and (A)dS). The corresponding Weyl and Ricci tensors are both of aligned type II, in general.

If $\boldsymbol{F}$ in (2) is required to obey the source-free Maxwell equations $\mathrm{d}\boldsymbol{F} = 0 = \mathrm{d}^*\boldsymbol{F}$, the following conditions are obtained

$$f_{[\alpha_2 \ldots \alpha_{p-1}, \alpha_1]} = 0, \qquad (\sqrt{\tilde{g}} \, f^{\beta \alpha_1 \ldots \alpha_{p-2}})_{,\beta} = 0, \tag{6}$$

where $\tilde{g} \equiv \det g_{\alpha\beta} = -\det g_{ab} \equiv -g$. These are Maxwell's equations for the $(p-1)$-form $\boldsymbol{f}$ in the $(n-2)$-dimensional Riemannian geometry associated with $g_{\alpha\beta}$, i.e., $\boldsymbol{f}$ must be *harmonic* (recall, however, that $\boldsymbol{f}$ can also depend on u). For $n = 3, 4$, it can be argued that to any $\boldsymbol{F}$ which is VSI, it can always be associated a solution of the Maxwell equations which is also VSI[1]. It is also interesting to observe that the effective Maxwell equations (6) are "immune" to adding a Chern-Simons term (except when this is linear, see[1] for details) to the full Maxwell equations.

2.2. *Universal solutions of generalized electrodynamics*

It was already known to Schrödinger[9,10] that all null Maxwell fields (with $n = 4 = 2p$) solve the equations for the electromagnetic field in any non-linear electrodynamics. More generally, we observe that a subset of the VSI Maxwell fields described above possesses a "universal" property, i.e., they solve simultaneously any electrodynamics whose field equations can be expressed as $\mathrm{d}\boldsymbol{F} = 0$, $*\mathrm{d}*\tilde{\boldsymbol{F}} = 0$, where $\tilde{\boldsymbol{F}}$

can be any p-form constructed from $\boldsymbol{F}$ and its covariant derivatives. For example, any VSI Maxwell $\boldsymbol{F}$ is universal if the background is a Kundt spacetime of Weyl and traceless-Ricci type III (aligned) with $DR = 0 = \delta_i R$ (an affine parameter and a frame parallelly transported along $\boldsymbol{\ell}$ are assumed – cf. [1] for the notation employed here). In particular, Ricci flat and Einstein Kundt spacetimes of Weyl type III/N/O can occur, the latter including Minkowski and (A)dS.

An explicit simple example is given by the Maxwell field

$$\boldsymbol{F} = e^{x/2}c(u)\mathrm{d}u \wedge \left(-\cos\frac{ye^u}{2}\mathrm{d}x + e^u \sin\frac{ye^u}{2}\mathrm{d}y \right), \tag{7}$$

defined in the type III vacuum spacetime found by Petrov (eq. (31.40) in [11])

$$\mathrm{d}s^2 = 2\mathrm{d}u\left[\mathrm{d}r + \frac{1}{2}\left(xr - xe^x\right)\mathrm{d}u\right] + e^x(\mathrm{d}x^2 + e^{2u}\mathrm{d}y^2). \tag{8}$$

More general results will be presented elsewhere.

3. Einstein-Maxwell solutions

3.1. *General field equations*

The previous discussion applies to VSI test fields, since we have not considered the consequences of the backreaction on the spacetime geometry. In the full Einstein-Maxwell theory with an arbitrary cosmological constant Λ, one finds that the metric functions entering (3) must satisfy the following set of equations (obtained in [1] refining the results of [12])

$$\mathcal{R}_{\alpha\beta} = \frac{2\Lambda}{n-2}g_{\alpha\beta} + \frac{1}{2}W_\alpha^{(1)}W_\beta^{(1)} - W_{(\alpha||\beta)}^{(1)}, \tag{9}$$

$$2H^{(2)} = \frac{\mathcal{R}}{2} - \frac{n-4}{n-2}\Lambda + \frac{1}{4}W^{(1)\alpha}W_\alpha^{(1)}, \tag{10}$$

$$W_{\alpha||\beta}^{(1)\ \beta} = \frac{1}{2}W^{(1)\beta}\left(3W_{\alpha||\beta}^{(1)} - W_{\beta||\alpha}^{(1)}\right) + W_\alpha^{(1)}\left(W^{(1)\beta}{}_{||\beta} - \frac{1}{2}W^{(1)\beta}W_\beta^{(1)} - \frac{2\Lambda}{n-2}\right), \tag{11}$$

$$2H_{,\alpha}^{(1)} = -g_{\alpha\beta,u}{}^{||\beta} + 2W_{[\alpha||\beta]}^{(0)\ \beta} - 2W^{(0)\beta}W_{\alpha||\beta}^{(1)} + (W^{(0)\beta}W_\beta^{(1)})_{,\alpha} + W_{\alpha,u}^{(1)} + 2(\ln\sqrt{\tilde{g}})_{,u\alpha}$$
$$+ W_\alpha^{(1)}\left[W^{(0)\beta}W_\beta^{(1)} - W^{(0)\beta}{}_{||\beta} + (\ln\sqrt{\tilde{g}})_{,u}\right] + \frac{4\Lambda}{n-2}W_\alpha^{(0)}, \tag{12}$$

$$\Delta H^{(0)} + W^{(1)\alpha}H_{,\alpha}^{(0)} + W^{(1)\alpha}{}_{||\alpha}H^{(0)} = W^{(0)\beta}W_\beta^{(0)}\left(\frac{1}{2}W^{(1)\alpha}{}_{||\alpha} - \frac{2\Lambda}{n-2}\right)$$
$$+ H^{(1)}\left[W^{(0)\alpha}{}_{||\alpha} - (\ln\sqrt{\tilde{g}})_{,u}\right] - \frac{1}{2}(W^{(0)\alpha}W_\alpha^{(1)})^2 + W^{(0)[\alpha||\beta]}W_{[\alpha||\beta]}^{(0)} + W_{\alpha,u}^{(0)\ ||\alpha}$$
$$- W^{(0)\beta}\left(2W^{(1)\alpha}W_{[\alpha||\beta]}^{(0)} + W_{\beta,u}^{(1)} - 2H_{,\beta}^{(1)}\right) - (\ln\sqrt{\tilde{g}})_{,uu} + \frac{1}{4}g^{\alpha\beta}{}_{,u}g_{\alpha\beta,u} - \kappa_0\mathcal{F}^2. \tag{13}$$

Heree $\mathcal{R}_{\alpha\beta}$, $\mathcal{R}$ and $||$ denote, respectively, the Ricci tensor, the Ricci scalar and the covariant derivative associated with $g_{\alpha\beta}$, $W^{(1)\alpha} \equiv g^{\alpha\beta}W_\beta^{(1)}$, Δ is the Laplace operator in the geometry of the transverse metric $g_{\alpha\beta}$, and κ_0 is a gravitational coupling constant. The Maxwell equations (6) must also be satisfied.

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The simplest examples one can construct are electromagnetic and gravitational "plane-fronted" waves (with $W_\alpha^{(0)} = 0$) propagating in a constant curvature background, giving rise to Kundt waves of Weyl type N (in four dimensions see, e.g.,[13,14]). More general (e.g., with $W_\alpha^{(0)} \neq 0$) degenerate Kundt metrics with null Maxwell fields are also known (see[11,15,16] and references therein for $n = 4$). The case of VSI p-form Maxwell fields in VSI and pp-waves spacetimes have been discussed in[1] (where further references can be found).

3.2. *Universal Einstein-Maxwell solutions*

Some of the universal Maxwell fields mentioned in section 2.2 can also be used to construct exact solutions of full general relativity, where the energy-momentum tensor T_{ab} associated with the electromagnetic field is determined in the generalized electrodynamics (in terms of $\boldsymbol{F}$ and its covariant derivatives – cf.[17] for an example). For example, as pointed out in[1], all VSI spacetimes with $L_{i1} = 0 = L_{1i}$ (i.e., the recurrent ones) coupled to an aligned VSI p-form field that solve the standard Einstein-Maxwell equations are also exact solutions of gravity coupled to generalized electrodynamics, provided $p > 1$ and $\delta_i F_{1j_1\ldots j_{p-1}} = 0$ (in an "adapted" parallely transported frame, i.e., such that $\overset{i}{M}_{jk} = 0$). Within this family, metrics of Weyl type N are necessarily pp-waves, for which such a universal property was pointed out in[18–20], at least for certain values of p. But metrics of Weyl type III are also permitted, including pp-waves ($L_{11} = 0$) and also genuinely recurrent ($L_{11} \neq 0$) spacetimes (for $n = 4$, $p = 3$ this was discussed in[21]). One explicit example of the latter solutions in 4D is given by the Maxwell field (7) with the metric

$$\mathrm{d}s^2 = 2\mathrm{d}u\left[\mathrm{d}r + \frac{1}{2}\left(xr - xe^x - 2\kappa_0 e^x c^2(u)\right)\mathrm{d}u\right] + e^x(\mathrm{d}x^2 + e^{2u}\mathrm{d}y^2), \qquad (14)$$

which is a modification of (8) taking into account the backreaction.

As in section 2.2, the above discussion applies to generalized electrodynamics with arbitrary higher-order derivative "corrections". A special instance of this result is the fact that Einstein-Maxwell solutions with aligned null electromagnetic fields (not necessarily VSI) are also solution of NLE coupled to gravity, as previously demonstrated in[22–24].

4. Further remarks

For certain purposes, it may be useful to observe that if $\boldsymbol{F}$ is VSI$_3$ then it is necessarily VSI (this follows from the proof of Theorem 2.1 given in[1]). For completeness, let us thus also give the necessary and sufficient conditions for a p-form $\boldsymbol{F}$ to be VSI$_1$ or VSI$_2$ (recall that VSI$_0$ means type N, as mentioned in section 1):

Proposition 4.1 (VSI$_1$ and VSI$_2$ p-forms[1]). *A p-form $\boldsymbol{F}$ is VSI$_1$ iff it is is of type N, $\pounds_{\boldsymbol{\ell}}\boldsymbol{F} = 0$, $\boldsymbol{\ell}$ is Kundt. It is VSI$_2$ iff it is is of type N, $\pounds_{\boldsymbol{\ell}}\boldsymbol{F} = 0$, $\boldsymbol{\ell}$ is Kundt and (at least) doubly aligned with the Riemann tensor.*

	vector $\boldsymbol{\ell}$	p-form $\boldsymbol{F}$	Riemann
VSI_0	null	N	III (N,O)
VSI_1	Kundt	N, $\mathscr{L}_{\boldsymbol{\ell}}\boldsymbol{F} = 0$, Kundt	(*) N, $\kappa = 0$, $\sigma\Psi_4 = \rho\Phi_{22}$
VSI_2	Kundt, Riem II	N, $\mathscr{L}_{\boldsymbol{\ell}}\boldsymbol{F} = 0$, Kundt, Riem II	III, Kundt
VSI_3	degKundt	N, $\mathscr{L}_{\boldsymbol{\ell}}\boldsymbol{F} = 0$, degKundt	"
$\vdots$	"	"	"
VSI	"	"	"

Fig. 1. Comparing various VSI_I conditions for a 1-form $\boldsymbol{\ell}$, a p-form $\boldsymbol{F}$[1] and the Riemann tensor[7,8,25]. The quotation marks " mean that the same conditions as in the next-upper row apply. The Riemann VSI_1 condition (*) needs some comments for two reasons. First, it has been investigated only in the case $n = 4$[25] (and indeed it is given above in the standard 4D NP notation). Additionally, it is sufficient but not necessary (contrary to the rest of the table): also Kundt spacetimes of Riemann type III are VSI_1 (but not "properly", i.e., they are in fact VSI, cf. the lower rows in the same column).

It is also interesting to observe that Theorem 2.1 and Proposition 4.1 apply also in the limiting case $p = 1$, i.e., when $\boldsymbol{F}$ is a vector field. In particular, when $p = 1$, Theorem 2.1 reduces to: *a vector field $\boldsymbol{\ell}$ is VSI in a spacetime with metric g_{ab} iff $\boldsymbol{\ell}$ is Kundt and affinely parameterized, and g_{ab} is a degenerate Kundt metric w.r.t $\boldsymbol{\ell}$.*

The above comments are summarized in the first two columns of Fig. 1. The last column gives corresponding results in the case of the Riemann tensors (i.e., for the VSI_I spacetimes)[7,8,25]. While certain conditions turn out to be similar, there is also an important difference: for the Riemann tensor one has already $VSI_2 \Rightarrow VSI$.

Acknowledgments

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