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Abstract

We prove that higher dimensional Einstein spacetimes which possess a geodesic, non-degenerate double Weyl aligned null direction (WAND) $\boldsymbol{\ell}$ must additionally possess a second double WAND (thus being of type D) if either: (a) the Weyl tensor obeys $C_{abc[d}\ell_{e]}\ell^c = 0$ ($\Leftrightarrow \Phi_{ij} = 0$, i.e., the Weyl type is II(abd)); (b) $\boldsymbol{\ell}$ is twistfree. Some comments about an extension of the Goldberg-Sachs theorem to six dimensions are also made.

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1 Introduction and summary of main results

The Petrov classification has been extended to higher dimensions using the notions of null alignment and boost weight [1] (cf. also the review [2]). It provides one with a useful tool to classify and construct exact solutions, as well as to understand geometric properties of spacetimes (see [2] for examples of certain applications and for references). A class of spacetimes of particular interest consists of those of Weyl type II. In any dimension $n \geq 4$, the Weyl type II condition of [1] can be expressed as [3]

$$\ell_{[e}C_{a]b[cd}\ell_{f]}\ell^b = 0, \tag{1}$$

where $\boldsymbol{\ell}$ is a null vector field that defines a *multiple Weyl aligned null direction* (mWAND). Equivalently, the Weyl type II can be defined by the existence of a null frame in which all positive boost weight (b.w.) components of the Weyl tensor vanish [1, 2]. In four dimensions this is equivalent to the Petrov type II, for which the highest b.w. components (i.e., b.w. 0) are represented by the complex scalar Ψ_2. However, in higher dimension n the Weyl tensor admits $\frac{1}{12}(n-3)(n-2)[n(n-3)+8]$ real b.w. 0 components. Thus the number of such components increases rapidly with the dimension of the spacetime and, not surprisingly, new subtypes

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of type II, not occurring in four dimensions, appear.[1] One of these is the subtype II(abd) (considered also in this paper), which can be defined by the existence of a null vector field $\boldsymbol{\ell}$ obeying

$$C_{abc[d}\ell_{e]}\ell^c = 0, \tag{2}$$

(which is stronger than (1)) or equivalently by the vanishing of the part of b.w. 0 Weyl components described by the matrix Φ_{ij} (introduced in (5)).

While for $n = 4$ large classes of type II Einstein spacetimes are known [4], there appear to exist various restrictions on solutions of genuine type II in more than four dimensions. First, if an Einstein spacetime admits a non-geodesic mWAND, it is necessarily of type D [5] (more precisely, of type D(d), and there exists an infinity of mWANDs, including a geodesic one). Additionally, if $\boldsymbol{\ell}$ is twistfree and non-degenerate (as defined in (10) below) and $\Phi_{ij} \neq 0$ (i.e., the type is *not* II(abd)), then again the type is necessarily D (more precisely type D(bd), see Proposition 4.1 of [6] and cf. also [7]; this case includes, e.g., Robinson-Trautman solutions with $\mu \neq 0$ [8]). In five dimensions, the same conclusion remains true even if one drops the twistfree assumption [9] (while $\Phi_{ij} \neq 0$ follows automatically for type II with $n = 5$ [10] and need not be assumed). In six dimensions, a similar result has been obtained in [11] in the Ricci-flat case (again without assuming the twistfree condition, but with an additional assumption on the asymptotic fall-off of the Weyl tensor – which implies $\Phi_{ij} \neq 0$ and which is automatically satisfied in five dimensions). Restrictions in the $n = 5$ degenerate case have been obtained in [12].

The purpose of the present contribution is to present some new results which hold in arbitrary higher dimensions for the case of a non-degenerate mWAND. First, we will prove the following

Proposition 1. *An $n > 5$ dimensional Einstein spacetime of type II(abd) with a geodesic mWAND $\boldsymbol{\ell}$ such that $\det \boldsymbol{\rho} \neq 0$ must be in fact of type D(abd). In a frame such that $L_{i1} = 0$, the null vector $\boldsymbol{n}$ is another mWAND.*

Examples of such spacetimes with $n > 6$ can be found in appendix F of [6] (in particular, in appendix F.4 in the twisting case). Other non-twisting examples (for $n \geq 6$) are given by the Robinson-Trautman solutions with $\mu = 0$ [8].

As a further restriction on type II spacetimes, we will additionally prove

Proposition 2. *An $n > 4$ dimensional Einstein spacetime that admits a non-degenerate, non-twisting mWAND $\boldsymbol{\ell}$ is necessarily of type D(bd) and, in a frame such that $L_{i1} = 0$, the null vector $\boldsymbol{n}$ is another mWAND. If $\Phi_{ij} \neq 0$, $\boldsymbol{\ell}$ must be shearfree [6] and $\boldsymbol{\ell}$ and $\boldsymbol{n}$ are the only WANDs. If $\Phi_{ij} = 0$, the type further specializes to D(abd) and, for $n > 6$, the spacetime may admit a continuous infinity of mWANDs.*

This is a refinement of Proposition 4.1 of [6]. More details are discussed in section 2.3 (including the conditions for having an infinite number of mWANDs in the $\Phi_{ij} = 0$ case). We observe that if the non-degeneracy assumption is dropped, examples of Einstein spacetimes of genuine type II can be easily constructed taking direct products or Brinkmann warps of known lower-dimensional type II solutions [2, 14].

As a by-product of the main calculations, we will also show that

Proposition 3. *In an $n > 4$ dimensional Einstein spacetime of type III or N, the mWAND is necessarily degenerate (i.e., $\det \boldsymbol{\rho} = 0$).*

This result was already known for the type N [13]. For the type III it was known in the following cases: (i) for $n = 5, 6$ [13, 15]; (ii) for a non-twisting mWAND (for any $n > 4$) [13]; (iii) for a twisting mWAND with $n > 6$ under certain additional "genericity" assumptions on the Weyl tensor [13]. (For (i)–(iii) and for the type N, the optical matrix must in fact have rank 2 [13, 15].)

The proofs of the above propositions are presented in section 2. In connection with these results, in section 3 we study some aspects of an extension of the Goldberg-Sachs theorem to six dimensions. For the case of non-degenerate mWANDS, we arrive at some restrictions on the optical matrix compatible with Einstein spacetimes of type II(abd) (section 3.1). Some of these results can also be extended to the general type II (section 3.2).

[1] The definitions of the Weyl subtypes, such as D(d) and II(abd), can be found in [1–3]. The notation and various symbols to be used in this paper are explained at the end of this section (see also [1, 2, 10, 13] for more details).

Notation Let us summarize the Newman-Penrose notation used in the present paper. The n frame vectors $\boldsymbol{m}_{(a)}$ consists of two null vectors $\boldsymbol{\ell} \equiv \boldsymbol{m}_{(0)}$, $\boldsymbol{n} \equiv \boldsymbol{m}_{(1)}$ and $n-2$ orthonormal spacelike vectors $\boldsymbol{m}_{(i)}$, with $a, b \ldots = 0, \ldots, n-1$ while $i, j \ldots = 2, \ldots, n-1$ [1,2]. The Ricci rotation coefficients are defined by [13]

$$L_{ab} = \ell_{a;b}, \qquad N_{ab} = n_{a;b}, \qquad \overset{i}{M}_{ab} = m^{(i)}_{a;b}, \tag{3}$$

and satisfy the identities $L_{0a} = N_{1a} = N_{0a} + L_{1a} = \overset{i}{M}_{0a} + L_{ia} = \overset{i}{M}_{1a} + N_{ia} = \overset{i}{M}_{ja} + \overset{j}{M}_{ia} = 0$.

Covariant derivatives along the frame vectors are denoted as

$$D \equiv \ell^a \nabla_a, \qquad \triangle \equiv n^a \nabla_a, \qquad \delta_i \equiv m^a_{(i)} \nabla_a. \tag{4}$$

For the non-zero components of a Weyl tensor of type II, we use the symbols [10,13]

$$\Phi_{ij} = C_{0i1j}, \qquad \Phi^S_{ij} = \Phi_{(ij)}, \qquad \Phi^A_{ij} = \Phi_{[ij]}, \qquad \Phi = \Phi_{ii}, \tag{5}$$

$$\Psi_i = C_{101i}, \qquad \Psi_{ijk} = \frac{1}{2} C_{1kij}, \qquad \Psi_{ij} = \frac{1}{2} C_{1i1j}, \tag{6}$$

which satisfy the identities $C_{01ij} = 2C_{0[i|1|j]} = 2\Phi^A_{ij}$, $2C_{0(i|1|j)} = 2\Phi^S_{ij} = -C_{ikjk}$, $2C_{0101} = -C_{ijij} = 2\Phi$, $\Psi_i = 2\Psi_{ijj}$, $\Psi_{\{ijk\}} = 0$ (curly brackets denoting cyclicity), $\Psi_{ijk} = -\Psi_{jik}$, $\Psi_{ij} = \Psi_{ji}$, and $\Psi_{ii} = 0$. Throughout the paper, the Einstein equations $R_{ab} = \frac{R}{n} g_{ab}$ hold, so that $R_{abcd;e} = C_{abcd;e}$. It will be convenient to define the rescaled Ricci scalar

$$\tilde{R} = \frac{R}{n(n-1)}. \tag{7}$$

2 Proof of Propositions 1–3

2.1 Preliminaries: Einstein spacetimes with a non-degenerate mWAND

Let us consider a n-dimensional Einstein spacetime of type II or more special ($n > 4$). It follows [5] that there exists a null vector field $\boldsymbol{\ell}$ that satisfies (1) while being geodesic and affinely parametrized, i.e.,

$$\ell_{a;b} \ell^b = 0. \tag{8}$$

The rank of the $(n-2) \times (n-2)$ *optical matrix* with components

$$\rho_{ij} \equiv L_{ij} = \ell_{a;b} m^a_{(i)} m^b_{(j)}, \tag{9}$$

is thus frame-independent [16], for any choice of a frame adapted to $\boldsymbol{\ell}$.

In this paper we consider the *non-degenerate* (or full-rank) case, i.e., from now on we assume

$$\det \boldsymbol{\rho} \neq 0. \tag{10}$$

We will find certain conditions under which the genuine type II cannot occur.

For convenience, we take our frame to be parallelly propagated along $\boldsymbol{\ell}$ [16], so that (with (8))

$$L_{i0} = 0, \qquad L_{10} = 0, \qquad \overset{i}{M}_{j0} = 0, \qquad N_{i0} = 0. \tag{11}$$

Furthermore, thanks to (10), we can perform a null rotation about $\boldsymbol{\ell}$ such that (cf. appendix D.2.6 of [17] and lemma 1 of [18])

$$L_{i1} = 0. \tag{12}$$

Given $\boldsymbol{\ell}$, this uniquely fixes the null direction defined by $\boldsymbol{n}$ (this choice will be useful in the following). We also define an affine parameter r such that

$$\boldsymbol{\ell} = \partial_r. \tag{13}$$

With the above assumptions, the Sachs equation $D\boldsymbol{\rho} = -\boldsymbol{\rho}^2$ [13,16] fixes the r-dependence of $\boldsymbol{\rho}$ (thanks to the simple identity $D(\boldsymbol{\rho}\boldsymbol{\rho}^{-1}) = 0$) [7,17,19]

$$\boldsymbol{\rho}^{-1} = r\boldsymbol{I} - \boldsymbol{b}, \tag{14}$$

where $\boldsymbol{I}$ is the identity matrix and $D\boldsymbol{b} = 0$. We can restrict ourselves to the case $\boldsymbol{b} \neq 0$, for the case $\boldsymbol{b} = 0$ reduces to the Robinson-Trautman metrics, already known to be of type D [8]. Note that $\boldsymbol{\ell}$ is twistfree iff $b_{[ij]} = 0$ [7]. For later purposes, it is useful to observe [7] that, for large r, eq. (14) gives

$$\rho_{ij} = \frac{1}{r}\delta_{ij} + \frac{1}{r^2}b_{ij} + O(r^{-3}). \tag{15}$$

2.2 Type II(abd) spacetimes: proof of Propositions 1 and 3

Let us now focus on Einstein spacetimes of type II(abd), i.e., such that

$$\Phi_{ij} = 0, \tag{16}$$

in the frame specified above (this case is of interest only for $n > 5$, since (16) means that all b.w. 0 Weyl components vanish when $n = 5$ [10]).

The r-dependence of the Weyl tensor can be determined by using the Bianchi equations containing D-derivatives. Using (16) and the fact that $\boldsymbol{\ell}$ is a mWAND, eq. (B.12, [13]) (cf. also (7, [7])) reduces to

$$DC_{ijkm} = -C_{ijkl}\rho_{lm} + C_{ijml}\rho_{lk}, \tag{17}$$

which, using (14), implies $D(C_{ijkm}\rho^{-1}_{kl}\rho^{-1}_{ms}) = 0$. By integration one gets

$$C_{ijkm} = \rho_{lk}\rho_{sm}c_{ijls}. \tag{18}$$

The integration functions $c_{ijls}(= -c_{jils} = c_{lsij})$ do not depend on r and, thanks to $C_{ijkj} = 0$ (which follows from (16)) and (18) with (15) (cf. also (27) and (22) of [10]), satisfy

$$c_{ikjk} = 0, \tag{19}$$

$$c_{ijkl}b_{(lj)} = 0, \qquad c_{ijkl}b_{[lj]} = 0. \tag{20}$$

Next, eqs. (B.1), (B.6), (B.9) and (B.4) of [13] (cf. also (16)–(18), (22) of [7]) reduce to

$$D\Psi_i = -2\Psi_s\rho_{si}, \tag{21}$$

$$2D\Psi_{ijk} = -2\Psi_{ijs}\rho_{sk} - \Psi_i\rho_{jk} + \Psi_j\rho_{ik}, \tag{22}$$

$$D\Psi_{jki} = 2\Psi_{[k|si}\rho_{s|j]} + \Psi_i\rho_{[jk]}, \tag{23}$$

$$2D\Psi_{ij} = -2\Psi_{is}\rho_{sj} + \delta_j\Psi_i + \Psi_i L_{1j} + \Psi_s \overset{s}{M}_{ij} . \tag{24}$$

Thanks to (14), eqs. (21)–(23) can be easily integrated, giving

$$\Psi_i = \rho_{lk}\rho_{ki}\psi_l, \tag{25}$$

$$\Psi_{ijk} = \rho_{l[i}\rho_{j]k}\psi_l + \rho_{lk}\hat{\psi}_{ijl}, \tag{26}$$

$$\Psi_{jki} = \rho_{nj}\rho_{mk}(\psi_{nmi} - b_{[nm]}\psi_l\rho_{li}), \tag{27}$$

where ψ_l, $\hat{\psi}_{ijl}$ and ψ_{nmi} are r-independent quantities. Comparing (26) and (27) (e.g., using (15)) immediately gives $\hat{\psi}_{ijl} = 0$. Using this, the trace of (26) with (25) additionally yields (for $n > 4$) $\psi_l = 0$, and therefore

$$\Psi_i = 0 = \Psi_{ijk}. \tag{28}$$

Thanks to these, eq. (24) gives

$$\Psi_{ij} = \rho_{kj}\psi_{ik}, \tag{29}$$

with $\psi_{[ik]} = 0 = \psi_{ii}$ and $D\psi_{ik} = 0$.

In order to proceed, we will also need (B.13, [13]) (cf. also (B11, [11])), which here becomes

$$-\triangle C_{ijkm} = 2\Psi_{im}\rho_{jk} + 4\Psi_{[j|k}\rho_{|i]m} - 2\Psi_{jm}\rho_{ik} + 2C_{ij[k|s} \overset{s}{M}_{|m]1} + 2C_{[i|skm} \overset{s}{M}_{|j]1} + 2C_{ij[k|s}N_{s|m]}. \tag{30}$$

The r-dependence of the last term can be fixed using the Ricci identity (11j, [16]), which with (11) and (16) reduces to $DN_{ij} = -N_{ik}\rho_{kj} - \tilde{R}\delta_{ij}$. This gives

$$N_{ij} = \rho_{kj}\left(-\frac{1}{2}\tilde{R}r^2\delta_{ik} + \tilde{R}rb_{ik} + n_{ik}\right), \tag{31}$$

where $Dn_{ik} = 0$. The trace of (30) (with (14), (18), (29), (31) and (19)) leads to $\psi_{ik} = 0$ (thanks to $n > 4$) and thus

$$\Psi_{ij} = 0. \tag{32}$$

We have thus proven that all Weyl components of negative b.w. vanish identically, i.e., $\boldsymbol{n}$ is also a mWAND. Therefore we have proven Proposition 1. In passing, the above calculations also prove Proposition 3.

2.3 Type II spacetimes with a twistfree mWAND: proof of Proposition 2

Using Proposition 1, as a refinement of Proposition 4.1 of [6] one additionally obtains Proposition 2.

Let us further observe here that, in that context, a *shearfree* $\boldsymbol{\ell}$ defines Robinson-Trautman spacetimes [8], for which the type is D(bd) or more special and $\boldsymbol{\ell}$ and $\boldsymbol{n}$ define two mWANDs (this follows from [8], cf. also [6,20]; $\boldsymbol{n}$ is fixed by the $L_{i1} = 0$ condition). The fact that for $\Phi_{ij} \neq 0$, there are no additional (m)WANDs follows from footnote 15 of [20] – where it is also observed that, instead, a Weyl tensor of type D(abd) can admit an infinity of mWANDs (this happens when the equation $C_{ijkl}z_l = 0$ admits a solution $z_l \neq 0$, cf. appendix F.4 of [6] for an example; however, in this case a non-zero C_{ijkl} requires $n > 6$[2]).

3 Partial extension of the Goldberg-Sachs theorem to six dimensions

Various results partially extending the Goldberg-Sachs theorem to higher dimensions have been obtained in recent years [5–7,10,13,15,17,21]. Here we point out some additional restrictions which apply to the spacetimes of section 2.2 (in section 3.1) as well as to general type II Einstein spacetimes (in section 3.2), at least when $n = 6$.

3.1 Type II(abd) spacetimes

Let us consider type II Einstein spacetimes and further assume (10) and (16). By Proposition 1, these spacetimes are in fact of type D(abd).

In addition to (19) and (20), the cyclicity $C_{i\{jkl\}} = 0$ (or (21, [10])) further gives

$$c_{i\{jkl\}} = 0, \qquad c_{is\{jk|}b_{s|m\}} = 0. \tag{33}$$

[2] When $n = 6$, the condition $C_{ijkl}z_l = 0$ with $C_{ikjk} = 0$ implies $C_{ijkl} = 0$ (this can be seen explicitly, e.g., using eqs. (34)–(36) given below, if one takes a frame such that $\boldsymbol{m}_{(2)}$ is parallel to z_i).

In any dimension $n > 5$, the algebraic equations (19), (20) and (33) can be used to constrain the permitted forms of b_{ij} and thus of the optical matrix ρ_{ij}. For the sake of definiteness, let us discuss explicitly here only the case of *six dimensions*. Before starting, let us observe that, in this case, a Weyl tensor of type D(abd) admits *precisely two* mWANDs $\boldsymbol{\ell}$ and $\boldsymbol{n}$,[3] which must therefore [5] be both *geodesics*. In an adapted frame, eqs. (B.7, [13]) and (B.9, [13]) reduce to $C_{ijkl}L_{i1} = 0 = C_{ijkl}N_{i0}$, which gives (since $n = 6$) $L_{i1} = 0 = N_{i0}$ (in agreement with Proposition 1).

Now, when $n = 6$, one can take as the only (ten) independent components of c_{ijkl}

$$W_{23} = c_{2323}, \qquad W_{25} = c_{2525}, \tag{34}$$

$$Z_{23} = c_{2434}, \qquad Z_{24} = c_{2343}, \qquad Z_{25} = c_{2353}, \qquad Z_{34} = c_{3242}, \qquad Z_{35} = c_{3252}, \qquad Z_{45} = c_{4252}, \tag{35}$$

$$Y_{23} = c_{2345}, \qquad Y_{24} = c_{2453}, \tag{36}$$

while all the remaining components can be determined using (19) and the first of (33).[4]

Let us perform an r-independent spin to align the frame vectors $\boldsymbol{m}_{(i)}$ to an eigenframe of $b_{(ij)}$ and denote by b_i its eigenvalues. From the first of (20) we obtain

$$(b_3 - b_4)W_{23} + (b_5 - b_4)W_{25} = 0, \qquad (b_2 - b_5)W_{23} + (b_4 - b_5)W_{25} = 0, \tag{37}$$

$$(b_5 - b_2)W_{23} + (b_3 - b_2)W_{25} = 0, \qquad (b_4 - b_3)W_{23} + (b_2 - b_3)W_{25} = 0, \tag{38}$$

$$(b_4 - b_5)Z_{23} = 0, \qquad (b_3 - b_5)Z_{24} = 0, \qquad (b_3 - b_4)Z_{25} = 0, \tag{39}$$

$$(b_2 - b_5)Z_{34} = 0, \qquad (b_2 - b_4)Z_{35} = 0, \qquad (b_2 - b_3)Z_{45} = 0, \tag{40}$$

while the second of (33) (using the second of (20) to get rid of terms containing $b_{[ij]}$) gives

$$(b_2 - b_4)Y_{23} + (b_2 - b_3)Y_{24} = 0, \qquad (b_3 - b_5)Y_{23} + (b_4 - b_5)Y_{24} = 0, \tag{41}$$

$$(b_2 - b_4)Y_{23} + (b_5 - b_4)Y_{24} = 0, \qquad (b_5 - b_3)Y_{23} + (b_2 - b_3)Y_{24} = 0, \tag{42}$$

$$(b_3 - b_2)Z_{23} = 0, \qquad (b_4 - b_2)Z_{24} = 0, \qquad (b_5 - b_2)Z_{25} = 0, \tag{43}$$

$$(b_4 - b_3)Z_{34} = 0, \qquad (b_5 - b_3)Z_{35} = 0, \qquad (b_5 - b_4)Z_{45} = 0. \tag{44}$$

It can be immediately seen from (37)–(44) that having (at least one) Z_{ij} non-vanishing requires that at least two pairs of b_i coincide, and the same conclusion holds also for W_{ij} and Y_{ij}. Thus, a non-zero Weyl tensor is compatible only with the two following multiplicities of the eigenvalues of $b_{(ij)}$: $\{a, a, a, a\}$ or $\{a, a, b, b\}$. These two cases give, respectively:

1. $b_{(ij)} = b_0\delta_{ij}$ (i.e., all the eigenvalues of $b_{(ij)}$ coincide), which is equivalent (when (10) holds) to the so called *optical constraint* [6, 17]. In this case, eqs. (37)–(44) are identically satisfied. One can shift the affine parameter $r \mapsto r + b_0$, so that $b_{(ij)} = 0$. Using r-independent spins, one can thus arrive at the canonical form (cf., e.g., [6, 11, 17, 22])

$$b = \text{diag}\left(\begin{bmatrix} 0 & b_{23} \\ -b_{23} & 0 \end{bmatrix}, \begin{bmatrix} 0 & b_{45} \\ -b_{45} & 0 \end{bmatrix}\right). \tag{45}$$

   For $b_{23} = 0 = b_{45}$, $\boldsymbol{\ell}$ is expanding, twistfree and shearfree, which corresponds to the Robinson-Trauman spacetimes with $\mu = 0$ [8].

2. $b_3 = b_2 \neq b_5 = b_4$, in which case the Weyl tensor must obey (thanks to (37)–(44))

$$W_{23} = Z_{24} = Z_{25} = Z_{34} = Z_{35} = Y_{23} = 0. \tag{46}$$

[3]This applies to any type D(abd) in six dimensions (i.e., even without assuming (10)), cf. the comments in section 2.3.

[4]Namely: $c_{2535} = -Z_{23}$, $c_{2545} = -Z_{24}$, $c_{2454} = -Z_{25}$, $c_{3545} = -Z_{34}$, $c_{3454} = -Z_{35}$, $c_{3435} = -Z_{45}$, $c_{3535} = -W_{23} - W_{25}$, $c_{3434} = W_{25}$, $c_{4545} = W_{23}$, $c_{2534} = -Y_{24} - Y_{23}$.

(Other possible cases trivially correspond to reordering the frame vectors – these need not be discussed separately.) Using (46), the second of (20) reads

$$-b_{24}W_{25} - b_{35}Y_{24} + b_{34}Z_{23} + b_{25}Z_{45} = 0, \qquad b_{25}W_{25} - b_{34}Y_{24} - b_{35}Z_{23} + b_{24}Z_{45} = 0, \tag{47}$$

$$b_{34}W_{25} - b_{25}Y_{24} + b_{24}Z_{23} - b_{35}Z_{45} = 0, \qquad b_{35}W_{25} + b_{24}Y_{24} + b_{25}Z_{23} + b_{34}Z_{45} = 0. \tag{48}$$

A non-zero Weyl tensor requires the determinant associated to the above system to vanish, i.e., $b_{35} = \mp b_{24}$ and $b_{34} = \pm b_{25}$. There are therefore two possible subcases.

(a) $b_{35} = b_{24} = b_{34} = b_{25} = 0$, in which case (47) and (48) are identically satisfied and

$$b = \mathrm{diag}\left(\begin{bmatrix} b_2 & b_{23} \\ -b_{23} & b_2 \end{bmatrix}, \begin{bmatrix} b_4 & b_{45} \\ -b_{45} & b_4 \end{bmatrix}\right). \tag{49}$$

The matrices $\boldsymbol{b}$ and $\boldsymbol{\rho}$ (recall (14)) are *normal*. Redefining r one can set $b_2 = 0$ (or $b_4 = 0$) or $b_2 + b_4 = 0$. In the case of zero twist ($b_{23} = 0 = b_{45}$), it has been shown that this case cannot occur [23].

(b) $b_{35} = \mp b_{24}$, $b_{34} = \pm b_{25}$, $b_{24}^2 + b_{25}^2 \neq 0$, in which case (47) and (48) further restrict the Weyl tensor, namely

$$Y_{24} = \pm W_{25}, \qquad Z_{45} = \mp Z_{23}. \tag{50}$$

In this case $\boldsymbol{b}$ and $\boldsymbol{\rho}$ are not normal matrices. It can be verified that the symmetric matrix $\boldsymbol{\rho} + \boldsymbol{\rho}^T$ admits two pairs of repeated eigenvalues (although it is not diagonal in the frame in use). By an r-independent spin in the plane (23) or (45), one can set $b_{24} = 0$ or $b_{25} = 0$, but not both (and, as above, redefining r one can set $b_2 = 0$ or $b_4 = 0$).

3.2 Comments for general type II

Proving an extension of the Goldbers-Sachs theorem in the case of general type II (i.e., without assuming (16)) would deserve a separate investigation. However, let us make here some preliminary comments, still assuming (10). The r-dependence of C_{ijkl} is not given anymore by (18), nevertheless for $r \to \infty$ one still has $C_{ijkl} = c_{ijkl}r^{-2} + \ldots$ with $c_{ikjk} = 0$, while $\Phi^S_{ij} = \tau_{ij}r^{-4} + \ldots$, $\Phi^A_{ij} = \pi_{ij}r^{-3} + \ldots$ (where $D\tau_{ij} = 0 = D\pi_{ij}$, $\tau_{[ij]} = 0 = \pi_{(ij)}$) [7, 24].[5] In the special case $c_{ijkl} = 0$, it has been already shown that the optical matrix ρ_{ij} must obey the optical constraint [7], i.e., (45) holds. In the generic case with both $c_{ijkl} \neq 0$ and $\Phi_{ij} \neq 0$ (so far unexplored), we can adapt some of the arguments of section 3.1 to obtain at least a weaker result which restricts the multiplicity of the eigenvalues of $b_{(ij)}$. We only sketch the line of the arguments, since the details would be similar to section 3.1.

In this case, the second relations of both (20) and (33) are replaced by $c_{ijkl}b_{[jl]} = (n-4)\pi_{ik}$ and $c_{im\{jk|}b_{m|l\}} = -2\pi_{\{jk}\delta_{l\}i}$, respectively (while the first of (20) and (33) are unchanged; cf. eqs. (21,22,27, [10]), (10, [7]) and [24]), nevertheless one can still arrive at (37)–(44) by simple algebraic manipulations. As argued above, it follows that either: (i) $b_{(ij)} = b_0\delta_{ij}$, i.e., the optical constraint holds or (ii) $b_3 = b_2 \neq b_5 = b_4$, in which case the Weyl tensor is constrained by $W_{23} = Z_{24} = Z_{25} = Z_{34} = Z_{35} = Y_{23} = 0$ (up to relabeling the frame vectors).

In both cases, $b_{(ij)}$ *possesses at least two pairs of repeated eigenvalues.*

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[5] In order to arrive at such results, one typically assumes that the various quantities have a power-like behaviour at the leading (and sometimes sub-leading) order, cf. [24] for more details.

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