GENERALIZED TRIGONOMETRIC FUNCTIONS IN COMPLEX DOMAIN

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(Received October 31, 2013)

Cordially dedicated to Professor Pavel Drábek on the occasion of his sixtieth birthday

Abstract. We study extension of p-trigonometric functions $\sin_p$ and $\cos_p$ to complex domain. For $p = 4, 6, 8, \ldots$, the function $\sin_p$ satisfies the initial value problem which is equivalent to

$$(*) \qquad -(u')^{p-2}u'' - u^{p-1} = 0, \quad u(0) = 0, \quad u'(0) = 1$$

in $\mathbb{R}$. In our recent paper, Girg, Kotrla (2014), we showed that $\sin_p(x)$ is a real analytic function for $p = 4, 6, 8, \ldots$ on $(-\pi_p/2, \pi_p/2)$, where $\pi_p/2 = \int_0^1 (1 - s^p)^{-1/p}$. This allows us to extend $\sin_p$ to complex domain by its Maclaurin series convergent on the disc $\{z \in \mathbb{C}\colon |z| < \pi_p/2\}$. The question is whether this extensions $\sin_p(z)$ satisfies $(*)$ in the sense of differential equations in complex domain. This interesting question was posed by Došlý and we show that the answer is affirmative. We also discuss the difficulties concerning the extension of $\sin_p$ to complex domain for $p = 3, 5, 7, \ldots$ Moreover, we show that the structure of the complex valued initial value problem $(*)$ does not allow entire solutions for any $p \in \mathbb{N}$, $p > 2$. Finally, we provide some graphs of real and imaginary parts of $\sin_p(z)$ and suggest some new conjectures.

Keywords: p-Laplacian; differential equations in complex domain; extension of $\sin_p$

MSC 2010: 33E30, 34B15, 34M05, 34M99

The research of the first author has been supported by the Czech Science Foundation, project No. 13-00863S. The research of the second author was partially supported by the Czech Foundation, projects No. P202/12/G061 and No. 13-00863S.

1. Introduction

The initial value problem

$$-(|u'|^{p-2}u')' - (p-1)|u|^{p-2}u = 0, \quad u(0) = 0, \quad u'(0) = 1 \tag{1.1}$$

arises in connection with nonlinear boundary value problems for $p > 1$ (see e.g. [2], [3], [7], [9]). The solution of (1.1) is known as $\sin_p$, see e.g. [2], and $\cos_p \stackrel{\text{def}}{=} \sin_p'$. Since the functions $\sin_p$ and $\cos_p$ satisfy the well-known p-trigonometric identity, see e.g. [3],

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1, \tag{1.2}$$

they are also known as the p-trigonometric and/or generalized trigonometric functions. Note that (1.2) is in fact the so-called first integral of (1.1) (see e.g. [3]). It follows from this identity (see e.g. [3]) that

$$\int_0^{\sin_p(x)} (1-s^p)^{-1/p}\,\mathrm{d}s = x$$

for $0 \leqslant x \leqslant \pi_p/2$, where $\sin_p(x) \geqslant 0$ and $\cos_p(x) \geqslant 0$ and where

$$\pi_p \stackrel{\text{def}}{=} 2\int_0^1 (1-s^p)^{-1/p}\,\mathrm{d}s.$$

Thus it is natural to define

$$\operatorname{arcsin}_p(x) \stackrel{\text{def}}{=} \int_0^x (1-s^p)^{-1/p}\,\mathrm{d}s \quad \text{for } 0 \leqslant x \leqslant 1, \tag{1.3}$$

and extend it to $[-1,1]$ as an odd function. The function $\sin_p$ is the inverse function to $\operatorname{arcsin}_p(x)$ on $[-\pi_p/2, \pi_p/2]$. Moreover, $\sin_p(x) = \sin_p(\pi_p - x)$ for $x \in (\pi_p/2, \pi_p]$ and $\sin_p(x) = -\sin_p(-x)$ for $x \in [-\pi_p, 0]$. Finally, $\sin_p(x) = \sin_p(x + 2\pi_p)$ for all $x \in \mathbb{R}$ (see [3] for details).

Smoothness of $\sin_p$ on $(-\pi_p/2, \pi_p/2)$ for $p > 1$ was studied in [4]. The most surprising result of [4] is that $\sin_p$ is a real analytic function on $(-\pi_p/2, \pi_p/2)$ for $p = 4, 6, 8, \ldots$, i.e., $\sin_p(x)$ equals its Maclaurin on $(-\pi_p/2, \pi_p/2)$ for $p = 4, 6, 8, \ldots$ This approach naturally allows to extend $\sin_p$ for $p = 4, 6, 8, \ldots$ to an open disk

$$\{z \in \mathbb{C} \colon |z| < \pi_p/2\}$$

in the complex domain using power series (cf. [7], where the convergence of the series is conjectured without proof). When our recent result was presented at the conference “Nonlinear Analysis Plzeň 2013”, O. Došlý posed an interesting question whether this extension satisfies (1.1) in the sense of differential equations in complex domain. This paper addresses his question. For $p = 4, 6, 8, \ldots$, the initial value problem (1.1) in $\mathbb{R}$ is equivalent to

$$-(u')^{p-2}u'' - u^{p-1} = 0, \quad u(0) = 0, \quad u'(0) = 1. \tag{1.4}$$

Note that for $p > 1$ real not being an even positive integer, we cannot get rid of the absolute values in (1.1). Thus the equation (1.1) does not make sense for general $p > 1$ in the complex domain. In this paper we consider the problem (1.4) in complex domain for integer $p > 2$. The complex valued ordinary differential equations are studied by means of power series (mostly Maclaurin series). Note that, by [4], Theorem 3.2 on page 5, $\sin_p^{(n)}(0)$ exists for $1 < n \leqslant p$, but $\sin_p^{(n)}(0)$ *does not exist* when $p \geqslant 3$ is an odd integer and $n > p$. Thus, by the formal Maclaurin series of $\sin_p(x)$, we mean a series calculated from the limits of the derivatives $\lim\limits_{x\to 0+} \sin_p^{(n)}(x)$, which were shown to exist in [4] for any $n \in \mathbb{N}$ and $p \geqslant 3$ an odd integer.

In Section 2, we prove that, for $p = 4, 6, 8, \ldots$, the function $\sin_p$ can be extended by its Maclaurin series to the disc $\{z \in \mathbb{C} : |z| < \pi_p/2\}$ and that this series solves the ordinary differential equation (1.4) in the sense of differential equations in the complex domain. On the other hand, in Section 3 we show that the complex valued formal Maclaurin series $M_{\sin_p}(z)$ of the real function $\sin_p(x)$ does not satisfy (1.4) in the sense of differential equations in the complex domain for odd powers $p = 3, 5, 7, \ldots$ In Section 4 we explain the relations between the real and imaginary components of the complex valued function $\sin_p(z)$ for $p = 2, 6, 10, \ldots$ and $p = 4, 8, 12, \ldots$, and also the complex valued formal Maclaurin series $M_{\sin_p}(z)$ of the real function $\sin_p(x)$ for $p = 3, 5, 7, \ldots$ In Section 5 we show that the fact that the function $\sin_p(z)$ cannot be extended as an entire function follows from an interesting connection between the p-trigonometric identity and some classical results from complex analysis. Finally, in Section 6 we visualize some of our result.

In the whole paper, the independent variable z stands for a complex number and the independent variable x stands for a real number. In the same spirit, $\sin_p(z)$ stands for a complex valued function and $\sin_p(x)$ stands for a function of one real variable.

2. Extension of $\sin_p$ for $p = 4, 6, 8, \ldots$ to complex domain

We assume that $p = 4, 6, 8, \ldots$ throughout this section unless specified differently. In [4], Theorem 3.3, we proved the following result.

Proposition 2.1 ([4], Theorem 3.3, page 6). *Let $p = 4, 6, 8, \ldots$ Then the Maclaurin series of* $\sin_p(x)$ *converges on* $(-\pi_p/2, \pi_p/2)$.

Let $M_{\sin_p}(x)$ denote the formal Maclaurin series of $\sin_p(x)$, $p = 3, 4, 5, 6, \ldots$ (any integer greater than 2). We also proved in the paper [4] that this Maclaurin series has the particular structure

$$M_{\sin_p}(x) = \sum_{k=0}^{\infty} \alpha_k x^{kp+1}, \tag{2.1}$$

where $\alpha_0 > 0$ and $\alpha_k \leqslant 0$.

The following result answers the question by O. Došlý in the affirmative way.

Theorem 2.1. *Let $p = 4, 6, 8, \ldots$, then the unique solution of the initial value problem* (1.4) *on $|z| < \pi_p/2$ is the Maclaurin series* (2.1).

In order to prove this result, we need to state several auxilliary results. First of all, let us note that the equation (1.4) is of second order. In order to apply the known theory, we rewrite (1.4) as an equivalent system. Using the substitution $u' = v$, we get the first order system

$$u' = v, \quad v' = -u^{p-1}/v^{p-2}, \quad u(0) = 0, \quad v(0) = 1. \tag{2.2}$$

To study systems of equations like (2.2) in complex domain, we need to use complex functions of several variables. We will often make use of the following result.

Proposition 2.2 ([6], Theorem 16, page 33). *Let f and g be holomorphic functions in an open set $M \subset \mathbb{C}^r$, $r \in \mathbb{N}$. Then the functions $f + g$, $f - g$ and fg are holomorphic in M. Moreover, if $g(z) \neq 0$ for all $z \in M$, then f/g is holomorphic on M.*

Let us consider the first order ODE system

$$\mathbf{y}' = \mathbf{f}(z, \mathbf{y}), \quad \mathbf{y}(z_0) = \mathbf{y}_0, \tag{2.3}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in \mathbb{C}^n$ and $\mathbf{f} = (f_1(z, \mathbf{y}), f_2(z, \mathbf{y}), \ldots, f_n(z, \mathbf{y}))^{\mathrm{T}} \in \mathbb{C}^n$ and the function $\mathbf{f}\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ is an analytic complex function of $n+1$ complex variables. The folowing result concerning existence and uniqueness of the initial values problem in the complex domain is crucial in our proofs.

Proposition 2.3 ([5], Theorem 9.1, page 76). *Let a function* $\mathbf{f}\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ *be analytic and bounded in the region*

$$R\colon\ |z - z_0| < \alpha, \quad \|\mathbf{w} - \mathbf{w_0}\| < \beta,$$

where $\alpha > 0$, $\beta > 0$, *and let*

$$\mu \overset{\text{def}}{=} \sup_{(z,\mathbf{w})\in R} \|\mathbf{f}(z, \mathbf{w})\|, \quad \gamma \overset{\text{def}}{=} \min\Big(\alpha, \frac{\beta}{\mu}\Big).$$

Then there exists in the disk D_0, $|z - z_0| < \gamma$ *a unique analytic function* $\mathbf{w}\colon \mathbb{C} \to \mathbb{C}^n$ *which is the solution of* (2.3).

Lemma 2.1. *There is* $\delta > 0$ *such that in* $U_0 \overset{\text{def}}{=} \{z \in \mathbb{C}\colon\ |z| < \delta\}$ *the initial value problem* (1.4) *has the unique solution* $u(z)$ *which is an analytic function in* U_0.

P r o o f. Consider (2.2) in complex domain. Let us denote

$$f_1(z, \xi, \eta) \overset{\text{def}}{=} \eta$$

and (recall $p = 4, 6, 8, \ldots$ by assumption of this section)

$$f_2(z, \xi, \eta) \overset{\text{def}}{=} -\frac{\xi^{2m+1}}{\eta^{2m}}, \quad \text{where } z, \xi, \eta \in \mathbb{C} \text{ and } m \in \mathbb{N}.$$

Naturally, the functions $f = \xi$ and $g = \eta$ are holomorphic in the entire complex plane. Thus by Proposition 2.2, functions $f_1(z, \xi, \eta)$ and $f_2(z, \xi, \eta)$ are holomorphic on some neighborhood of $[0, 0, 1]$. Let R denote the maximal closed subset of this neigborhood. Then the functions f_1 and f_2 are holomorphic on the closed domain R and so they are continuous on R. Hence they are bounded on R (see [6], page 37). Therefore, the system (2.2) has a unique solution by Proposition 2.3. □

The previous lemma yields a local solution $u(z)$ of (1.4) in a small neighborhood U_0 of 0 in $\mathbb{C}$. Since $u(z)$ is analytic in U_0, it can be written as a power series $u(z) = \sum_{k=0}^{\infty} a_k z^k$, where this power series converges towards $u(z)$ for all $z \in U_0$. Our next aim is to show that the series corresponding to $u(z)$ has the same coefficients as the series corresponding to $\sin_p(x)$, which is the unique solution to the real-valued initial value problem (1.1). For this purpose, we will use the following result concerning the sums of two powers series.

Proposition 2.4 ([8], Theorem 16.6, page 352). *If the sums of two power series in the variable $z - z_0$ coincide on a set of points E for which z_0 is a limit point and $z_0 \notin E$, then identical powers of $z - z_0$ have identical coefficients, i.e., there is a unique power series in the variable $z - z_0$ with the given sum on the set E.*

Now we are ready to prove the main result of this section.

Proof of Theorem 2.1. By Lemma 2.1, $u(z) = \sum_{k=0}^{\infty} a_k z^k$ is the unique solution of (1.4) at any point $z \in U_0$. Observe that the solution $u(z) = \sum_{k=0}^{\infty} a_k z^k$ solves also the real-valued Cauchy problem (1.4) in the sense of real analysis. On the other hand, $\sin_p$ is the unique solution of the real-valued Cauchy problem (1.4). Since the Maclaurin series (2.1) of $\sin_p$ converges towards $\sin_p$ in $(-\pi_p/2, \pi_p/2)$ under the assumption of this section, we find that (2.1) satisfies (1.4) in $(-\pi_p/2, \pi_p/2)$. Moreover, convergence of (2.1) on $(-\pi_p/2, \pi_p/2)$ implies convergence of $\sum_{k=0}^{\infty} \alpha_k z^{kp+1}$ for all $z \in \mathbb{C}$, $|z| < \pi_p/2$. Therefore,

$$\sum_{j=0}^{\infty} a_j z^j = \sum_{k=0}^{\infty} \alpha_k z^{kp+1} \quad \text{for all } z \in U_0 \cap (-\pi_p/2, \pi_p/2).$$

Now we consider the set of points $z_n = \delta/(n+1)$, $n \in \mathbb{N}$. From the previous equation, we have

$$\sum_{j=0}^{\infty} a_j z_n^j - \sum_{k=0}^{\infty} \alpha_k z_n^{kp+1} = 0 = \sum_{j=0}^{\infty} 0 \cdot z_n^j.$$

By Proposition 2.4, we find that these two series must coincide on U_0. Hence the Maclaurin series (2.1) satisfies (1.4) on U_0. Let u be given by the series (2.1). Then u'', $(u')^{p-2}$, u^{p-1} have the radius of convergence $\pi_p/2$ for $p > 2$, $p \in \mathbb{N}$. Since any power series converges absolutely within the radius of its convergence, we see from (1.4) that

$$-\left[\left(\sum_{k=0}^{\infty} \alpha_k z_n^{kp+1}\right)'\right]^{p-2} \left(\sum_{k=0}^{\infty} \alpha_k z_n^{kp+1}\right)'' - \left(\sum_{k=0}^{\infty} \alpha_k z_n^{kp+1}\right)^{p-1} = 0 = \sum_{j=0}^{\infty} 0 \cdot z_n^j$$

for all $z_n = \delta/(n+1)$, $n \in \mathbb{N}$. Thus, by Proposition 2.4, u given by the series (2.1) is the solution of (1.4) on the disc $D = \{z \in \mathbb{C} \colon |z| < \pi_p/2\}$. $\square$

3. Obstacles for extension of $\sin_p$ for $p = 3, 5, 7, \dots$ to complex domain

Lindqvist [7] proposed an alternative definition of $\sin_p$ as the solution of

$$\frac{\mathrm{d}}{\mathrm{d}z}(w')^{p-1} + w^{p-1} = 0, \quad w(0) = 0, \quad w'(0) = 1 \tag{3.1}$$

in complex domain for $p > 1$ (considered only formally). In [7], Section 7, he conjectures the possibility that solutions to (3.1) and the real Cauchy problem

$$(|u'|^{p-2}u')' + |u|^{p-2}u, \quad u(0) = 0, \quad u'(0) = 1 \tag{3.2}$$

could produce different solutions on $\mathbb{R}$. We address this question in this section. However, we have definitions of π_p and $\sin_p$ in this paper different from those in [7]. Turning to our definitions of π_p and $\sin_p$, we get an equation corresponding to (3.1):

$$\frac{\mathrm{d}}{\mathrm{d}z}(w')^{p-1} + (p-1)w^{p-1} = 0, \quad w(0) = 0, \quad w'(0) = 1 \tag{3.3}$$

which is equivalent to (1.4), which is equivalent to (2.2). Since the $(p-1)$-st power is a multivalued complex function, we will limit ourselves to $p \in \mathbb{N}$, $p > 1$, in order to be able to perform rigorous analysis. The question is whether (3.3) produces a solution which is different from the solution (1.1) on $\mathbb{R}$. In the previous section we proved that for $p = 4, 6, 8, \dots$ (and of course for $p = 2$) the solutions of (3.3) and (1.1) are identical. Now we show that for $p = 3, 5, 7, \dots$ the solutions are different for negative arguments.

This proposition is crucial for the proof of the main result of this section.

Proposition 3.1 ([4], Theorem 3.4, page 6)**.** *Let $p = 3, 5, 7, \dots$ Then the formal Maclaurin series of $\sin_p(x)$—the solution of the Cauchy problem (1.1)—converges on $(-\pi_p/2, \pi_p/2)$. Moreover, the formal Maclaurin series of $\sin_p(x)$ converges towards $\sin_p(x)$ on $[0, \pi_p/2)$, but does not converge towards $\sin_p(x)$ on $(-\pi_p/2, 0)$.*

Now we are ready to formulate the main result of this section.

Theorem 3.1. *Let $p = 3, 5, 7, \dots$ Then the unique solution $u(z)$ of the complex initial value problem (1.4) differs from the solution $\sin_p(x)$ of the Cauchy problem (1.1) for $z = x \in (-\pi_p/2, 0)$.*

P r o o f. Let us recall that (3.3) is equivalent to (2.2). There exists a unique solution of (2.2) on some nonempty open disc in $\mathbb{C}$ containing 0 by Proposition 2.3.

In the same way as in the proof of Theorem 2.1 (with obvious modifications), it follows that $M_{\sin_p}(z)$ solves (3.3) on the open disc $|z| < \pi_p/2$ and it is the unique solution on this disc. Since $\sin_p(x)$ is the unique solution of (1.1), $\sin_p(x) \neq M_{\sin_p}(x)$ for $x \in (-\pi_p/2, 0)$ by Proposition 3.1, we see that (1.1) and (3.3) produce different solutions on $\mathbb{R}$. □

4. Relations between real and imaginary parts

Let us mention an interesting relationship between real and imaginary parts of $\sin_p(z)$ for $p = 4, 8, 12, \ldots$ One can see in Figure 1 that the graph of the imaginary part of $\sin_4(z)$ is the graph of the real part, rotated by $-\pi/2$.

Theorem 4.1. *Let $p = 4, 8, 12, \ldots$ Then*

$$\Re[\sin_p(z)] = \Im[\sin_p(\mathrm{i}z)]$$

for all $z \in \mathbb{C}$, $|z| < \pi_p/2$.

Proof. Note that by (2.1)

$$\sin_p(z) = \sum_{k=0}^{\infty} \alpha_k z^{kp+1} = z \sum_{k=0}^{\infty} \alpha_k z^{kp}$$

for $z \in \mathbb{C}$, $|z| < \pi_p/2$. We assume $p = 4l$ where $l = 1, 2, 3, \ldots$ and thus

$$\sin_p(z) = z \sum_{k=0}^{\infty} \alpha_k z^{4kl}.$$

Substituting $\mathrm{i}z$ into this formula we find

$$\sin_p(\mathrm{i}z) = \mathrm{i}z \sum_{k=0}^{\infty} \alpha_k (\mathrm{i}z)^{4kl} = \mathrm{i} \sum_{k=0}^{\infty} \alpha_k z^{4kl+1} = \mathrm{i}\sin_p(z).$$

Now the result easily follows from comparison of the real and imaginary parts of $\sin_p(z)$ and $\mathrm{i}\sin_p(\dot{z})$. This completes the proof. □

Theorem 4.2. *Let $p = 2, 6, 10, 14, \ldots$ Then for all $\varphi \in [0, 2\pi)$ there exists $z \in \mathbb{C}$, $|z| < \pi_p/2$ such that*

$$\Re[\sin_p(z)] \neq \Im[\sin_p(\mathrm{e}^{\mathrm{i}\varphi} z)].$$

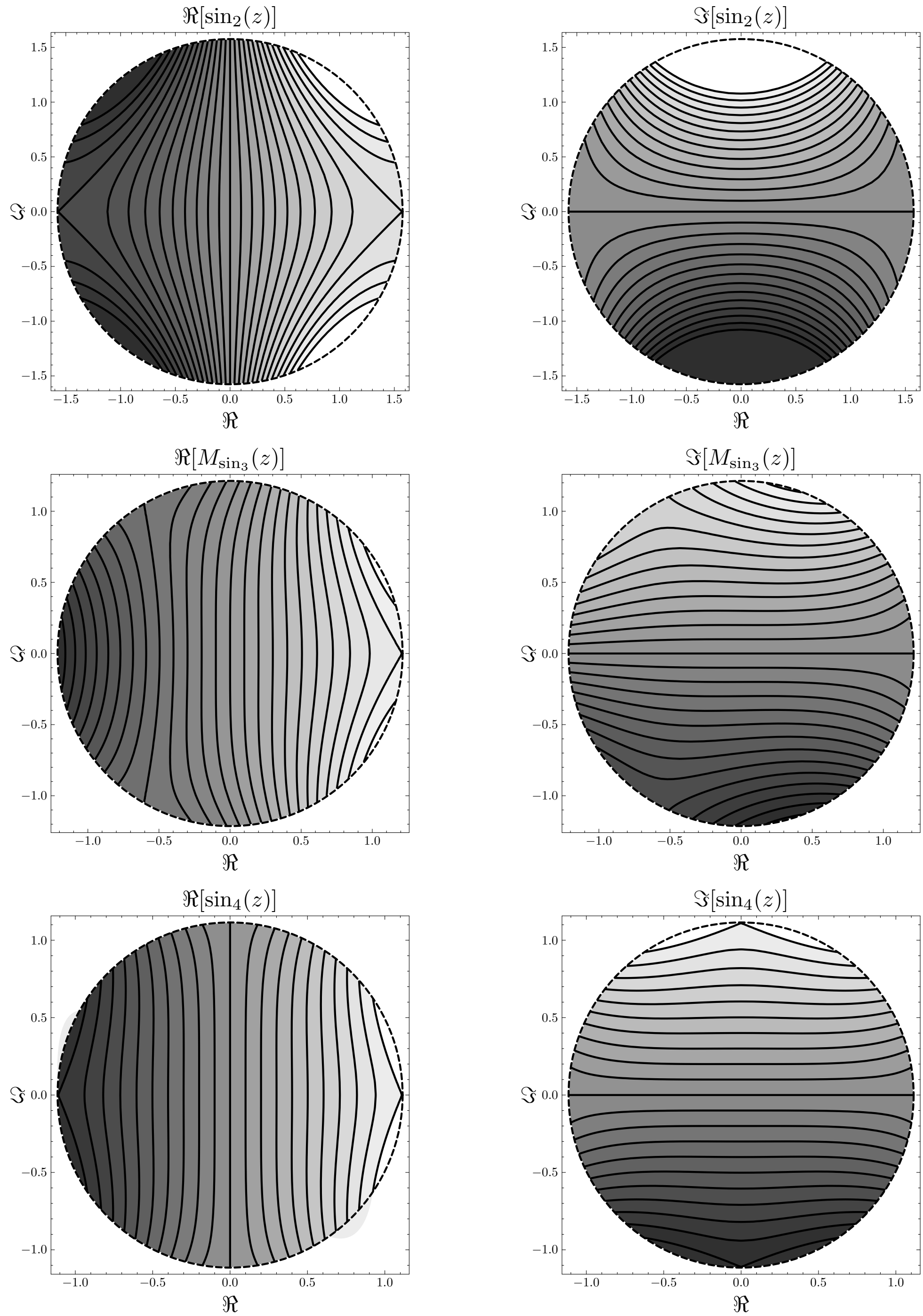

Figure 1. Continued on the next page.

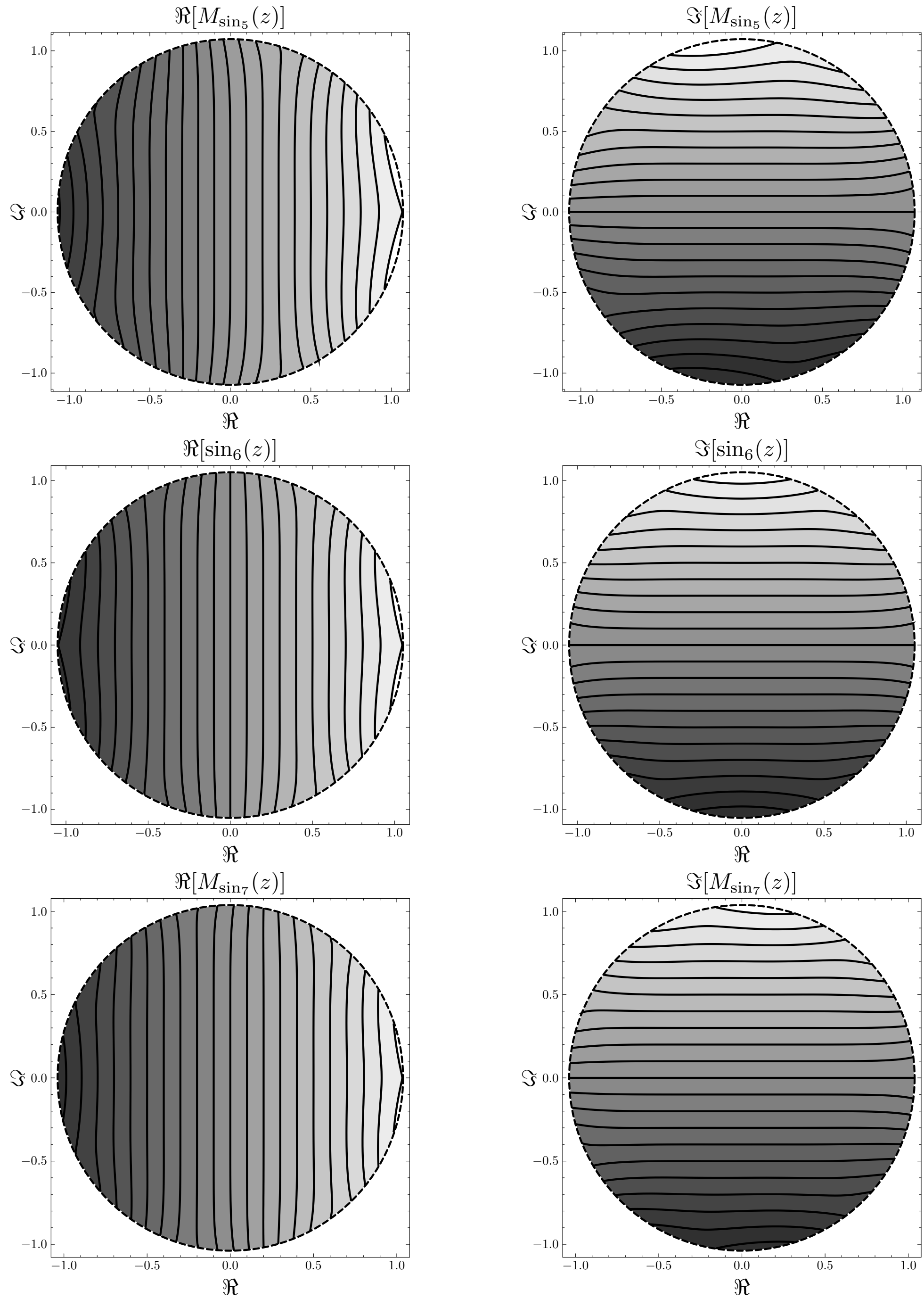

Figure 1. Contourlines of the real and imaginary parts of $\sin_p(z)$ for $p = 2, 4, 6$ and $M_{\sin_p}(z)$ for $p = 3, 5, 6$. Note that the imaginary part of $\sin_4(z)$ is its real part rotated by $-\pi/2$.

P r o o f. It is known from [4] that the series $M_{\sin_p}(z)$ has the form

$$M_{\sin_p}(z) = \sum_{k=0}^{\infty} \alpha_k z^{kp+1}.$$

First we show that $\alpha_0 = 1$ and $\alpha_1 = -1/(p(p+1)) < 0$ (cf e.g. [7]). In fact, evaluating the integral in (1.3) we see that

$$\operatorname{arcsin}_p(x) = \int_0^x (1-s^p)^{-1/p}\,\mathrm{d}s = {}_2\mathrm{F}_1\Big(\frac{1}{p}, \frac{1}{p}, 1+\frac{1}{p}, x^p\Big)x \quad \text{for } 0 \leqslant x \leqslant 1,$$

where ${}_2\mathrm{F}_1$ is the Gauss hypergeometric function. Using the known series

$${}_2\mathrm{F}_1(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k z^k}{(c)_k k!} \quad \text{for } |z| < 1,$$

where $(a)_k = \prod_{j=0}^{k} (a+k-1)$ for any $a \in \mathbb{R}$ stands for the rising factorial, we find

$$\operatorname{arcsin}_p(w) = w \sum_{k=0}^{\infty} \frac{(1/p)_k^2 w^{kp}}{(1+1/p)_k k!} \quad \text{for } 0 < w < 1.$$

Hence

$$\operatorname{arcsin}_p(w) = w + \frac{1}{p(p+1)} w^{p+1} + O(w^{2p+1}) \quad \text{for } 0 < w < 1.$$

Denoting $w = \sin_p(x)$, we find

$$x = w + \frac{1}{p(p+1)} w^{p+1} + O(w^{2p+1}),$$

which yields

$$w = x - \frac{1}{p(p+1)} w^{p+1} + O(w^{2p+1}). \tag{4.1}$$

Substituting (4.1) into itself we obtain

$$w = x - \frac{1}{p(p+1)}\Big(x - \frac{1}{p(p+1)} + O(w^{2p+1})\Big)^{p+1} + O(w^{2p+1}).$$

Hence

$$\sin_p(x) = x - \frac{1}{p(p+1)} x^{p+1} + O(w^{2p+1}), \tag{4.2}$$

which gives the desired formulas for $\alpha_1 = 1$ and $\alpha_2 = -1/p(p+1)$. With this at hand, we can write

$$M_{\sin_p}(z) = z - \frac{1}{p(p+1)} z^{p+1} + \sum_{m=2}^{\infty} \alpha_m z^{mp+1} \tag{4.3}$$
$$= z - \frac{z^{p+1}}{p(p+1)} - z^{2p+1} \sum_{m=0}^{\infty} \alpha_{m+2} z^{mp}.$$

Let $z = a$, $a \in \mathbb{R}$, $0 < a < \pi_p/2$ for simplicity. Then $\varphi_0 = \pi/2$ is the unique angle in $[0, 2\pi)$ such that $\Re[z] = \Im[e^{i\varphi_0} z]$. The assumption on p of this theorem is that there exists $l \in \mathbb{N} \cup \{0\}$ such that $p = 4l + 2$. Thus $\Re[z^{p+1}] = \Re[z^{4l+3}] = \Re[a^{4l+3}]$. On the other hand, $\Im[(e^{i\varphi_0} z)^{p+1}] = \Im[(ia)^{4l+3}] = -a^{4l+3}$ for $\varphi_0 = \pi/2$. Inserting $z = a$ and $z = ia$ into (4.3), taking the real and imaginary part, respectively, and subtracting, we get

$$\Re[M_{\sin_p}(a)] - \Im[M_{\sin_p}(ia)] = \tag{4.4}$$
$$= -\frac{2a^{p+1}}{p(p+1)} + a^{2p+1} \left(\Re\left[\sum_{m=0}^{\infty} \alpha_{m+2} a^{mp} \right] - \Im\left[i^{2p+1} \sum_{m=0}^{\infty} \alpha_{m+2} (ia)^{mp} \right] \right).$$

Since the series on the right-hand side are convergent on the disc $\{z \in \mathbb{C} \colon |z| < \pi_p/2\}$,

$$A \overset{\text{def}}{=} \max_{\{z \in \mathbb{C} \colon |z| \leqslant \pi_p/4\}} \left| \left(\Re\left[\sum_{m=0}^{\infty} \alpha_{m+2} z^{mp} \right] - \Im\left[i^{2p+1} \sum_{m=0}^{\infty} \alpha_{m+2} (iz)^{mp} \right] \right) \right| < \infty$$

exists and from (4.4) we find

$$\left| \frac{\Re[M_{\sin_p}(a)] - \Im[M_{\sin_p}(ia)]}{a^{p+1}} - \frac{2}{p(p+1)} \right| \leqslant A a^p.$$

Taking $0 < a < \min\{\pi_p/4, (Ap(p+1))^{-1/p}\}$, we see that $\Re[M_{\sin_p}(a)] - \Im[M_{\sin_p}(ia)] \neq 0$. This concludes the proof. $\square$

5. Consequence of complex p-trigonometric identity

As was mentioned earlier, the maximal possible radius of convergence for the (formal) Maclaurin series for functions $\sin_p$ and $\cos_p$ is $\pi_p/2$. This fact was anticipated in [7] and studied in detail in [4]. In this section we explain that there was no hope for these series to have their radius of convergence infinite for $p = 3, 4, 5, 6, \ldots$ Contrary to what one would think, we will show that it is not the absolute value in (1.1) that produces the main difficulty. It is a complex analogy of the p-trigonometric identity that produces the impossibility of $\sin_p$ to be an entire complex functions for $p = 3, 4, 5, 6, \ldots$

Let us reconsider (1.4), i.e.,

$$-(u')^{p-2}u'' - u^{p-1} = 0, \quad u(0) = 0, \quad u'(0) = 1,$$

now for any $p = 3, 4, 5, 6, \ldots$ in the complex domain. Let us assume that u is a solution which is a holomorphic function on some neighborhood U_0 of 0. Multiplying the equation of (1.4) by u' and integrating from 0 to $z \in U_0$, we obtain

$$(u'(z))^p - (u'(0))^p + (u(z))^p - (u(0))^p = 0.$$

Now using the initial conditions of (1.4) we get

$$(u'(z))^p + (u(z))^p = 1, \tag{5.1}$$

which is the first integral of (1.4), and we can think of it as a complex p-trigonometric identity for holomorphic solutions of (1.4) for $p = 3, 4, 5, 6, \ldots$

Now we state the very classical result from complex analysis.

Proposition 5.1 ([1], Theorem 12.20, page 433). *Let f and g be entire functions satisfying for some positive integer n the identity*

$$f^n + g^n = 1.$$

(i) *If $n = 2$, then there is an entire function h such that $f = \cos \circ h$, $g = \sin \circ h$.*
(ii) *If $n > 2$, then both f and g are constants.*

It follows from this result that a holomorphic solution u of (1.4) cannot be an entire function for any $p = 3, 4, 5, 6, \ldots$, since the derivative of an entire function is an entire function as well and u and u' must satisfy (5.1). Thus by Proposition 5.1 u and u' are constant, which contradicts $u'(0) = 1$.

In particular, for $p = 4, 6, 8, \ldots$, with $u(z) = \sin_p(z)$ and $u'(z) = \cos_p(z)$ this becomes

$$\cos_p^p(z) + \sin_p^p(z) = 1$$

and we see that $\sin_p$ and $\cos_p$ cannot be entire functions.

Note that there was an interesting internet discussion [10] that called our attention towards this connection between complex analysis (including the classical reference [1], Theorem 12.20) and p-trigonometric functions. It seems to us that this connection has been overlooked by the "p-trigonometric community".

6. Visualization of $\sin_p(z)$ and their Maclaurin series

In this section we visualize graphs of the extensions of $\sin_p(z)$ by its Maclaurin series for $p = 4, 6$ and the formal Maclaurin series for $p = 3, 5, 7$ and compare them with the classical result $\sin_p(z) = \sin_2(z)$, see Figure 2. To the best of our knowledge, these figures in complex domain are new and we believe that they will help to stimulate discussion on this topic. We also formulate some conjectures in the caption of Figure 3.

A c k n o w l e d g e m e n t . The authors would like to thank O. Marichev from Wolfram Research, Inc. for his valuable advice concerning series representation of functions and their inverses in the software package Mathematica during the Wolfram Technology Conference (2011), and for sending us some materials concerning computation of p, q-trigonometric functions after the Wolfram Technology Conference (2012).

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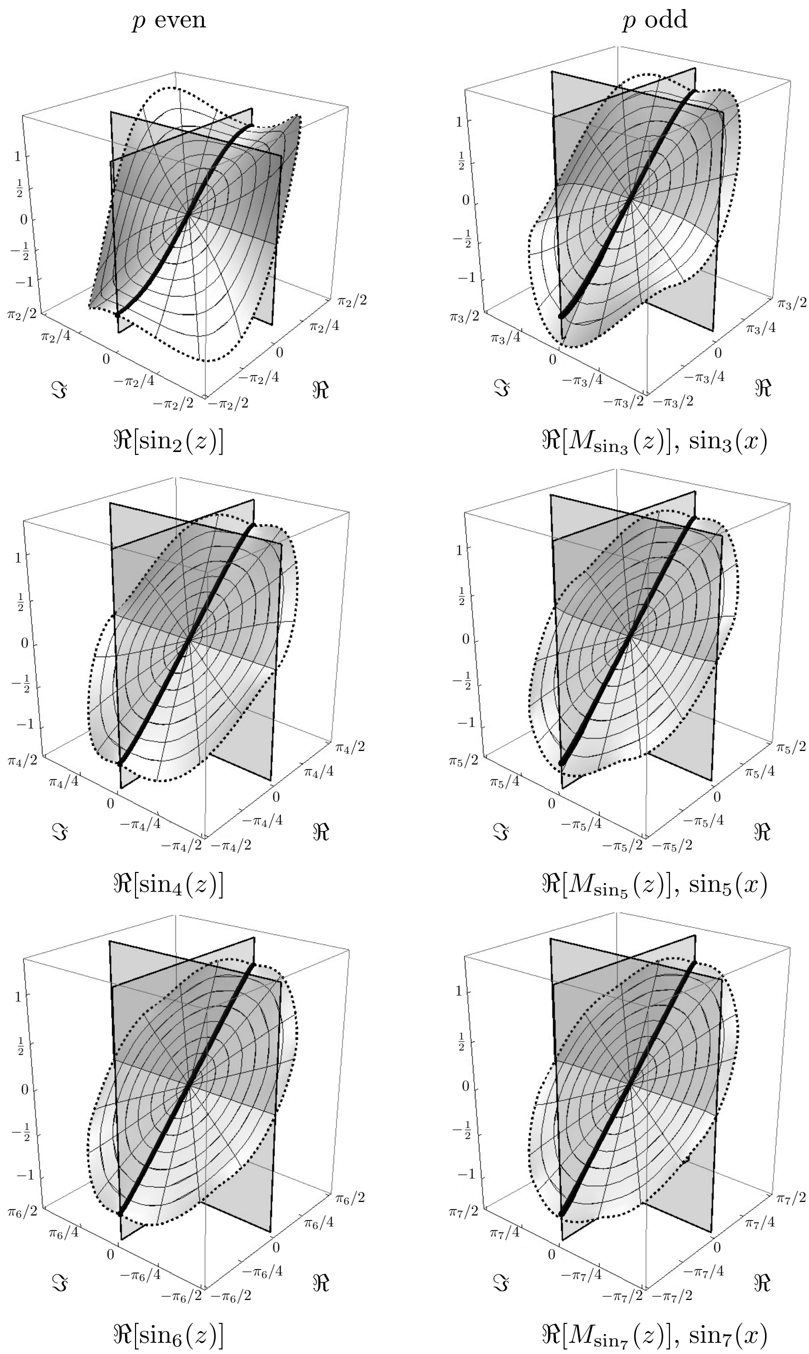

Figure 2. Comparison of real parts of $\sin_p(z)$ for p even (extended by the Maclaurin series) and the real parts of the formal Maclaurin series $M_{\sin_p}(z)$ and the real function $\sin_p(x)$ for p odd.

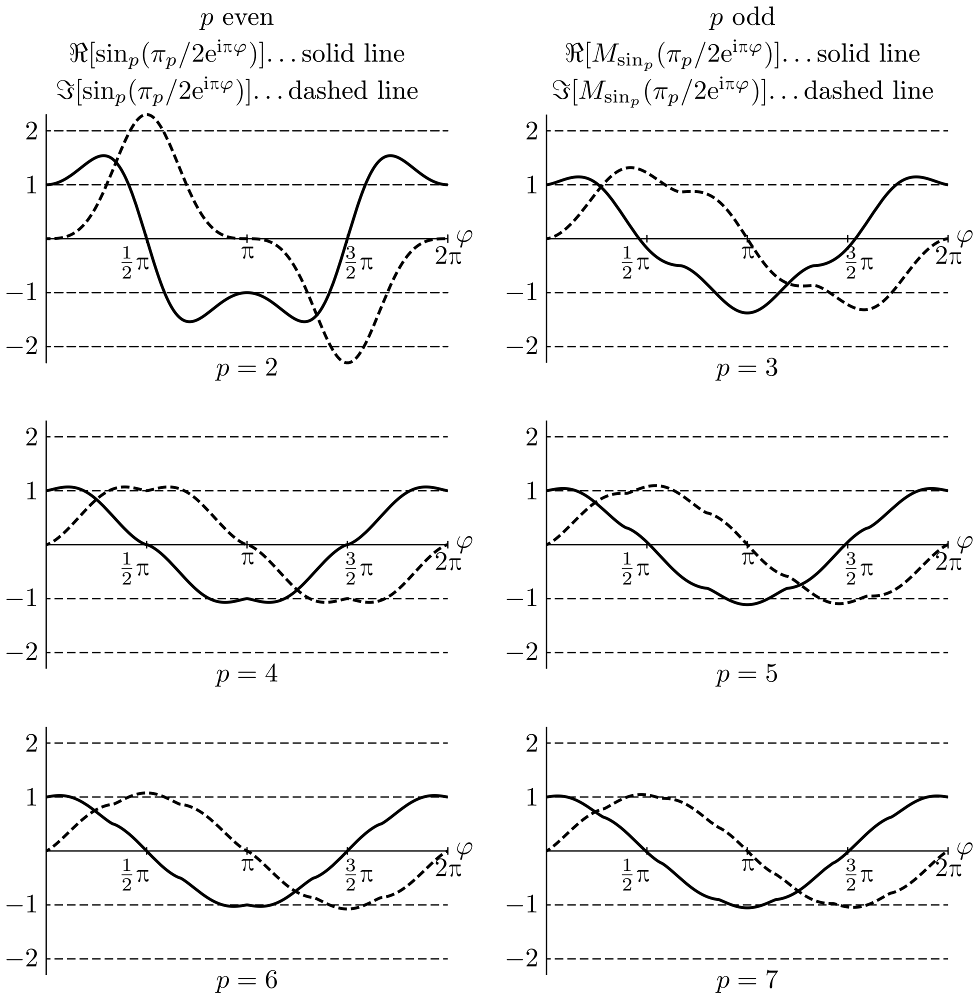

Figure 3. Numerical comparison of the real and imaginary parts of $\sin_p(\pi_p/2\mathrm{e}^{\mathrm{i}\pi\varphi})$ for $p = 2, 4, 6$ (extended by Maclaurin series) and the real and imaginary parts of $M_{\sin_p}(\pi_p/2\mathrm{e}^{\mathrm{i}\pi\varphi})$ for $p = 3, 5, 7$. Note that these graphs are only an illustration, because we know nothing about the convergence of the series for $z \in \mathbb{C}$, $|z| = \pi_p/2$. From these pictures we conjecture this convergence. It is interesting to note in these pictures that for larger p, the graph of the real part is a small perturbation of $\pi_p/2 \cos\varphi$ and the graph of the imaginary part is a small perturbation of $\pi_p/2 \sin\varphi$. We conjecture that this phenomena occur due to the fact that the Maclaurin series is $M_{\sin_p}(z) = z - z^{p+1}/(p(p+1)) + O(z^{2p+1})$ and for large p the higher order terms are negligible. Moreover, $\lim_{p\to\infty} \pi_p/2 = 1$. Thus we conjecture that these graphs tend to graphs of $\sin\varphi$ and $\cos\varphi$ for $p \to \infty$, respectively.

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