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Circuits and Matchings in Graphs

Komise pro obhajoby doktorských disertací
v oboru Matematické struktury

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Contents

List of articles included in this work

The articles in the list below are included in the thesis. They are listed in the order of appearance of the corresponding sections.

[**I**] H. Broersma, G. Fijavž, T. Kaiser, R. Kužel, Z. Ryjáček and P. Vrána. Contractible subgraphs, Thomassen's conjecture and the dominating cycle conjecture for snarks. *Discrete Math.* **308** (24) (2008), 6064–6077.

[**II**] T. Kaiser and P. Vrána. Hamilton cycles in 5-connected line graphs. *European J. Combin.* **33** (5) (2012), 924–947.

[**III**] T. Kaiser. A short proof of the tree-packing theorem. *Discrete Math.* **312** (10) (2012), 1689–1691.

[**IV**] T. Kaiser, Z. Ryjáček and P. Vrána. On 1-Hamilton-connected claw-free graphs. *Discrete Math.* **321** (2014), 1–11.

[**V**] J. Ekstein, P. Holub, T. Kaiser, L. Xiong and S. Zhang. Star subdivisions and connected even factors in the square of a graph. *Discrete Math.* **312** (17) (2012), 2574–2578.

[**VI**] S. Jendrol', T. Kaiser, Z. Ryjáček and I. Schiermeyer. A Dirac theorem for trestles. *Discrete Math.* **312** (12–13) (2012), 2000–2004.

[**VII**] T. Kaiser, R. Kužel, H. Li and G. Wang. A note on k-walks in bridgeless graphs. *Graphs Combin.* **23** (3) (2007), 303–308.

[**VIII**] R. Čada, T. Kaiser, M. Rosenfeld and Z. Ryjáček. Disjoint Hamilton cycles in the star graph. *Inform. Process. Lett.* **110** (1) (2009), 30–35.

[**IX**] H. Broersma, J. Fiala, P. A. Golovach, T. Kaiser, D. Paulusma and A. Proskurowski. Linear-time algorithms for scattering number and Hamilton-connectivity of interval graphs. *J. Graph Theory* (Early View 2014). DOI: 10.1002/jgt.21832.

[**X**] T. Kaiser, D. Král' and S. Norine. Unions of perfect matchings in cubic graphs. In: *Topics in discrete mathematics*. Vol. 26. Algorithms Combin. Springer, Berlin, 2006, 225–230.

[**XI**] T. Kaiser and A. Raspaud. Perfect matchings with restricted intersection in cubic graphs. *European J. Combin.* **31** (5) (2010), 1307–1315.

[**XII**] T. Kaiser, D. Král', B. Lidický, P. Nejedlý and R. Šámal. Short cycle covers of graphs with minimum degree three. *SIAM J. Discrete Math.* **24** (1) (2010), 330–355.

[**XIII**] T. Kaiser, D. Král' and R. Škrekovski. A revival of the girth conjecture. *J. Combin. Theory Ser. B* **92** (1) (2004), 41–53.

[**XIV**] T. Kaiser, D. Král', R. Škrekovski and X. Zhu. The circular chromatic index of graphs of high girth. *J. Combin. Theory Ser. B* **97** (1) (2007), 1–13.

[**XV**] T. Kaiser and R. J. Kang. The distance-t chromatic index of graphs. *Combin. Probab. Comput.* **23** (1) (2014), 90–101.

[**XVI**] T. Kaiser, M. Montassier and A. Raspaud. Covering a graph by forests and a matching. *SIAM J. Discrete Math.* **25** (4) (2011), 1804–1811.

1 Introduction

1.1 Motivation

Much of this work is motivated by two long-standing open problems which we recall in this section. The necessary definitions are given in the remaining part of the present chapter.

The first of these problems is a conjecture of Thomassen [69] on Hamilton circuits in line graphs, which we simply call 'the Thomassen Conjecture' in this text. While graphs of arbitrarily high connectivity may easily fail to be Hamiltonian, the conjecture states that for line graphs, being 4-connected is sufficient:

Conjecture 1.1.1 (Thomassen Conjecture). *Every 4-connected line graph is Hamiltonian.*

An interesting feature of the Thomassen Conjecture is that it has an unexpected number of equivalent formulations, some of which are seemingly much stronger. (See Chapter 2.) We remark that according to [69], Conjecture 1.1.1 was inspired by a conjecture of Chvátal on the Hamiltonicity of tough graphs, discussed in Section 3.6.

The second problem, which we regard as one of the most beautiful in graph theory, was first posed by Fulkerson [28] and is also attributed to C. Berge:

Conjecture 1.1.2 (Berge–Fulkerson Conjecture). *Every bridgeless cubic graph admits a list of 6 perfect matchings covering each edge exactly twice.*

Conjecture 1.1.2 is trivially true for 3-edge-colourable cubic graphs (to get the required list of perfect matchings, take each colour class twice). In fact, the conjecture would be established if one could prove it for special non-3-edge-colourable cubic graphs known as *snarks* (see Section 2.2). In this respect, the situation is similar with the Thomassen Conjecture: it would also be sufficient to prove it for line graphs of snarks. These observations highlight the prominent place of snarks in the present thesis.

Thus, we conclude this section with a kind of emblematic illustration (Figure 1.1), showing how the above two conjectures hold for the most ubiquitous of snarks, the Petersen graph. The right hand part of Figure 1.1 is taken from [28, Figure 12] (and is essentially unique).
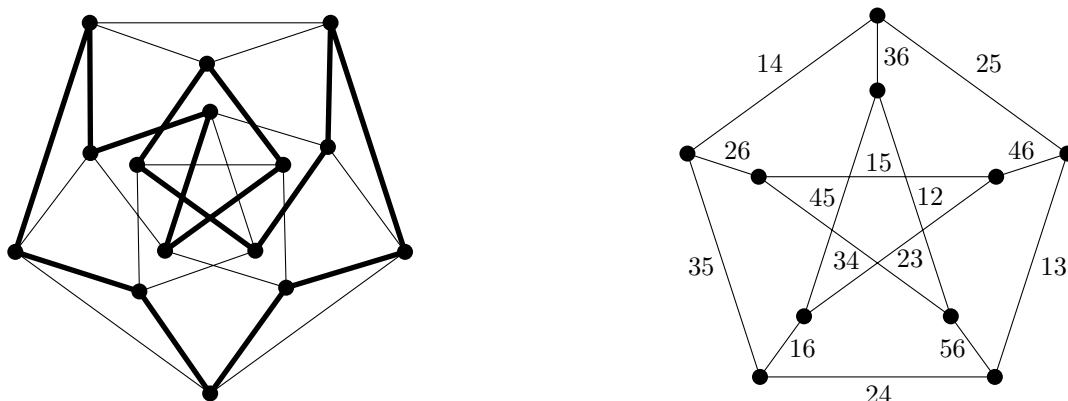
Figure 1.1: Left: a Hamilton circuit (bold) in the line graph of the Petersen graph. Right: a list of six perfect matchings in the Petersen graph satisfying the Berge–Fulkerson Conjecture. The two labels at each edge e indicate the two perfect matchings containing e.

1.2 An outline of this thesis

This thesis comprises the papers listed on page v, which are referred to by Roman numerals such as [**I**]. The choice of these papers is disjoint from the set of papers constituting my habilitation thesis [41].

In Sections 1.3–1.8, we recall basic notions and results used in the subsequent chapters, providing the necessary background.

Chapters 2–5 each start with a brief general account of the topic. Each section of these chapters is devoted to a problem studied in one or two papers contained in this thesis, which are referred to in the margin of the section. We try to provide a clear and accurate statement of each result, introduce relevant definitions, embed the results in the broader context, and highlight conceptual links to other sections. Results from papers contained in the thesis are marked with an asterisk (e.g., Theorem* 2.2.2).

Chapter 2 deals with problems related to Conjecture 1.1.1, discussing Hamilton circuits in line graphs or dominating circuits in cubic graphs. Chapter 3 collects further results on the existence of Hamiltonian structures in graphs, such as k-walks or trestles.

Chapter 4 shifts the focus from circuits to matchings as it examines several results related to the Berge–Fulkerson Conjecture (Conjecture 1.1.2). Finally, Chapter 5 discusses graph decompositions involving matchings, such as edge colourings or decompositions into acyclic subgraphs and matchings.

1.3 Graphs

We use standard terminology of graph theory as defined, e.g., in the monographs [10] or [17], to which we also refer readers interested in additional background.

Unless stated otherwise, our graphs are undirected, they may contain parallel edges but they do not contain loops. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Graphs without parallel edges are called *simple*. A graph is *k-regular* if all vertices have degree k. We refer to 3-regular graphs as *cubic*.

There are several competing conventions regarding the term cycle, circuit and Eulerian graph. In this commentary, we try to be consistent in using the term *circuit* for a connected 2-regular graph, *cycle* for an edge-disjoint union of circuits, and *Eulerian graph* for a connected cycle. The *girth* of a graph G is the length of a shortest circuit in G.

A *walk* in a graph G is a sequence $v_0, e_0, v_1, e_1, \ldots, v_k$, where each v_i ($0 \leq i \leq k$) is a vertex and each e_i ($0 \leq i \leq k-1$) is an edge with endvertices v_i and v_{i+1}. The walk is *closed* if $v_k = v_0$. It is *spanning* if it contains all the vertices of G. A *trail* is a walk where each edge of G appears at most once.

A subgraph F of G is a *factor* (or a *spanning subgraph*) if $V(F) = V(G)$. A factor which is a circuit is called a *Hamilton circuit*; any graph admitting such a factor is *Hamiltonian*.

A subgraph H of G is *induced* if each edge of G with both endvertices in $V(H)$ is present in H.

A *vertex cut* in a graph G is a set S of vertices such that $G - S$ is disconnected. The graph is *k-connected* if it contains no vertex cuts of size at most $k - 1$ and has more than k vertices. The *connectivity* of G is the largest k such that G is k-connected.

Similarly, an *edge cut* in G is a set C of edges such that $G - C$ is disconnected, and G is *k-edge-connected* if it contains no edge cuts of size at most $k - 1$. A graph is said to be *bridgeless* if it contains no edge cut of size 1. Thus, any connected bridgeless graph is 2-edge-connected.

1.4 Line graphs

Let G be a graph. The *line graph* $L(G)$ of G is the graph with vertex set $E(G)$ where two vertices e_1, e_2 are joined by an edge if e_1, e_2 are incident edges of G. Graphs of the form $L(G)$ for some G are just called *line graphs*.

Not every graph is a line graph, the simplest example being the claw $K_{1,3}$. Since the class of line graphs is closed under taking induced subgraphs, no graph containing $K_{1,3}$ as an induced subgraph is a line graph. Beineke [8] was able to characterise the class of line graphs in terms of nine forbidden induced subgraphs, one of which is $K_{1,3}$.

The claw and K_3 are distinct graphs with the same line graph. However, this is the only such pair (see, e.g. [32]): for any line graph H other than K_3, there is a unique graph G such that $H = L(G)$, called the *preimage* of H.

Various properties of the line graph $L(G)$ can be expressed in terms of properties of the graph G. For instance, the vertex-connectivity of $L(G)$ translates to the so-called essential edge-connectivity of G which we define next.

An edge cut in a graph G is *trivial* if it consists of all edges incident with a single vertex. The graph G is *essentially k-edge-connected* if each edge cut in G of size less than k is trivial[1]. The *essential edge-connectivity* of G is the largest k such that G is essentially k-edge-connected. We have:

Observation 1.4.1. *For a graph G and an integer k, $L(G)$ is k-connected if and only if G is essentially k-edge-connected and $|E(G)| > k$.*

Likewise, the Hamiltonicity of $L(G)$ has a counterpart in G. Let us call a subgraph H of G *dominating* if each edge of G is incident with a vertex of H. The following characterisation is proved in [33]:

Theorem 1.4.2. *Let G be a graph. The line graph $L(G)$ of G is Hamiltonian if and only if G admits a dominating (connected) Eulerian subgraph H and $|E(G)| \geq 3$.*

A dominating Eulerian subgraph is also traditionally called a *dominating closed trail*.

1.5 Claw-free graphs

A graph is *claw-free* if it contains no $K_{1,3}$ as an induced subgraph. By the discussion in Section 1.4, all line graphs are claw-free. Hence the following conjecture of Matthews and Sumner [53] is formally stronger than Conjecture 1.1.1:

Conjecture 1.5.1. *Every 4-connected claw-free graph is Hamiltonian.*

The conjectures are in fact equivalent, as shown by Ryjáček in [62]. The key tool in the proof is the (claw-free) closure which transforms a claw-free graph to a line graph containing it, while preserving its Hamiltonicity properties. The *closure* $cl(G)$ of a claw-free graph G is obtained by the following procedure: as long as the graph contains a vertex v such that the induced subgraph on its neighbourhood $N(v)$ is connected (an *eligible* vertex), add edges between all neighbours of v. The procedure stops when there is no vertex v with this property.

It is shown in [62] that $cl(G)$ is independent of the order in which the eligible vertices are chosen, and therefore is well-defined. Furthermore, the following holds:

Theorem 1.5.2. *Let G be a claw-free graph. Then $cl(G)$ is a line graph of a triangle-free graph. Furthermore, the length of a longest circuit is the same in G and in $cl(G)$.*

The equivalence of Conjectures 1.1.1 and 1.5.1 is obvious from this theorem.

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[1]We note that the definition in [10] requires that, in addition, G is $(k-1)$-edge-connected.

1.6 Chordal graphs

A graph is *chordal* if it contains no induced circuit of length greater than 3. Chordal graphs are interesting from the algorithmic viewpoint, since for a number of problems considered computationally hard, there are efficient algorithms in the class of chordal graphs [30, Section 4.7].

Two interesting subclasses of the class of chordal graphs are those of split graphs and interval graphs. A graph G is *split* if it contains a complete subgraph K such that $G - V(K)$ is an independent set.

To define the class of interval graphs, let $\mathcal{I}$ be a finite family of (closed bounded) intervals on the real line. The *intersection graph* $G(\mathcal{I})$ of $\mathcal{I}$ has vertex set $\mathcal{I}$ and two vertices I, J are joined by an edge if they intersect. A graph G is said to be an *interval graph* if there exists a family of intervals $\mathcal{I}$ such that G is isomorphic to $G(\mathcal{I})$. In such a case, $\mathcal{I}$ is referred to as an *interval representation* of G.

1.7 Colouring

A *(proper) colouring* of a graph G with k colours is an assignment of colours from $\{1, \ldots, k\}$ to the vertices of G such that adjacent vertices are assigned distinct colours. The *chromatic number* $\chi(G)$ of G is the least k allowing for a colouring of G with k colours. It is easy to characterise graphs with chromatic number 1 or 2, but already determining whether the chromatic number of a graph is at most 3 is a well-known NP-hard problem.

A most basic upper bound to the chromatic number is given by the following theorem of Brooks:

Theorem 1.7.1 (Brooks Theorem). *If a connected graph G has maximum degree Δ, then $\chi(G) \leq \Delta$ unless G is a complete graph or an odd circuit.*

An *edge colouring* of a graph G is defined as a colouring of the line graph $L(G)$. In other words, it is an assignment of colours to the edges of G such that incident edges are assigned distinct colours. The *chromatic index* $\chi'(G)$ of G is defined as the chromatic number of $L(G)$.

Note that for any edge colouring of G with k colours, the set of edges assigned any specific colour in $\{1, \ldots, k\}$ is a matching. Thus, edge colourings can be viewed as edge-decompositions into matchings.

A fundamental result about edge colouring is the Vizing Theorem, relating $\chi'(G)$ to the maximum degree of G:

Theorem 1.7.2 (Vizing Theorem). *If G is a simple graph with maximum degree Δ, then*

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$

Simple graphs G with $\chi'(G) = \Delta$ are referred to as *Class I* graphs, while the other graphs are *Class II*. In the cubic case, Class II graphs are closely related to *snarks* (see Section 2.2).

1.8 Tree packing

Tutte [71] and Nash-Williams [57] characterised graphs containing k edge-disjoint connected factors — or, equivalently, k edge-disjoint spanning trees:

Theorem 1.8.1 (Tree Packing Theorem)**.** *Let $k \geq 1$. A graph G contains k disjoint spanning trees if and only if for every partition $\mathcal{P}$ of $V(G)$, there are at least $k(|\mathcal{P}|-1)$ edges with endvertices in distinct classes of $\mathcal{P}$.*

By a simple counting argument, Theorem 1.8.1 implies the following:

Corollary 1.8.2. *A $2k$-edge-connected graph contains k edge-disjoint spanning trees, for any $k \geq 1$.*

This result has many applications in structural graph theory and the theory of nowhere-zero flows. For instance, Jaeger [36] used it to obtain the first positive result related to the 5-flow conjecture of Tutte [70].

2 Hamilton circuits in line graphs

2.1 Introduction

This chapter discusses our contributions related to the Hamiltonicity of line graphs. In Section 2.2, we present an equivalent form of the Thomassen Conjecture (Conjecture 1.1.1) in terms of dominating circuits in snarks.

The focus of Section 2.3 is a result which confirms the Thomassen Conjecture for 5-connected line graphs with minimum degree at least 6. This is the closest approximation to Conjecture 1.1.1 available to date, and the method devised for this purpose (which we call the *method of iterated partitions*) is one of the most substantial contributions in this thesis.

The following section briefly discusses a spinoff from the above project, namely an alternative short proof of Theorem 1.8.1 obtained by an application of the same general approach.

Finally, in Section 2.5, we discuss another application of the method of iterated partitions, namely to the 1-Hamilton-connectivity of line graphs.

2.2 Snarks and the Dominating Circuit Conjecture

Conjecture 1.1.1 has a surprising number of equivalent formulations. As mentioned in Section 1.5, it is equivalent to the conjecture, due to Matthews and Sumner [53], that all 4-connected claw-free graphs are Hamiltonian.

On the other hand, by Observation 1.4.1 and Theorem 1.4.2, Conjecture 1.1.1 is equivalent to the statement that all essentially 4-edge-connected graphs admit dominating Eulerian subgraphs.

Let us now restrict the latter statement to cubic graphs. An Eulerian subgraph of a cubic graph G is just a circuit. To rephrase the condition that G is essentially 4-edge-connected in a way better suited for a cubic graph, recall that G is said to be *cyclically k-edge-connected* if there is no edge cut C in G such that $|C| < k$ and more than one component of $G - C$ contains a circuit. Since it is not hard to see that G is cyclically 4-edge-connected if and only if it is essentially 4-edge connected, the sought restriction is:

Conjecture 2.2.1. *Every cyclically 4-edge-connected cubic graph contains a dominating circuit.*

As shown by Fleischner and Jackson [26], this restriction is actually still equivalent to Conjecture 1.1.1. In fact, Kochol [46] proved that it is equivalent to the statement that every cyclically 4-edge-connected, non-3-edge-colourable cubic graph contains a dominating circuit.

Continuing this development, we showed that the above conjectures are equivalent to a further restriction to snarks. While there are competing definitions of snarks, a widely accepted one says that a *snark* is a cyclically 4-edge-connected, non-3 edge-colourable cubic graph without circuits of length less than 5. (The reason for this condition is that circuits of length 2 to 4 make it easy to reduce the snark to a smaller one.) In [**I**], we proved:

Theorem[*] **2.2.2.** *Conjecture 2.2.1 (and thus also Conjecture 1.1.1) is equivalent to the statement that every snark admits a dominating circuit.*

2.3 Hamilton circuits in 5-connected line graphs

[**II**]   The first general result in the direction of Conjecture 1.1.1 was proved by Zhan [72] in 1991:

Theorem 2.3.1. *Every 7-connected line graph is Hamiltonian.*

Zhan's proof is based on the use of Theorem 1.8.1 and a counting argument, which together imply that essentially 7-edge-connected graphs admit dominating Eulerian subgraphs. The theorem then follows by Observation 1.4.1 and Theorem 1.4.2.

This method fails for 6-connected line graphs, where examples show that Theorem 1.8.1 is in general not applicable. In the paper [**II**], we presented a new method of dealing with the problem and we succeeded in relaxing the connectivity assumption even a bit further:

Theorem[*] **2.3.2.** *Every 5-connected line graph with minimum degree at least 6 is Hamiltonian.*

Broadly speaking, the proof method is still based on finding a structure that resembles disjoint spanning trees. However, the concept has to be modified in substantial ways, such as replacing graphs with hypergraphs of rank 3, or relaxing the notion of connectedness. The key technical ingredient of the paper is the 'Skeletal Lemma' [**II**, Lemma 17] whose applicability is more general.

2.4 Tree packing revisited

[**III**]   In view of the use of the Tree Packing Theorem to prove Theorem 2.3.1, and the fact that the proof of the stronger Theorem 2.3.2 is self-contained, it is not surprising that the method introduced in [**II**] provides a new proof of the Tree Packing Theorem.

In such a proof, the setting is in various ways simpler than that of [**II**]. As a result, the argument (published in [**III**]) is much shorter and easier to follow, and can in fact be recommended as a preparatory reading for anyone wishing to study the paper [**II**].

As pointed out by D. Král' (personal communication), the approach of [**III**] can be used, on a more general level, to get an alternative proof of the Base Packing Theorem [19] for matroids.

2.5 1-Hamilton-connectedness

Although Theorem 2.3.2 is stated for line graphs, it extends to claw-free graphs: [IV] that is, all 5-connected claw-free graphs of minimum degree at least 6 are Hamiltonian. This follows by a simple application of the closure theorem (Theorem 1.5.2).

In another direction, Theorem 2.3.2 can be extended to Hamilton-connectedness. Recall that a graph is said to be *Hamilton-connected* if for every pair of distinct vertices u, v, there is a Hamilton path starting at u and ending at v. The general statement proved in [**II**] is as follows:

Theorem[*] **2.5.1.** *All* 5*-connected claw-free graphs with minimum degree at least* 6 *are Hamilton-connected.*

As expected, this result is first proved for line graphs, and then extended to claw-free graphs by means of a closure result. However, the claw-free closure and Theorem 1.5.2 cannot be used for this purpose. Instead, we applied the so-called multigraph closure — introduced in [63] — which behaves well with respect to Hamilton-connectedness in the same way as the claw-free closure works for Hamiltonicity.

A natural strengthening of Hamilton-connectedness is the notion of k-Hamilton-connectedness. A graph is said to be k-*Hamilton-connected* if the removal of any k vertices produces a Hamilton-connected graph. Using a modification of the argument of [**II**] in combination with a suitable closure concept, we managed to strengthen Theorem 2.5.1 to 1-Hamilton-connected graphs [**IV**]:

Theorem[*] **2.5.2.** *Every* 5*-connected claw-free graph with minimum degree at least* 6 *is* 1*-Hamilton-connected.*

In this case, the closure used was the 1HC-closure [64] which preserves the (non)-1-Hamilton-connectedness of claw-free graphs.

3 Hamiltonian structures in graphs

3.1 Introduction

This chapter is intended to demonstrate some of the techniques used when dealing with Hamiltonian structures in graphs of various types. In the first three sections, the structures are not Hamilton circuits as such, but are obtained by relaxing their definition in one way or another. For instance, if we view a Hamilton circuit in a graph G as an Eulerian factor of maximum degree 2, we can be interested in the existence of Eulerian factors of maximum degree $2s$ $(s \geq 1)$. This is the case of Section 3.2, where this problem is studied from the point of view of forbidden induced subgraphs.

In the setting of Section 3.3, Hamilton circuits are viewed as 2-connected factors with maximum degree 2. Allowing higher maximum degree, say k, we obtain the definition of k-trestle. The result of this section is a trestle variation on Dirac's classical degree condition for Hamiltonicity [18].

Yet another Hamiltonian-type structure appears in Section 3.4: a Hamilton circuit can also be regarded as a closed walk visiting each vertex once. If we allow it to visit a vertex up to k times, the result is the notion of k-walk. Naturally, one is interested in the existence of k-walks where k is as small as possible; we present a result bounding k in terms of the maximum degree for bridgeless graphs.

Returning to Hamilton circuits as such, Section 3.5 focuses on the Hamiltonicity of network topologies, in particular certain Cayley graphs known as star graphs. While these graphs are known to be Hamiltonian, fault tolerance in network design makes it desirable to have multiple edge-disjoint Hamilton circuits. Our result is a lower bound on the number of such circuits in the class of star graphs.

Research reported in the last section of this chapter is motivated by another long-standing open problem, the conjecture of Chvátal about the Hamiltonicity of tough graphs. We investigate Hamiltonian properties of interval graphs and relate them to the so-called scattering number.

3.2 Star subdivisions and the square of a graph

The results in this section are motivated by a well-known result of Fleischner [27] [V] from 1971 concerning Hamilton circuits in the square of a graph. Recall that the *square* G^2 of G is the graph with the same vertex set and edges joining vertices at

distance 1 or 2 in G. Fleischner showed the following:

Theorem 3.2.1. *The square of every 2-connected graph is Hamiltonian.*

The assumption of 2-connectedness in Theorem 3.2.1 cannot be relaxed, as shown by the claw with each edge subdivided once (denoted by $S(K_{1,3})$). On the other hand, Gould and Jacobson [31] conjectured that if a *connected* graph G contains no induced copy of $S(K_{1,3})$, then G^2 is Hamiltonian. This conjecture was proved by Hendry and Vogler [34] and the result was further extended in [1].

As noted in Section 3.1, a Hamilton circuit may be regarded as an Eulerian factor with maximum degree 2. Allowing maximum degree $2s$ ($s \geq 1$) instead, we obtain the notion of $[2, 2s]$-*factor*. Thus, Hamilton circuits coincide with $[2, 2]$-factors.

We found in [**V**] that Hendry and Vogler's result generalises in a natural way to a sufficient condition for the existence of a $[2, 2s]$-factor in G^2. The condition involves $S(K_{1,2s+1})$, the subdivision of the $(2s + 1)$-star, as a forbidden induced subgraph:

Theorem* 3.2.2. *Let G be a connected graph containing no induced copy of the graph $S(K_{1,2s+1})$, where $s \geq 1$. Then G^2 contains a $[2, 2s]$-factor.*

3.3 A Dirac theorem for trestles

[**VI**]   A number of sufficient conditions for Hamiltonicity involve vertex degrees — well-known examples include conditions due to Dirac, Ore or Chvátal. Dirac's result [18] reads as follows:

Theorem 3.3.1 (Dirac Theorem). *If G is a simple graph with $n \geq 3$ vertices and minimum degree at least $n/2$, then G is Hamiltonian.*

Searching for degree conditions for more general types of Hamiltonian structures, we focused on the notion of k-trestle. For $k \geq 2$, a k-*trestle* in a graph G is a 2-connected factor of G with maximum degree at most k. Clearly, a 2-trestle is just a Hamilton circuit. The concept originated in [6] and has mainly been studied in the context of planar graphs or graphs on other surfaces.

In [**VI**], we succeeded in proving the following generalisation of the Dirac Theorem to trestles:

Theorem* 3.3.2 ([**VI**]). *Every 2-connected simple graph with n vertices and minimum degree at least $2n/(k + 2)$, where $k \geq 2$, contains a k-trestle.*

Note that the $k = 2$ case implies the Dirac Theorem, since a simple graph with n vertices and minimum degree at least $n/2$ is easily seen to be 2-connected.

3.4 k-walks in graphs

A natural way of looking at a Hamilton circuit in a graph G is as a closed walk visiting each vertex of G exactly once. If we allow multiple visits, we obtain the definition of a *k-walk* in G — a closed walk visiting each vertex at least once and at most k times.

There is a close connection between k-walks and k-trees, defined (in this context) as spanning trees of maximum degree at most k. Indeed, a depth-first traversal of a k-tree produces a k-walk. In the converse direction, if we follow a k-walk starting at an arbitrary vertex, and mark the current edge whenever it leads to a vertex that has not yet been visited, then the resulting set of marked edges constitutes a $(k+1)$-tree.

There is an interesting problem relating k-walks to toughness. Jackson and Wormald [35] proved that for $k \geq 3$, every $1/(k-2)$-tough graph admits a k-walk, and made the following conjecture:

Conjecture 3.4.1. *For $k \geq 2$, every $1/(k-1)$-tough graph admits a k-walk.*

Even the $k = 2$ case of this conjecture is an appealing open problem.

In [**VII**], we relate the existence of k-walks in graphs to the maximum degree. Suppose that a graph G has maximum degree Δ. If G is connected, then any spanning tree is a Δ-tree, and therefore G admits a Δ-walk. The star on $\Delta + 1$ vertices shows that we cannot expect a k-walk with $k < \Delta$ in this case. On the other hand, we prove in [**VII**] that an improvement is possible if G is 2-edge-connected:

Theorem* 3.4.2. *Any 2-edge-connected graph with maximum degree Δ admits a $\lceil (\Delta + 1)/2 \rceil$-walk.*

A construction in the paper shows that this result is optimal.

3.5 Hamilton circuits in network topologies

Network topologies are graphs with properties suitable for interconnection net- works, with vertices representing processors and edges representing data links between them. The relevant properties include, e.g., maximum degree, connectivity, diameter, Hamiltonicity or Hamilton connectedness depending on the situation.

A classical example of a network topology is the *n-hypercube* whose vertices are binary strings of length n and whose edges connect strings that differ in a single coordinate. Explicit Hamilton circuits in the n-hypercube, and more generally in other combinatorially defined networks, are traditionally known as *Gray codes* (see the comprehensive survey [65]).

Not only is the n-hypercube Hamiltonian; it is even Hamilton-decomposable [5]. (We recall that a regular graph of degree d is said to be *Hamilton-decomposable* if it admits $\lfloor d/2 \rfloor$ edge-disjoint Hamilton circuits.) This is a favourable property for

a network, since up to $\lfloor d/2 \rfloor - 1$ links can be allowed to be faulty and a Hamilton circuit is still guaranteed to exist.

In the paper [**VIII**], we study related problems for another class of network topologies, namely the star graphs. These graphs were introduced by Akers et al. [3] and are regarded as alternatives to hypercube networks with slightly better parameters in some respects. Before we define star graphs, we recall the definition of a Cayley graph.

Let Γ be a group and $X \subseteq \Gamma$ a subset containing the inverse x^{-1} of each $x \in X$. The *Cayley graph on Γ generated by X*, denoted by $Cay(\Gamma, X)$, has vertex set Γ and edges joining g to gx for each $g \in \Gamma$ and $x \in X$.

The *star graph* $\Sigma(n)$ is obtained by taking Γ to be the symmetric group S_n on n elements (say $0, \ldots, n-1$) and taking X to be the set of transpositions $\{(0, 1), \ldots, (0, n-1)\}$. It is known to be Hamiltonian; in fact, the same is true for every Cayley graph $Cay(S_n, X)$ where X is a set of transpositions generating S_n [47]. It is an open problem whether such graphs are Hamilton-decomposable, and the answer is not known even for $\Sigma(n)$.

In [**VIII**], we obtain a partial result toward this question:

Theorem* 3.5.1. *The graph $\Sigma(n)$ contains a family of $\Omega(n/\log\log n)$ edge-disjoint Hamilton circuits. For prime n, the bound improves to $\lfloor n/8 \rfloor$.*

Since $\Sigma(n)$ is $(n-1)$-regular, proving it Hamilton-decomposable would require finding $\lfloor (n-1)/2 \rfloor$ edge-disjoint Hamilton circuits. Our lower bound has been improved in [16] roughly by a factor of two (to $\lfloor n/5 \rfloor$ in the prime case).

3.6 Hamilton circuits in interval graphs

[**IX**]   Chvátal [14] defined the *toughness* $\tau(G)$ of a graph G as follows:

$$\tau(G) = \min_{S} \frac{|S|}{c(G - S)},$$

where S ranges over all vertex cuts of G and $c(G - S)$ is the number of components of $G - S$; in the case of a complete graph, we set $\tau(K_n) = \infty$. The graph G is said to be *t-tough* if $\tau(G) \geq t$.

Every Hamiltonian graph is easily seen to be 1-tough. Chvátal [14] conjectured a partial converse:

Conjecture 3.6.1. *There exists t_0 such that every t_0-tough graph is Hamiltonian.*

If the constant t_0 in Conjecture 3.6.1 exists, it is known [7] that $t_0 \geq 9/4$.

Much of the research of Conjecture 3.6.1 has been directed at the class of chordal graphs, where the conjecture is known to hold [13] with $t_0 = 18$ and where the bound was recently improved to 10 by Kabela and Kaiser [40]. Subclasses for which the optimal value of t_0 is known include the class of split graphs [49] or interval graphs. For the latter, Keil [43] provided an algorithm that finds a Hamilton circuit in

an interval graph whenever it exists. According to [13], this algorithm implicitly contains the result that an interval graph is Hamiltonian if and only if it is 1-tough.

In [**IX**], we continued the study of Hamilton circuits in interval graphs in connection with a parameter related to toughness, the *scattering number* $sc(G)$ of G, defined in [39]:

$$sc(G) = \max_{S} c(G - S) - |S|.$$

Since $sc(G) \leq 0$ if and only if $\tau(G) \geq 1$, Keil's result implies that interval graphs G with $sc(G) \leq 0$ are Hamiltonian. As for positive values of $sc(G)$, it is known [15] that for an interval graph G and $k \geq 1$, $sc(G) \leq k$ is equivalent to the existence of k vertex-disjoint paths covering $V(G)$. In [**IX**], we found the following interpretation of the negative values of $sc(G)$ which complements the above results in a natural way. Recall that a graph is said to be *t-Hamilton-connected* if the removal of any set of vertices of cardinality at most t yields a Hamilton-connected graph.

Theorem* 3.6.2. *For $k \leq -1$ and an interval graph G, $sc(G) \leq k$ if and only if G is $-(k+1)$-Hamilton-connected.*

Furthermore, we give an $O(n + m)$ time algorithm for finding the scattering number of an interval graph with n vertices and m edges, improving a previously known $O(n^3)$ time algorithm [48]. (For background on computational complexity, we refer the reader to the monographs [29] or [67].)

Lastly, our paper [**IX**] includes an explicit proof that Keil's algorithm finds a Hamilton circuit in any 1-tough interval graph (more precisely, an explicit construction of a set showing that the toughness is below 1 if the algorithm fails).

4 Around the Berge–Fulkerson Conjecture

4.1 Introduction

This chapter brings matchings into play as it centers around the Berge–Fulkerson Conjecture:

Conjecture (Berge–Fulkerson; stated earlier as **Conjecture 1.1.2**)**.** *Any bridgeless cubic graph admits a set of* 6 *perfect matchings covering each edge exactly twice.*

On the other hand, this does not take us too far away from circuits in graphs. Recall that a *cycle* in a graph is defined as an edge-disjoint union of circuits. By considering the complements of the six perfect matchings in Conjecture 1.1.2, we obtain an equivalent statement in terms of cycles:

Conjecture 4.1.1. *Every bridgeless cubic graph admits a set of* 6 *cycles covering each edge exactly* 4 *times.*

This formulation is reminiscent of the another fundamental problem, due to Szekeres [68] and Seymour [66]:

Conjecture 4.1.2 (Cycle Double Cover Conjecture)**.** *Every bridgeless graph admits a set of cycles covering each edge exactly twice.*

In fact, the stronger Five Cycle Double Cover Conjecture [11, 61] parallels Conjecture 4.1.1 even more closely; for emphasis, we state the conjecture in its equivalent form for cubic graphs:

Conjecture 4.1.3 (Five Cycle Double Cover Conjecture)**.** *Every bridgeless cubic graph admits a set of* 5 *cycles covering each edge exactly twice.*

Given the simplicity of the formulation of the Berge–Fulkerson Conjecture, it is surprising how little is known about it. In the absence of techniques needed to attack the conjecture itself, much attention has been given to special cases, connections to other open questions and modifications of the problem.

The Berge–Fulkerson Conjecture, if true, implies that the edge set of any bridgeless cubic graph can be covered by five perfect matchings. This suggests the question about the proportion of the edges that can be covered by k perfect matchings. In Section 4.2, we recall related results and problems.

Another consequence of this conjecture is that every bridgeless cubic graph contains three perfect matchings with empty intersection, as conjectured by Fan and Raspaud [24]. A weakening of this statement, motivated by the concept of Fano colouring of cubic graphs, is discussed in Section 4.3.

Section 4.4 focuses on short cycle covers of graphs, a topic closely related to the Berge–Fulkerson Conjecture and other problems involving cubic graphs. We review these connections as well as known bounds on the length of a shortest cycle cover.

4.2 Unions of perfect matchings

[X] Assuming the truth of the Berge–Fulkerson Conjecture, consider a *Berge–Fulkerson covering* of a bridgeless cubic graph G — that is, a set of six perfect matchings satisfying the condition stated by the conjecture. Any five out of these perfect matchings still cover $E(G)$. What is the maximum number of edges coverable by $k < 5$ perfect matchings, relative to the size of G?

It is convenient to restate the question using the following notation. For a bridgeless cubic graph G and an integer k, define $m_k(G)$ as the maximum ratio

$$\frac{|M_1 \cup \cdots \cup M_k|}{|E(G)|},$$

where $M_1, \ldots, M_k$ range over perfect matchings of G. Furthermore, let m_k be the infimum of $m_k(G)$ over all bridgeless cubic graphs G. By the above, the Berge–Fulkerson Conjecture implies that $m_5 = 1$. The latter statement, conjectured by Berge, was in fact shown to be equivalent to the conjecture [54].

As for the values of m_k with $k < 5$, it is trivial that $m_1 = 1/3$. By considering the Petersen graph, we find that

$$m_2 \le 3/5, m_3 \le 4/5 \text{ and } m_4 \le 14/15. \tag{4.1}$$

In [**X**], we determined the exact value of m_2 and provided a lower bound for m_3:

Theorem* 4.2.1. *It holds that $m_2 = 3/5$ and $m_3 \ge 27/35$.*

The main tool in the proof is the Perfect Matching Polytope Theorem of Edmonds [20].

By an easy counting argument (cf. [59]), the Berge–Fulkerson Conjecture would imply that all the inequalities in (4.1) are in fact equalities. It is worth noting that Esperet and Mazzuoccolo [23] recently showed that $m_4 = 14/15$ would imply $m_3 = 4/5$, and the latter would imply the Fan–Raspaud Conjecture discussed in Section 4.3.

4.3 Sparsely intersecting perfect matchings

Let $\mathcal{C}$ be a Berge–Fulkerson covering of a bridgeless cubic graph G. Since every edge is only contained in two perfect matchings of $\mathcal{C}$, the intersection of any three perfect matchings of $\mathcal{C}$ is empty. Fan and Raspaud [24] stated this consequence as a separate conjecture: [XI]

Conjecture 4.3.1 (Fan–Raspaud Conjecture). *In any bridgeless cubic graph, there are three perfect matchings with empty intersection.*

A natural relaxation of Conjecture 4.3.1 replaces perfect matchings with *joins*, defined — in cubic graphs — as factors with all vertex degrees odd (thus, any perfect matching is a join):

Theorem 4.3.2. *In any bridgeless cubic graph, there are three joins with empty intersection.*

Theorem 4.3.2 is easily seen to be equivalent to Jaeger's 8-Flow Theorem [36]. (To point out a connection to an earlier chapter, we mention that the main tool used to prove the theorem is Theorem 1.8.1, discussed in Section 2.4.)

Máčajová and Škoviera [51] observed a connection between Theorem 4.3.2 and the so-called Fano colourings which they introduced. A *Fano colouring* of a cubic graph G is an assignment of points of the Fano plane to the edges of G such that any three edges incident with a common vertex are assigned three distinct collinear points. The existence of Fano colourings in all cubic bridgeless graphs is equivalent to Theorem 4.3.2.

Define a *k-line Fano colouring* of G to be one in which the collinear point triples appearing at the vertices of G correspond to only k of the lines of the Fano plane. Thus, every bridgeless cubic graph admits a 7-line Fano colouring. At the other extreme, the existence of a 1-line Fano colouring is (clearly) equivalent to G being 3-edge-colourable. As shown in [51], every bridgeless cubic graph actually admits a 6-line Fano colouring. Furthermore, the existence of a 4-line Fano colouring in a cubic bridgeless graph is shown there to be equivalent to the existence of three perfect matchings with empty intersection as in Conjecture 4.3.1.

In [**XI**], we studied the existence of 5-line Fano colourings in a class of cubic graphs that are, in a sense, close to being 3-edge-colourable. We recall that the *oddness* of a bridgeless cubic graph G is defined as the minimum number of odd circuits in a 2-factor of G. The main result of [**XI**] is:

Theorem* 4.3.3. *Every bridgeless cubic graph of oddness 2 admits a 5-line Fano colouring.*

Our result was subsequently improved by Máčajová and Škoviera [50] who established the validity of the Fan–Raspaud Conjecture (and thus, the existence of 4-line Fano colourings) for bridgeless cubic graphs of oddness 2.

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4.4 Short circuit covers

A *circuit cover* of a graph G is a set of circuits covering each edge at least once. (Note that a graph is bridgeless if and only if it admits a circuit cover.) The *length* of a circuit cover is the sum of the lengths of all the circuits it contains. The length of a shortest circuit cover, relative to the total number of edges, is the subject of several results and conjectures.

Let $s(G)$ denote the length of a shortest circuit cover of a graph G, and let $m(G)$ denote the number of edges of G. An upper bound on $s(G)$ in terms of $m(G)$ is provided by the following result [4, 9]:

Theorem 4.4.1. *For any bridgeless graph G,*

$$s(G) \leq \frac{5m(G)}{3} \approx 1.667m(G).$$

For cubic bridgeless graphs, the bound was improved by Fan [25] to $s(G) \leq 44m(G)/27 \approx 1.630m(G)$. In [**XII**], we provide a further improvement:

Theorem* 4.4.2. *For any cubic bridgeless graph,*

$$s(G) \leq \frac{34m(G)}{21} \approx 1.619m(G).$$

Very recently, a proof of the bound $s(G) \leq 1.6m(G)$ was announced by R. Lukoťka.

Short circuit covers are closely related to the Cycle Double Cover Conjecture (Conjecture 4.1.2): as shown by Jamshy and Tarsi [38], the latter conjecture would follow if one could show that $s(G) \leq 1.4m(G)$ for every cubic bridgeless graph G. The constant 1.4 is attained for the Petersen graph.

A proof of the Berge–Fulkerson Conjecture would have significant consequences for the shortest circuit cover problem: as proved in [24], it would imply that $s(G) \leq 22m(G)/15 \approx 1.467m(G)$ for every bridgeless cubic graph. Note that the constant is only $1/15$ away from the value that would suffice to prove the Cycle Double Cover Conjecture.

5 Matchings and decompositions

5.1 Introduction

This chapter deals with graph decomposition problems involving matchings. Included among them is edge colouring, which is just a decomposition into matchings.

By the Vizing Theorem (Theorem 1.7.2), the chromatic index of simple cubic graphs is 3 or 4. Cubic graphs which do not admit any 3-edge-colouring play an important rôle in previous chapters (particularly snarks, whose definition includes additional nontriviality contraints). Thus, edge colouring underlies much of the material presented in this thesis.

Section 5.2 is related to a conjecture of Jaeger and Swart that the girth of snarks is bounded. Although this conjecture is known to be false, we show that its natural counterpart involving circular edge colouring holds. An extension of the result for higher maximum degree is presented in the same section.

A more restrictive variant of edge colouring, the strong edge colouring, is studied in Section 5.3. It has proved difficult to upper bound the strong chromatic index in terms of the maximum degree (in the spirit of the Vizing Theorem). We consider a generalisation of strong edge colouring, the so-called distance-t colouring, and extend Molloy and Reed's upper bound for the strong chromatic index to this more general case.

Many decomposition problems deal with decompositions into other 'simple' subgraphs than matchings. A particularly natural choice are acyclic subgraphs; the parameter corresponding to the chromatic index is called arboricity and is quite well-understood. In Section 5.4, we examine a less well-understood combination of the two problems, where the sought decomposition is into a number of acyclic subgraphs and a matching.

5.2 Snarks and the circular chromatic index

The problem of determining whether a given cubic graph is 3-edge-colourable is [XIII] a well-known NP-complete problem. Thus, there seems to be little hope of finding a good characterisation of snarks. On the other hand, there are a number of known or conjectural necessary conditions. For instance, it is known that every snark contains a subdivision of the Petersen graph (the proof of this conjecture of Tutte was finished in [22]). A bold conjecture was made by Jaeger and Swart [37] in 1979:

Conjecture 5.2.1. *There exists an integer k such that every snark contains a circuit of length at most k.*

Recall that the *girth* of a graph G is defined as the minimum length of a circuit in G. Thus, the conjecture asserts that the girth of snarks is bounded. Kochol [45] used his superposition method to construct snarks of arbitrarily large girth, thus disproving Conjecture 5.2.1. (We remark, however, that it is still an intriguing open problem whether the cyclic connectivity of snarks is bounded.)

In the paper [**XIII**], we studied a relaxation of Conjecture 5.2.1 involving the so-called circular chromatic index. For positive integers p, q, a *circular p/q-edge colouring* of a graph G is an assignment of colours in $\{0, \dots, p-1\}$ to the edges of G, such that for incident edges, the difference modulo p of their colours does not fall in the interval $\{-(q-1), -(q-2), \dots, q-1\}$. The *circular chromatic index* $\chi'_c(G)$ of G is defined as the minimum p/q such that there exists a circular p/q-edge colouring of G. (The minimum is known to be attained for every graph G.) The circular chromatic index of G is a lower bound to $\chi'(G)$ since it can be shown that $\chi'(G) = \lceil \chi'_c(G) \rceil$.

We showed in [**XIII**] that Conjecture 5.2.1 is, in a sense, true for circular edge colouring:

Theorem* 5.2.2. *For each $\varepsilon > 0$, there is g such that for every cubic bridgeless graph G with girth at least g, $\chi'_c(G) < 3 + \varepsilon$.*

[**XIV**] In the follow-up paper [**XIV**], we extended the result to graphs of higher maximum degree:

Theorem* 5.2.3. *For each $\varepsilon > 0$ and $\Delta \geq 1$, there is g such that for every graph G with maximum degree Δ and girth at least g, $\chi'_c(G) < \Delta + \varepsilon$.*

5.3 The strong chromatic index

[**XV**] A k-edge colouring of a graph G is said to be *strong* if the edges of each colour form an induced matching in G. That is, if edges e and f have the same colour, then there is no edge in G incident with both e and f. The least k for which a graph G admits a strong k-edge colouring is called the *strong chromatic index* of G and denoted by $\chi'_2(G)$.

Erdős and Nešetřil proposed in 1985 the problem of determining the maximum possible strong chromatic index of a graph with maximum degree Δ. They observed that for even Δ, the graph obtained by blowing up each vertex of a 5-cycle to an independent set of size $\Delta/2$ (in such a way that edges of the 5-cycle correspond to complete bipartite subgraphs) has maximum degree Δ and strong chromatic index $5\Delta^2/4$. They conjectured that this lower bound is optimal.

A trivial upper bound to $\chi'_2(G)$ for graphs G of maximum degree Δ, namely

$$\chi'_2(G) \leq 2\Delta(\Delta - 1), \tag{5.1}$$

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follows from the Brooks Theorem. Although the coefficient 2 at the Δ^2 term in (5.1) is far from the conjectured value of $5/4$, it has proved hard to improve it even by a tiny margin. The best general bound is due to Molloy and Reed [55]:

Theorem 5.3.1. *For any graph G with sufficiently large maximum degree Δ, $\chi'_2(G) \leq 1.992\Delta^2$.*

The theorem is proved by a probabilistic argument.

Together with R. Kang, we studied in [**XV**] a variant of strong colouring with a more restrictive condition for edges of the same colour. Given $t \geq 1$, a *distance-t colouring* of a graph G is an edge colouring where no two edges of the same colour are at distance less than t. (The distance of two edges is measured as the number of vertices in a shortest path joining an endvertex of one to an endvertex of the other.) Thus, distance-1 colouring is just proper edge colouring, and distance-2 colouring is strong edge colouring.

For any t, there is the trivial upper bound $\chi'_t \leq 2\Delta^t$. As shown by examples constructed in [42], the term Δ^t cannot be improved to an asymptotically lesser one. As we found in [**XV**], the coefficient 2 can be decreased by a universal constant, valid not only for all graphs but (somewhat surprisingly) also for all t:

Theorem* 5.3.2. *Let $t \geq 2$. For any graph G with sufficiently large maximum degree Δ, $\chi'_t(G) \leq 1.99992 \cdot \Delta^t$.*

For graphs of large girth, the upper bound can be improved. In the case of the strong chromatic index, Mahdian [52] showed that if a graph G contains no circuits of length 4, then $\chi'_2(G) = O(\Delta^2/\log \Delta)$. For general distance-$t$ colouring, we obtain the following result [**XV**]:

Theorem* 5.3.3. *Let $t \geq 2$. For any graph of girth at least $2t + 1$ and maximum degree Δ, $\chi'_t(G) = O(\Delta^t/\log \Delta)$.*

5.4 Decomposing graphs into forests and a matching

Edge colouring is the most usual kind of an edge-decomposition problem where graphs are decomposed into 'simply structured' subgraphs (in this case matchings). Another very natural problem of this type deals with decompositions into acyclic subgraphs and leads to a parameter known as arboricity. [XVI]

Let G be a graph. The *arboricity* of G, denoted by $\Upsilon(G)$, is the least k such that there exists an edge-decomposition of G into k acyclic subgraphs (forests). An exact min-max relation, expressing the arboricity in terms of the density of subgraphs, is given by a theorem of Nash-Williams [58]:

Theorem 5.4.1. *For any graph G,*

$$\Upsilon(G) = \max_H \left\lceil \frac{|E(H)|}{|V(H) - 1|} \right\rceil,$$

$$(5.2)$$

where the maximum is taken over all subgraphs H of G.

A very short proof of Theorem 5.4.1 was given in [12]. The arboricity of a graph can be found in polynomial time, using algorithms for matroid partitioning (see, e.g., [21]). This is a substantial difference from the chromatic index which is NP-hard to determine.

A kind of combination of the two problems was considered by Montassier et al. [56]. Removing the rounding from the right-hand side of (5.2), we obtain the definition of *fractional arboricity*, first introduced by Payan [60] and denoted by Υ_f.

Suppose that $\Upsilon(G) = k + 1$ and $\Upsilon_f(G)$ is close to k. All we know from Theorem 5.4.1 is that G can be decomposed into $k + 1$ forests. On the other hand, it is natural to ask (as in [56]) whether the information on $\Upsilon_f(G)$ allows us to place an additional restriction on some of the forests — for instance, we could require that one of them has bounded maximum degree or bounded component size. The following conjecture (called the Nine Dragon Tree Conjecture[1]) was stated in [56]:

Conjecture 5.4.2. *Let k and d be positive integers. If G is a graph such that*

$$\Upsilon_f(G) \leq k + \frac{d}{k + 1 + d},$$

then $E(G)$ can be decomposed into $k + 1$ forests, one of which has maximum degree at most d.

Two cases of Conjecture 5.4.2 proved in [56] were $k = 1, d = 1$ (decomposition into a forest and a matching) and $k = 1, d = 2$ (decomposition into a forest and a linear forest).

In [**XVI**], we sought to extend the first result to decompositions into a number of forests and a matching. We proved the following fractional arboricity bound implying the existence of such a decomposition:

Theorem* **5.4.3.** *If G is a graph such that*

$$\Upsilon_f(G) \leq k + \frac{1}{3k + 2},$$

then G admits an edge-decomposition into $k + 1$ forests, one of which is a matching.

The proof is an interesting application of a topological extension of the Matroid Intersection Theorem, due to Aharoni and Berger [2].

Kim et al. [44] established Conjecture 5.4.2 for $k = 1$ and $d \leq 6$, and proved several strong partial results, one of which improves the bound in Theorem 5.4.3 to $k + 1/(2k + 2)$. (Compare this to the conjectured value of $k + 1/(k + 2)$.) It remains an interesting problem to see if one can match this bound, or even the bound in Conjecture 5.4.2, using the topological tool employed in [**XVI**].

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[1]This exotic name was chosen in reference to a giant banyan tree in Kaohsiung, Taiwan.

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Résumé

This thesis is based on a collection of 16 papers about the properties of circuits and matchings in graphs. It includes a commentary divided into five chapters that provides a background for the reported results.

Chapter 1 describes some of the motivating questions and recalls basic definitions and results used in the commentary.

The following two chapters (Chapters 2– 3) belong to Hamiltonian graph theory: the questions under study primarily concern the existence of Hamilton circuits and related structures in various types of graphs. This includes results on the Hamiltonian properties of line graphs (in relation to a conjecture of Thomassen that 4-connected line graphs are Hamiltonian), as well as those involving k-walks, trestles or Hamilton circuits depending on degree conditions or forbidden subgraph assumptions. Also included is a theorem on disjoint Hamilton circuits in a particular class of network topologies known as star graphs. Concluding the first part, we present a result relating the Hamiltonian properties of interval graphs to their scattering number, and providing algorithmic procedures to decide these properties.

In the last two chapters (Chapters 4–5), the structure of matchings comes to the fore in a discussion of two contributions related to the Berge–Fulkerson conjecture. One of them determines the best possible fraction of edges of a cubic bridgeless graph coverable by two perfect matchings; the other one confirms the existence of restricted Fano colourings in cubic graphs of oddness two. The topic is intimately connected to short cycle covers, the subject of the following result which improves known upper bounds for the length of the shortest cycle cover of a bridgeless graph with minimum degree at least 3.

The part devoted to matchings continues with decompositions and edge colourings. While it is known that the girth of non-3-edge-colourable graphs is unbounded (contrary to a conjecture of Jaeger and Swart), we show that an analogue of the conjecture for circular edge-colouring holds true, and we extend it to a Vizing-type theorem for circular edge-colouring, valid for graphs of arbitrary maximum degree. Another result extends a bound for the strong chromatic index (in terms of the maximum degree) to distance-t colourings. Finally, we present a contribution regarding a more general type of a decomposition problem, namely on decompositions into a number of acyclic subgraphs and one matching, related to the notion of fractional arboricity.