Convergence of graphons and structuredness order

Martin Doležal
(joint work with J. Grebík, J. Hladký, I. Rocha, V. Rozhoň)

Institute of Mathematics of the Czech Academy of Sciences

Workshop Graph limits in Bohemian Switzerland
March 28, 2018

Limits of dense graph sequences

Motivation:

Limits of dense graph sequences

Motivation:

Find a compactification of the space of finite graphs

Limits of dense graph sequences

Motivation:

Find a compactification of the space of finite graphs (so that every sequence of finite graphs has a convergent subsequence).

Limits of dense graph sequences

Motivation:

Find a compactification of the space of finite graphs (so that every sequence of finite graphs has a convergent subsequence).

Borgs, Chayes, Lovász, Sós, Szegedy, Vesztergombi, 2006:

Limits of dense graph sequences

Motivation:

Find a compactification of the space of finite graphs (so that every sequence of finite graphs has a convergent subsequence).

Borgs, Chayes, Lovász, Sós, Szegedy, Vesztergombi, 2006:

The elements of the compactification are *graphons*

Limits of dense graph sequences

Motivation:

Find a compactification of the space of finite graphs (so that every sequence of finite graphs has a convergent subsequence).

Borgs, Chayes, Lovász, Sós, Szegedy, Vesztergombi, 2006:

The elements of the compactification are *graphons* = symmetric Lebesgue measurable functions from $[0,1]^2$ to $[0,1]$

Limits of dense graph sequences

Motivation:

Find a compactification of the space of finite graphs (so that every sequence of finite graphs has a convergent subsequence).

Borgs, Chayes, Lovász, Sós, Szegedy, Vesztergombi, 2006:

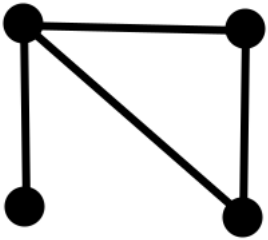
The elements of the compactification are *graphons* = symmetric Lebesgue measurable functions from $[0,1]^2$ to $[0,1]$ (or more generally from Ω^2 to $[0,1]$ where Ω is a given probability space).

Graphons

How do we represent a graph by a graphon?

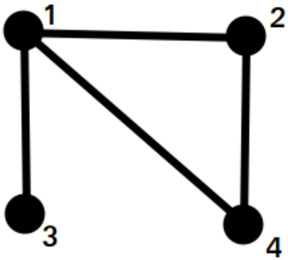
Graphons

How do we represent a graph by a graphon?



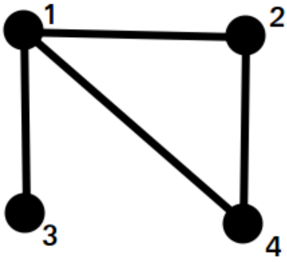
Graphons

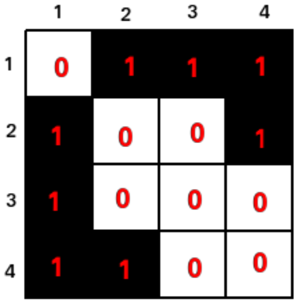
How do we represent a graph by a graphon?



Graphons

How do we represent a graph by a graphon?





Basic example

$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n

Basic example

$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n

There are many possible representations of $K_{n,n}$.

Basic example

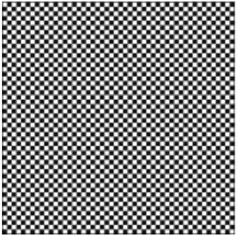
$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n

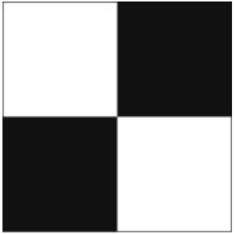
There are many possible representations of $K_{n,n}$. Here are two of them:

Basic example

$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n

There are many possible representations of $K_{n,n}$. Here are two of them:

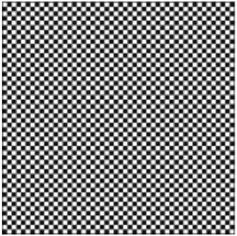


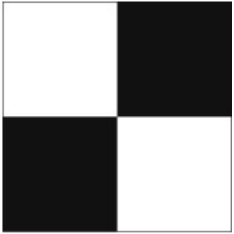


Basic example

$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n

There are many possible representations of $K_{n,n}$. Here are two of them:



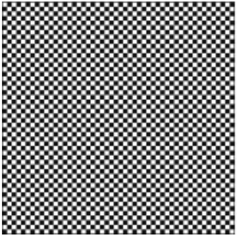
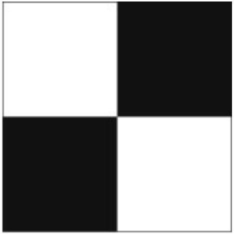


When n is large then the chessboard on the left looks like the constant graphon $W \equiv \frac{1}{2}$.

Basic example

$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n

There are many possible representations of $K_{n,n}$. Here are two of them:

When n is large then the chessboard on the left looks like the constant graphon $W \equiv \frac{1}{2}$. But the chessboard on the right does not depend on n at all!

Cut-norm and cut-distance

The cut norm

Cut-norm and cut-distance

The cut norm compares the density of edges inside any vertex set:

Cut-norm and cut-distance

The cut norm compares the density of edges inside any vertex set:

$$d_{\square}(U, W) := \sup_{A \subset [0,1]} \left| \int_{A \times A} (U(x,y) - W(x,y)) \right| .$$

Cut-norm and cut-distance

The cut norm compares the density of edges inside any vertex set:

$$d_{\square}(U,W) := \sup_{A \subset [0,1]} \left| \int_{A \times A} (U(x,y) - W(x,y)) \right|.$$

The cut-distance

Cut-norm and cut-distance

The cut norm compares the density of edges inside any vertex set:

$$d_{\square}(U, W) := \sup_{A \subset [0,1]} \left| \int_{A \times A} (U(x,y) - W(x,y)) \right| .$$

The cut-distance allows any permutations of the vertex sets:

Cut-norm and cut-distance

The cut norm compares the density of edges inside any vertex set:

$$d_{\square}(U,W) := \sup_{A \subset [0,1]} \left| \int_{A \times A} (U(x,y) - W(x,y)) \right|.$$

The cut-distance allows any permutations of the vertex sets:

$$\delta_{\square}(U,W) := \inf_{\varphi} d_{\square}(U, W^{\varphi})$$

Cut-norm and cut-distance

The cut norm compares the density of edges inside any vertex set:

$$d_{\square}(U,W) := \sup_{A\subset[0,1]} \left| \int_{A\times A} (U(x,y) - W(x,y)) \right|.$$

The cut-distance allows any permutations of the vertex sets:

$$\delta_{\square}(U,W) := \inf_{\varphi} d_{\square}(U, W^{\varphi})$$

where the infimum is taken over all measure preserving bijections $\varphi\colon [0,1] \to [0,1]$ and $W^{\varphi}(x,y) := W(\varphi(x),\varphi(y))$.

Compactness of the cut-distance

Recall the cut-distance:

$$\delta_\square(U, W) := \inf_{\varphi} d_\square(U, W^\varphi).$$

Compactness of the cut-distance

Recall the cut-distance:

$$\delta_\square(U, W) := \inf_\varphi d_\square(U, W^\varphi).$$

We say that two graphons are equivalent if their cut-distance is 0.

Compactness of the cut-distance

Recall the cut-distance:

$$\delta_\square(U, W) := \inf_{\varphi} d_\square(U, W^\varphi).$$

We say that two graphons are equivalent if their cut-distance is 0. Then $\delta_\square$ gives us a metric on the space of all equivalence classes.

Compactness of the cut-distance

Recall the cut-distance:

$$\delta_\square(U, W) := \inf_\varphi d_\square(U, W^\varphi).$$

We say that two graphons are equivalent if their cut-distance is 0. Then $\delta_\square$ gives us a metric on the space of all equivalence classes.

Theorem (Lovász & Szegedy, 2006)

The metric $\delta_\square$ on the equivalence classes of graphons is compact.

Proofs of the Lovász-Szegedy theorem

Known proofs of the Lovász-Szegedy theorem:

Proofs of the Lovász-Szegedy theorem

Known proofs of the Lovász-Szegedy theorem:

- Lovász & Szegedy, 2006: using Szemerédis regularity lemma

Proofs of the Lovász-Szegedy theorem

Known proofs of the Lovász-Szegedy theorem:

- Lovász & Szegedy, 2006: using Szemerédis regularity lemma
- Elek & Szegedy, 2012: using ultraproducts

Proofs of the Lovász-Szegedy theorem

Known proofs of the Lovász-Szegedy theorem:

- Lovász & Szegedy, 2006: using Szemerédis regularity lemma
- Elek & Szegedy, 2012: using ultraproducts
- Diaconis & Janson and (independently) Austin, 2008: using Aldous-Hoover theorem on exchangeable arrays (1981)

Proofs of the Lovász-Szegedy theorem

Known proofs of the Lovász-Szegedy theorem:

- Lovász & Szegedy, 2006: using Szemerédis regularity lemma
- Elek & Szegedy, 2012: using ultraproducts
- Diaconis & Janson and (independently) Austin, 2008: using Aldous-Hoover theorem on exchangeable arrays (1981)
- Our proof: using the weak* convergence

Proofs of the Lovász-Szegedy theorem

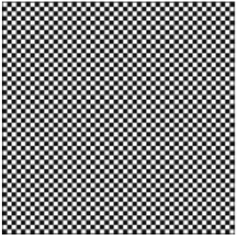
Known proofs of the Lovász-Szegedy theorem:

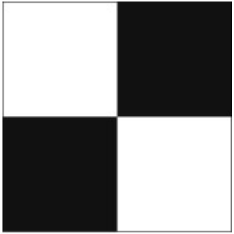
- Lovász & Szegedy, 2006: using Szemerédis regularity lemma
- Elek & Szegedy, 2012: using ultraproducts
- Diaconis & Janson and (independently) Austin, 2008: using Aldous-Hoover theorem on exchangeable arrays (1981)
- Our proof: using the weak* convergence

Definition

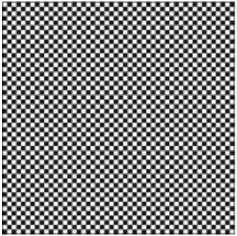
A sequence $(W_n)_n$ of graphons *weak* converges* to a graphon W if for every $A \subset [0,1]$ it holds $\lim_{n\to\infty} \int_{A\times A} W_n(x,y) = \int_{A\times A} W(x,y)$.

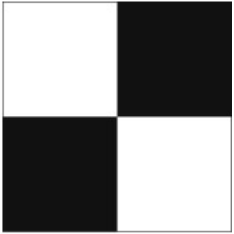
Basic example again





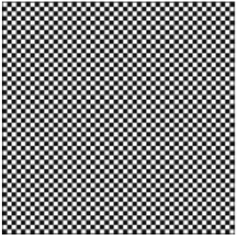
Basic example again

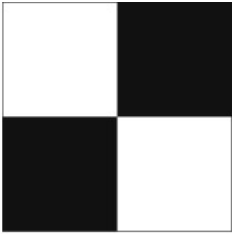




When $n \to \infty$ then the chessboards on the left weak* converge to the constant graphon $W \equiv \frac{1}{2}$

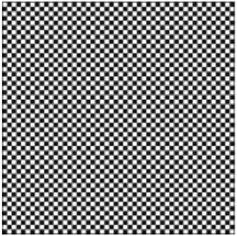
Basic example again

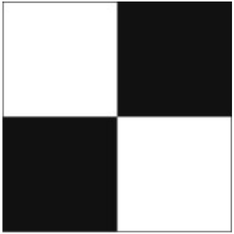




When $n \to \infty$ then the chessboards on the left weak* converge to the constant graphon $W \equiv \frac{1}{2}$ but not in the cut-distance!

Basic example again





When $n \to \infty$ then the chessboards on the left weak* converge to the constant graphon $W \equiv \frac{1}{2}$ but not in the cut-distance! The cut-distance limit exists as well but equals the graphon on the rigth!

Comparing the three convergence notions

$$W_n \stackrel{w*}{\to} W \;\Leftrightarrow\; \sup_{A\subset[0,1]} \lim_{n\to\infty} \left| \int_{A\times A} (W_n(x,y) - W(x,y)) \right| = 0$$

Comparing the three convergence notions

$$W_n \stackrel{w*}{\to} W \;\Leftrightarrow\; \sup_{A\subset[0,1]} \lim_{n\to\infty} \left| \int_{A\times A} (W_n(x,y) - W(x,y)) \right| = 0$$

$$W_n \stackrel{d_\square}{\to} W \;\Leftrightarrow\; \lim_{n\to\infty} \sup_{A\subset[0,1]} \left| \int_{A\times A} (W_n(x,y) - W(x,y)) \right| = 0$$

Comparing the three convergence notions

$$W_n \stackrel{w*}{\to} W \;\Leftrightarrow\; \sup_{A\subset[0,1]} \lim_{n\to\infty} \left| \int_{A\times A} (W_n(x,y) - W(x,y)) \right| = 0$$

$$W_n \stackrel{d_\square}{\to} W \;\Leftrightarrow\; \lim_{n\to\infty} \sup_{A\subset[0,1]} \left| \int_{A\times A} (W_n(x,y) - W(x,y)) \right| = 0$$

Therefore if $W_n \stackrel{d_\square}{\to} W$ then $W_n \stackrel{w*}{\to} W$ as well.

Comparing the three convergence notions

$$W_n \overset{w*}{\to} W \;\Leftrightarrow\; \sup_{A\subset[0,1]} \lim_{n\to\infty} \left| \int_{A\times A} (W_n(x,y) - W(x,y)) \right| = 0$$

$$W_n \overset{d_\square}{\to} W \;\Leftrightarrow\; \lim_{n\to\infty} \sup_{A\subset[0,1]} \left| \int_{A\times A} (W_n(x,y) - W(x,y)) \right| = 0$$

Therefore if $W_n \overset{d_\square}{\to} W$ then $W_n \overset{w*}{\to} W$ as well.

$W_n \overset{\delta_\square}{\to} W \;\Leftrightarrow\;$ there are measure preserving bijections $\varphi_n \colon [0,1] \to [0,1]$ such that $W_n^{\varphi_n} \overset{d_\square}{\to} W$

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons.

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find a cut-distance accumulation point W of $(W_n)_n$.

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find a cut-distance accumulation point W of $(W_n)_n$. We already know that for every such W there are measure preserving bijections $\varphi_n\colon [0,1] \to [0,1]$ such that W is a weak* accumulation point of $(W_n^{\varphi_n})_n$.

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find a cut-distance accumulation point W of $(W_n)_n$. We already know that for every such W there are measure preserving bijections $\varphi_n \colon [0,1] \to [0,1]$ such that W is a weak* accumulation point of $(W_n^{\varphi_n})_n$.

$$\mathrm{ACC}_{w*}((W_n)_n) := \{W\colon \text{ there are measure preserving bijections } \varphi_n \colon [0,1] \to [0,1] \text{ such that } W \text{ is a weak* accumulation point of } (W_n^{\varphi_n})_n\}$$

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find a cut-distance accumulation point W of $(W_n)_n$. We already know that for every such W there are measure preserving bijections $\varphi_n\colon [0,1] \to [0,1]$ such that W is a weak* accumulation point of $(W_n^{\varphi_n})_n$.

$$\begin{aligned}\mathrm{ACC}_{w*}((W_n)_n) := \{W\colon \;& \text{there are measure preserving} \\ & \text{bijections } \varphi_n\colon [0,1] \to [0,1] \text{ such that } W \\ & \text{is a weak* accumulation point of } (W_n^{\varphi_n})_n\}\end{aligned}$$

Note that $\mathrm{ACC}_{w*}((W_n)_n)$ is nonempty by Banach-Alaoglu theorem.

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find a cut-distance accumulation point W of $(W_n)_n$. We already know that for every such W there are measure preserving bijections $\varphi_n \colon [0,1] \to [0,1]$ such that W is a weak* accumulation point of $(W_n^{\varphi_n})_n$.

$$\mathrm{ACC}_{w*}((W_n)_n) := \{W\colon \text{ there are measure preserving bijections } \varphi_n \colon [0,1] \to [0,1] \text{ such that } W \text{ is a weak* accumulation point of } (W_n^{\varphi_n})_n\}$$

Note that $\mathrm{ACC}_{w*}((W_n)_n)$ is nonempty by Banach-Alaoglu theorem. We want to take the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$ and prove that it is a cut-distance accumulation point of $(W_n)_n$.

Our proof of compactness

But the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$ does not need to exist!

Our proof of compactness

But the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$ does not need to exist!

Recall that

$$\mathrm{ACC}_{w*}((W_n)_n) = \{W\colon \text{ there are measure preserving bijections } \varphi_n\colon [0,1] \to [0,1] \text{ such that } W \text{ is a weak* accumulation point of } (W_n^{\varphi_n})_n\}$$

Our proof of compactness

But the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$ does not need to exist!

Recall that

$$\mathrm{ACC}_{w*}((W_n)_n) = \{W\colon \text{ there are measure preserving bijections } \varphi_n\colon [0,1]\to[0,1] \text{ such that } W \text{ is a weak* accumulation point of } (W_n^{\varphi_n})_n\}$$

and define

$$\mathrm{LIM}_{w*}((W_n)_n) := \{W\colon \text{ there are measure preserving bijections } \varphi_n\colon [0,1]\to[0,1] \text{ such that } W \text{ is a weak* limit of } (W_n^{\varphi_n})_n\}.$$

Our proof of compactness

But the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$ does not need to exist!

Recall that

$$\mathrm{ACC}_{w*}((W_n)_n) = \{W\colon \text{ there are measure preserving bijections } \varphi_n\colon [0,1] \to [0,1] \text{ such that } W \text{ is a weak* accumulation point of } (W_n^{\varphi_n})_n\}$$

and define

$$\mathrm{LIM}_{w*}((W_n)_n) := \{W\colon \text{ there are measure preserving bijections } \varphi_n\colon [0,1] \to [0,1] \text{ such that } W \text{ is a weak* limit of } (W_n^{\varphi_n})_n\}.$$

Unfortunately, $\mathrm{LIM}_{w*}((W_n)_n)$ can be empty.

Our proof of compactness

Key Theorem A

For every sequence $(W_n)_n$ *of graphons there is a subsequence* $(W_{n_k})_k$ *of* $(W_n)_n$ *such that*

$$\mathrm{ACC}_{w*}((W_{n_k})_k) = \mathrm{LIM}_{w*}((W_{n_k})_k).$$

Our proof of compactness

Key Theorem A

For every sequence $(W_n)_n$ *of graphons there is a subsequence* $(W_{n_k})_k$ *of* $(W_n)_n$ *such that*

$$\mathrm{ACC}_{w*}((W_{n_k})_k) = \mathrm{LIM}_{w*}((W_{n_k})_k).$$

Key Theorem B

For every sequence $(W_n)_n$ *of graphons the following conditions are equivalent:*

Our proof of compactness

Key Theorem A

For every sequence $(W_n)_n$ *of graphons there is a subsequence* $(W_{n_k})_k$ *of* $(W_n)_n$ *such that*

$$\mathrm{ACC}_{w*}((W_{n_k})_k) = \mathrm{LIM}_{w*}((W_{n_k})_k).$$

Key Theorem B

For every sequence $(W_n)_n$ *of graphons the following conditions are equivalent:*

- $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$,

Our proof of compactness

Key Theorem A

For every sequence $(W_n)_n$ *of graphons there is a subsequence* $(W_{n_k})_k$ *of* $(W_n)_n$ *such that*

$$\mathrm{ACC}_{w*}((W_{n_k})_k) = \mathrm{LIM}_{w*}((W_{n_k})_k).$$

Key Theorem B

For every sequence $(W_n)_n$ *of graphons the following conditions are equivalent:*

- $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$,
- $(W_n)_n$ *is cut-distance Cauchy.*

Our proof of compactness

Key Theorem A

For every sequence $(W_n)_n$ of graphons there is a subsequence $(W_{n_k})_k$ of $(W_n)_n$ such that

$$\mathrm{ACC}_{w*}((W_{n_k})_k) = \mathrm{LIM}_{w*}((W_{n_k})_k).$$

Key Theorem B

For every sequence $(W_n)_n$ of graphons the following conditions are equivalent:

- $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$,
- $(W_n)_n$ *is cut-distance Cauchy.*

If one of these conditions holds then $(W_n)_n$ converges in the cut-distance to the most structured element of $\mathrm{ACC}_{w*}((W_n)_n)$.

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$?

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$?

For every graphon W we define the envelope of W as $\langle W \rangle := \mathrm{LIM}_{w*}((W)_n)$.

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$?

For every graphon W we define the envelope of W as $\langle W \rangle := \mathrm{LIM}_{w*}((W)_n)$.

We say that U is at most as structured as W, $U \preceq W$, if $\langle U \rangle \subset \langle W \rangle$.

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$?

For every graphon W we define the envelope of W as $\langle W \rangle := \mathrm{LIM}_{w*}((W)_n)$.

We say that U is at most as structured as W, $U \preceq W$, if $\langle U \rangle \subset \langle W \rangle$.

It turns out that the mapping $W \mapsto \langle W \rangle$ is a homeomorphism of $(\mathcal{W}, \delta_\square)$ onto a closed subset of the hyperspace of all weak* compact subsets of graphons.

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$?

For every graphon W we define the envelope of W as $\langle W\rangle := \mathrm{LIM}_{w*}((W)_n)$.

We say that U is at most as structured as W, $U \preceq W$, if $\langle U\rangle \subset \langle W\rangle$.

It turns out that the mapping $W \mapsto \langle W\rangle$ is a homeomorphism of $(\mathcal{W}, \delta_\square)$ onto a closed subset of the hyperspace of all weak* compact subsets of graphons. As the hyperspace is compact, $(\mathcal{W}, \delta_\square)$ is compact as well.

How to find the most structured graphons?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.

How to find the most structured graphons?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$. Is there an easy way to tell which $W \in \mathrm{ACC}_{w*}((W_n)_n)$ is the most structured element of $\mathrm{ACC}_{w*}((W_n)_n)$?

How to find the most structured graphons?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$. Is there an easy way to tell which $W \in \mathrm{ACC}_{w*}((W_n)_n)$ is the most structured element of $\mathrm{ACC}_{w*}((W_n)_n)$?

Yes!

How to find the most structured graphons?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$. Is there an easy way to tell which $W \in \mathrm{ACC}_{w*}((W_n)_n)$ is the most structured element of $\mathrm{ACC}_{w*}((W_n)_n)$?

Yes!

Fix an arbitrary strictly concave function $f\colon [0,1] \to \mathbb{R}$.

How to find the most structured graphons?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$. Is there an easy way to tell which $W \in \mathrm{ACC}_{w*}((W_n)_n)$ is the most structured element of $\mathrm{ACC}_{w*}((W_n)_n)$?

Yes!

Fix an arbitrary strictly concave function $f\colon [0,1] \to \mathbb{R}$. Define

$$\mathrm{INT}_f(W) := \int_{[0,1]^2} f(W(x,y)).$$

How to find the most structured graphons?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$. Is there an easy way to tell which $W \in \mathrm{ACC}_{w*}((W_n)_n)$ is the most structured element of $\mathrm{ACC}_{w*}((W_n)_n)$?

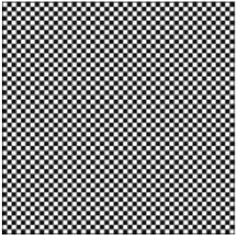
Yes!

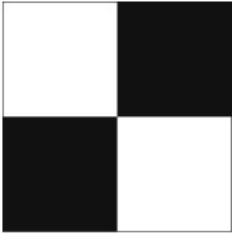
Fix an arbitrary strictly concave function $f\colon [0,1] \to \mathbb{R}$. Define

$$\mathrm{INT}_f(W) := \int_{[0,1]^2} f(W(x,y)).$$

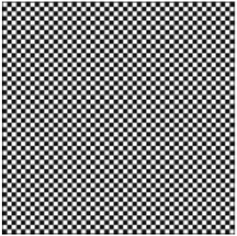
Then the most structured W is that one which minimizes INT_f.

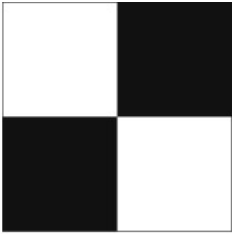
Basic example once more





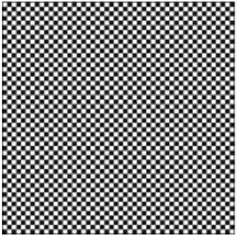
Basic example once more

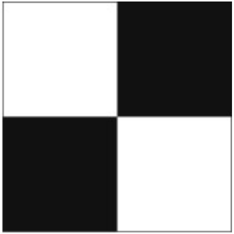




Let (W_n) be the sequence of the chessboards on the left.

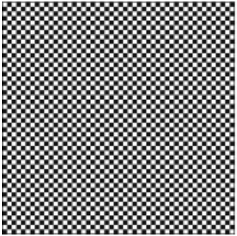
Basic example once more

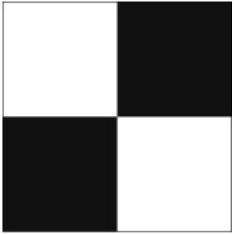




Let (W_n) be the sequence of the chessboards on the left. Then $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.

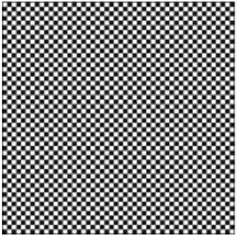
Basic example once more

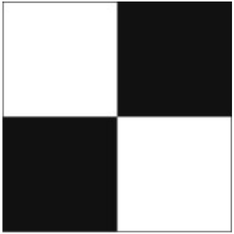




Let (W_n) be the sequence of the chessboards on the left. Then $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$. The constant graphon $W \equiv \frac{1}{2}$ and the graphon U on the rigth are both elements of $\mathrm{ACC}_{w*}((W_n)_n)$.

Basic example once more





Let (W_n) be the sequence of the chessboards on the left. Then $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$. The constant graphon $W \equiv \frac{1}{2}$ and the graphon U on the rigth are both elements of $\mathrm{ACC}_{w*}((W_n)_n)$. The graphon U on the right is more structured than the constant graphon $W \equiv \frac{1}{2}$ as

$$\mathrm{INT}_f(W) = f\left(\tfrac{1}{2}\right) > \tfrac{1}{2}\left(f(0) + f(1)\right) = \mathrm{INT}_f(U).$$

Thank you for your attention!