On the problem of singular limit of the Navier-Stokes-Fourier system coupled with radiation or with electromagnetic field

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Univ. of Ohio, 2015

- Diffusion limits in a model of radiative flow (B. Ducomet, S.N.)
- Low Mach number limit in a model of radiative flow (B. Ducomet, S.N.)
- Low Mach number limit for a model of accretion disk (D.Donatelli, B.Ducomet,S.N.)
- Inviscid incompressible limits on expanding domains (E.Feireisl, S.N., Y.Sun)

General questions

Compressible vs. incompressible

Is air compressible? Is it important?
Is the physical space bounded or unbounded?

Viscous vs. inviscid

What is turbulence?
Do extremely viscous fluids exhibit turbulent behavior?

Effect of rotation

Does it matter that the Earth rotate?
Is the rotation fast or slow? Is it important?

The scaling effect

Characteristic values and scaling

$$X \approx \frac{X}{X_{\mathrm{char}}}$$

Scaling of derivatives

$$\partial_t \approx \frac{1}{T_{\mathrm{char}}} \partial_t$$

$$\partial_x \approx \frac{1}{L_{\mathrm{char}}} \partial_x$$

Gaseous stars in astrophysics

The effect of coupling between the macroscopic description of the fluid and the statistical character of the motion of the massless photons.

Thermostatic variable

- **mass density** $\varrho = \varrho(t, x)$

Motion

- **macroscopic velocity** $\mathbf{u} = \mathbf{u}(t, x)$

(compressible) Navier-Stokes system

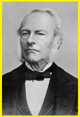
Claude Louis Marie Henri Navier [1785-1836]

Mass conservation

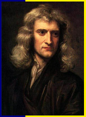
$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}$$



George Gabriel Stokes [1819-1903]



Isaac Newton [1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Compressible Navier-Stokes system with radiation

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Equation of motion

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mu \Delta_x \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u} + \vec{S}_F$$

Radiative transfer equation

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S,\ \ c \text{ is speed of light}$$

$$S = \sigma_a(B - I) + \sigma_s\left(\frac{1}{4\pi}\int_{S^2} I \,\mathrm{d}\vec{\omega} - I\right)$$

$$\vec{S}_F = (\sigma_a + \sigma_s)\int_0^\infty \int_{S^2} \vec{\omega} I \,\mathrm{d}\vec{\omega}\,\mathrm{d}\nu.$$

Radiative intensity

the radiative intensity $I = I(t, x, \omega, \nu)$, depending on the direction $\omega \in \mathcal{S}^2$,
$\mathcal{S}^2 \subset \mathbb{R}^3$ the unit sphere,
the frequency $\nu \geq 0$.

Hypothesis

- **Isotropy.** The coefficients σ_a, σ_s are independent of $\vec{\omega}$.
- **Grey hypothesis** The coefficients σ_a, σ_s are independent of ν.

$B = B(\nu, \varrho)$ measures the departure from equilibrium

is a barotropic equivalent of the Planck function

b the frequency average of $B(\nu, \varrho)$

$$b(\varrho) := \int_0^\infty B(\nu, \varrho)\, d\nu.$$

boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0$$

$$I(t, x, \vec{\omega}, \nu) = 0 \text{ for } (x, \vec{\omega}) \in \left\{ (x, \vec{\omega}) \;\middle|\; (x, \vec{\omega}) \in \partial\Omega \times S^2,\ \vec{\omega} \cdot \vec{n} \leq 0 \right\},$$

Scaled equations

Scaling

$$X \approx \frac{X}{X_{\mathrm{char}}}$$

Mass conservation

$$[\mathrm{Sr}]\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$[\mathrm{Sr}]\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \left[\frac{1}{\mathrm{Ma}^2}\right] \nabla_x p(\varrho)$$

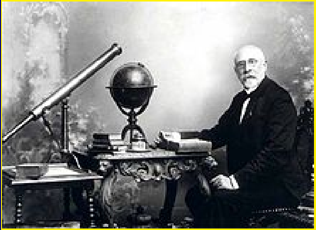
$$= \left[\frac{1}{\mathrm{Re}}\right] (\Delta_x \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}) + (\text{external forces})$$

Transport of radiative intensity

$$\frac{Sr}{\mathcal{C}}\partial_t I + \omega \cdot \nabla_x I = S =$$

$$= \mathcal{L}\sigma_a (B - I) + \mathcal{L}\mathcal{L}_s\sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I \,\mathrm{d}\omega - I \right).$$

Characteristic numbers - Strouhal number



Čeněk Strouhal
[1850-1922]

Strouhal number

$$[\mathrm{Sr}] = \frac{\mathrm{length}_{\mathrm{char}}}{\mathrm{time}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}}$$

Scaling by means of Strouhal number is used in the study of the long-time behavior of the fluid system, where the characteristic time is large

Mach number

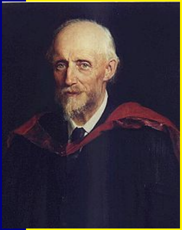
Ernst Mach [1838-1916]

Mach number

$$[\mathrm{Ma}] = \frac{\mathrm{velocity}_{\mathrm{char}}}{\sqrt{\mathrm{pressure}_{\mathrm{char}}/\mathrm{density}_{\mathrm{char}}}}$$

Mach number is the ratio of the characteristic speed to the speed of sound in the fluid. Low Mach number limit, where, formally, the speed of sound is becoming infinite, characterizes incompressibility

Reynolds number



Osborne Reynolds
[1842-1912]

Reynolds number

$$[\mathrm{Re}] = \frac{\mathrm{density}_{\mathrm{char}}\,\mathrm{velocity}_{\mathrm{char}}\,\mathrm{length}_{\mathrm{char}}}{\mathrm{viscosity}_{\mathrm{char}}}$$

High Reynolds number is attributed to turbulent flows, where the viscosity of the fluid is negligible

Radiation dimensionless numbers

$$\mathcal{C} = \frac{c}{U_{ref}}$$

$$\mathcal{L} = L_{ref}\,\sigma_{a,ref}\,,\;\; \mathcal{L}_s = \frac{\sigma_{s,ref}}{\sigma_{a,ref}}$$

$$\mathcal{P} = \frac{L_{ref}\,\nu_{ref}\,S_{ref}}{c\;\;\rho_{ref}\,U_{ref}^2},$$

Target system

Incompressible limit

Low Mach number $\Rightarrow$ compressible $\rightarrow$ incompressible

Fast rotation

Low Rossby number $\Rightarrow$ 3D motion $\rightarrow$ 2D motion

Inviscid limit

High Reynolds number $\Rightarrow$ viscous flow $\rightarrow$ inviscid flow

Diffusion limit

compressible Navier-Stokes system with radiation $\rightarrow$ compressible Navier-Stokes system with diffusion

Diffusion limit

Mass conservation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) +$$
$$\left(\varepsilon \sigma_a + \frac{1}{\varepsilon} \sigma_s \right) \int_0^\infty \int_{\mathcal{S}^2} \omega I \ d\omega \ d\nu.$$

Transport of radiative intensity

$$\varepsilon \partial_t I + \omega \cdot \nabla_x I = \varepsilon \sigma_a \left(B - I \right) + \frac{1}{\varepsilon} \sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I \, \mathrm{d}\omega - I \right)$$

Asymptotic limit(formal)

$$c \approx \frac{1}{\varepsilon}, \ \sigma_a \approx \varepsilon \sigma_a(\varrho), \ \sigma_s \approx \frac{1}{\varepsilon} \sigma_s(\varrho),$$

Limit system

Continuity equation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \left(p(\varrho) + \frac{1}{3} N \right) = \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla_x \operatorname{div}_x \mathbf{u}$$

Diffusion equation

$$\partial_t N - \operatorname{div}_x\left(\frac{1}{\sigma_s(\varrho)} \nabla_x N\right) = \sigma_a(\varrho)(b(\varrho) - N),$$

$$b(\varrho) = \int_0^\infty B(\varrho, \nu) \, \mathrm{d}\nu$$

Preparing the initial data

Ill prepared initial data

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}$$

$$\left\{ \varrho^{(1)}_{0,\varepsilon} \right\}_{\varepsilon>0} \text{ bounded in } L^2 \cap L^\infty$$

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2$$

Well prepared initial data

$$\varrho^{(1)}_{0,\varepsilon} \to 0 \text{ in } L^2 \text{ as } \varepsilon \to 0$$

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2 \text{ as } \varepsilon \to 0, \ \mathrm{div}_x \mathbf{u}_0 = 0$$

Fundamental issues

Solvability of the primitive system

The primitive system should admit (global) in time solutions for any choice of the scaling parameters and any admissible initial data

Solvability of the target system

The target system should admit solutions, at least locally in time; the solutions are regular

Stability

The family of solutions to the primitive system should be stable with respect to the scaling parameters

Assumptions

Assumption on the pressure

- p is a C^1 function on $[0,\infty)$ such that $p(0)=0$,
- $p'(\rho)>0$ for all $\rho>0$, such that

$$\frac{p'(z)}{\rho^{\gamma-1}} = p_\infty > 0,\ \gamma > \frac{3}{2}.$$

Assumptions on radiative quantities

$$0 \le \sigma_s(\varrho),\ \ \sigma_a(\varrho) \le c_1,$$

$$\sigma_a(\varrho)B^m(\nu,\varrho) \le h(\nu),\ \ h \in L^1(0,\infty) \ \ \text{for } m = 1, 2,$$

for any $\varrho \ge 0$.

Weak formulation

Renormalized continuity equation

$$\int_0^T \int_\Omega \left(\Big(\varrho + \beta(\varrho) \Big) \partial_t \psi \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \left(\Big(\varrho + \beta(\varrho) \Big) \mathbf{u} \cdot \nabla_x \psi + \Big(\beta(\varrho) - \beta'(\varrho)\varrho \Big) \mathrm{div}_x \mathbf{u} \psi \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$= - \int_\Omega \Big(\varrho_0 + \beta(\varrho_0) \Big) \psi(0, \cdot) \ \mathrm{d}x$$

satisfied for any $\psi \in C_c^\infty([0,\infty) \times \overline{\Omega})$, and any $\beta \in C^\infty[0,\infty)$, $\beta' \in C_c^\infty[0,\infty)$.

The momentum equation

$$\int_\Omega \varrho \mathbf{u}(\tau,\cdot)\phi(\tau,\cdot)\ dx - \int_\Omega \varrho_0 \mathbf{u}_0 \phi(0,\cdot)\ dx$$

$$= \int_0^\tau \int_\Omega \varrho \mathbf{u} \cdot \partial_t \phi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \phi + p \mathrm{div}_x \phi - \mathbb{S} : \nabla_x \phi - \vec{S}_F \cdot \phi \ dx\ dt,$$

for any $\phi \in C^1([0,T] \times \overline{\Omega}; \mathbb{R}^3)$ with $\phi|_{\partial\Omega} = 0$, any $\tau \in [0,T]$.

Definition of weak solution of primitive system

Weak solution

- the density ϱ is a non negative measurable function,
-
$$\rho \in C_{\mathrm{weak}}(0, T; L^{\gamma}(\Omega))$$
-
$$\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega)),$$
-
$$\varrho\mathbf{u} \in C_{\mathrm{weak}}(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^3)),$$
-
$$p \in L^1((0, T) \times \Omega),$$
-
$$I \in L^{\infty}((0, T) \times \Omega \times \mathcal{S}^2 \times (0, \infty)),$$
-
$$I \in L^{\infty}(0, T; L^1(\Omega \times \mathcal{S}^2 \times (0, \infty))),$$

Existence of primitive system

Existence of weak solution

- $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu > 0$.
- Assumptions on p, the transport coefficients σ_a, σ_s and the equilibrium function B are satisfied
- Let $(\varrho, , \mathbf{u}, I)$ be a weak solution to radiative Navier-Stokes system for $(t, x) \in [0, T] \times \Omega$, and $(\omega, \nu) \in \mathcal{S}^2 \times R_+$ in the sense of previous definition

Class of regularity of primitive system

- the density ϱ is a non negative measurable function,

$$\rho \in C_{\mathrm{weak}}(0, T; L^{\gamma}(\Omega))$$

- $$\mathbf{u} \in L^2(0, T; W^{1,2}(\Omega)),$$

- $$\varrho \mathbf{u} \in C_{\mathrm{weak}}(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^3)),$$

- $$p \in L^1((0, T) \times \Omega),$$

- $$I \in L^{\infty}((0, T) \times \Omega \times \mathcal{S}^2 \times (0, \infty)),$$

- $$I \in L^{\infty}(0, T; L^1(\Omega \times \mathcal{S}^2 \times (0, \infty))),$$

A finite energy weak solution

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2(\tau) + \Pi(\varrho)(\tau) + E^R(\tau)\right]\ dx$$

$$+\int_0^\tau \int_\Omega \left[\mu|\nabla_x \mathbf{u}|^2 + (\lambda+\mu)|\mathrm{div}_x\ \mathbf{u}|^2\right]\ dx\ \mathrm{d}t$$

$$\leq \int_\Omega \left[\frac{1}{2}\frac{|\vec{q}|^2}{\varrho_0} + \Pi(\varrho_0) + E_0^R\right]\ dx\ \mathrm{d}t + \int_0^T \int_\Omega S_F \mathbf{u}$$

,

$$E^R(t,x) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(t,x,\omega,\nu)\ \mathrm{d}\omega\ \mathrm{d}\nu$$

$$\Pi(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\ dz$$

for a.e. $\tau \in (0,T)$.

satisfying the integral identities for the continuity equation and the momentum equations and the transport of radiative intensity

Global weak solutions

Barotropic case

P. L. Lions (98)

$p(\varrho) = \varrho^{\gamma}$, $\gamma \geq 9/5$

Generalization to a larger class of exponents $\gamma > 3/2$

E. Feireisl, A. Novotný and H. Petzeltová

Stability result

Main Theorem(B.Ducomet, Š.N.)

- $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$
- Assumptions on p,radiative quantities
- $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, I_\varepsilon)$ be a weak solution of rescaled system of equations
- $$\varrho_{0,\varepsilon} \to \varrho_0 \text{ in } L^\gamma(\Omega),$$
- $$\int_\Omega \frac{(\varrho \mathbf{u})_{0,\varepsilon}}{\varrho_{0,\varepsilon}} \, \mathrm{d}x \leq c,$$
- $$|I_{0,\varepsilon}(\cdot,\nu)| \leq h(\nu), \;\; h \in L^1 \cap L^\infty(0,\infty).$$

Convergence

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \text{ in } C([0,T]; L^1(\Omega)) \text{ and in } C_{\mathrm{weak}}([0,T]; L^\gamma(\Omega)),$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)),$$

$$I_\varepsilon \to I \text{ weakly in } {*}L^\infty(0,T; \Omega \times \mathcal{S}^2 \times (0,\infty))$$

Limit system

where $\varrho, \mathbf{u}, I$ is a weak solution satisfying

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \left(p(\varrho) + \frac{1}{3} N \right) = \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla_x \mathrm{div}_x \mathbf{u}$$

$$\partial_t N - \mathrm{div}_x \left(\frac{1}{\sigma_s(\varrho)} \nabla_x N \right) = \sigma_a(\varrho)(b(\varrho) - N), \; b(\varrho) = \int_0^\infty B(\varrho, \nu) \, \mathrm{d}\nu.$$

- **Pomraning**
- **Mihalas and Weibel-Mihalas** in the framework of special relativity.
- astrophysics, laser applications (in the relativistic and inviscid case) by **Lowrie, Morel and Hittinger, Buet and Després**
- with a special attention to asymptotic regimes **Dubroca and Feugeas, Lin, Lin, Coulombel and Goudon**
  a simplified version of the system (non conducting fluid at rest) - investigated by **Golse and Perthame** , where global existence was proved by means of the theory of nonlinear semi-groups under very general hypotheses.

Full system

The continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \quad \text{in } (0,T) \times \Omega,$$

The momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \mathbf{S}_F \qquad \text{in } (0,T) \times \Omega,$$

The energy equation

$$\partial_t \left(\varrho \left(\frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \right) + \mathrm{div}_x \left(\varrho \left(\frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \mathbf{u} \right)$$

$$+ \mathrm{div}_x \Big(p\mathbf{u} + \mathbf{q} - \mathbb{S}\mathbf{u} \Big) = -S_E \quad \text{in } (0,T) \times \Omega,$$

The radiative intensity

$$\frac{1}{c}\partial_t I + \omega \cdot \nabla_x I = S \quad \text{in } (0,T) \times \Omega \times (0,\infty) \times \mathcal{S}^2.$$

Sources

$S := S_{a,e} + S_s$, where

$$S_{a,e} = \sigma_a \Big(B(\nu, \vartheta) - I \Big), \; S_s = \sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I(\cdot, \omega) \, \mathrm{d}\omega - I \right),$$

$$S_E = \int_{\mathcal{S}^2} \int_0^\infty S(\cdot, \nu, \omega) \, \mathrm{d}\nu \, \mathrm{d}\omega,$$

$$\vec{S}_F(t,x) = \frac{1}{c} \int_0^\infty \int_{\mathcal{S}^2} \omega S \, \mathrm{d}\omega \, \mathrm{d}\nu,$$

Full system

Maxwell's equation

$$\frac{\partial e}{\partial \varrho} = \frac{1}{\varrho^2}\left(p(\varrho, \vartheta) - \vartheta \frac{\partial p}{\partial \vartheta} \right).$$

Stress tensor

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right) + \eta \, \mathrm{div}_x \mathbf{u} \; \mathbb{I},$$

the shear viscosity coefficient $\mu = \mu(\vartheta) > 0$ the bulk viscosity coefficient $\eta = \eta(\vartheta) \geq 0$ are effective functions of the temperature

Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta,$$

the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$

Radiation quantities

the absorption coefficient $\sigma_a = \sigma_s(\nu, \vartheta) \geq 0$,
the scattering coefficient $\sigma_s = \sigma_s(\nu, \vartheta) \geq 0$

$$B(\nu, \vartheta) = 2h\nu^3 c^{-2} \left(e^{\frac{h\nu}{k\vartheta}} - 1 \right)^{-1}$$

– the radiative equilibrium function
h and k are the Planck and Boltzmann constants,

Hypothesis on pressure

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ a > 0,$$

$P : [0, \infty) \to [0, \infty)$

$$P \in C^1[0, \infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0,$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0,$$

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0.$$

$\frac{a}{3}\vartheta^4$- “equilibrium” radiation pressure.

Assumptions on viscosities

$$0 < c_1(1+\vartheta) \le \mu(\vartheta),\ \ \mu'(\vartheta) < c_2,\ \ 0 \le \eta(\vartheta) \le c(1+\vartheta),$$

Assumption on heat conductivity coefficient

$$0 < c_1(1+\vartheta^3) \le \kappa(\vartheta) \le c_2(1+\vartheta^3)$$

for any $\vartheta \ge 0$.

Assumptions on radiation quantities

$$0 \le \sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), |\partial_\vartheta \sigma_a(\nu,\vartheta)|, |\partial_\vartheta \sigma_s(\nu,\vartheta)| \le c_1,$$

$$0 \le \sigma_a(\nu,\vartheta) B(\nu,\vartheta), |\partial_\vartheta \{\sigma_a(\nu,\vartheta) B(\nu,\vartheta)\}| \le c_2,$$

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), \sigma_a(\nu,\vartheta) B(\nu,\vartheta) \le h(\nu),\ \ h \in L^1(0,\infty).$$

Weak formulation of renormalized continuity equation

$$\int_0^T \int_\Omega \left(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi\right)\, \mathrm{d}x\, \mathrm{d}t$$

$$+\int_0^T \int_\Omega \left(\left(b(\varrho) - b'(\varrho)\varrho\right)\mathrm{div}_x\mathbf{u}\varphi\right)\, \mathrm{d}x\mathrm{d}t$$

$$= -\int_\Omega b(\varrho_0)\varphi(0,\cdot)\, \mathrm{d}x$$

satisfied for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, and any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$

Weak formulation of the momentum equation

$$\int_0^T \int_\Omega \left(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p \operatorname{div}_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega \mathbb{S} : \nabla_x \varphi + \mathbf{S}_F \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\varrho \mathbf{u})_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0, T) \times \Omega; \mathbb{R}^3)$.

$$\mathbf{u} \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3)),$$

Entropy inequality

$$\int_0^T \int_\Omega \left(\varrho s \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq - \int_\Omega (\varrho s)_0 \varphi(0, \cdot) \, \mathrm{d}x$$

$$- \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^T \int_\Omega \frac{1}{\vartheta} \Big(\mathbf{u} \cdot \mathbf{S}_F - S_E \Big) \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0, T) \times \overline{\Omega})$, $\varphi \geq 0$.

The total energy balance

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) + E^R \right)(\tau,\cdot)\ \mathrm{d}x$$

$$+ \int_0^\tau \int\int_{\partial\Omega\times\mathcal{S}^2,\ \omega\cdot\mathbf{n}\geq 0} \int_0^\infty \omega\cdot\mathbf{n} I(t,x,\omega,\nu)\ \mathrm{d}\nu\ \mathrm{d}\omega\ \mathrm{d}S_x\ \mathrm{d}t$$

$$= \int_\Omega \left(\frac{1}{2\varrho_0}|(\varrho\mathbf{u})_0|^2 + (\varrho e)_0 + E_{R,0} \right)\ \mathrm{d}x,$$

$$E^R(t,x) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(t,x,\omega,\nu)\ \mathrm{d}\omega\ \mathrm{d}\nu.$$

$$E_{R,0} = \int_{\mathcal{S}^2}\int_0^\infty I_0(\cdot,\omega,\nu)\ \mathrm{d}\omega\ \mathrm{d}\nu$$

Definition of weak solution

We say that $\varrho, \mathbf{u}, \vartheta, I$ *is a weak solution of problem if*

$$\varrho \geq 0,\ \vartheta > 0 \text{ for a.a. } (t,x)\times\Omega,\ I \geq 0 \text{ a.a. in } (0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty),$$

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega)),\ \vartheta \in L^\infty(0,T;L^4(\Omega)),$$

$$\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),\ \vartheta \in L^2(0,T;W^{1,2}(\Omega)),$$

$$I \in L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),\ I(t,\cdot) \in L^\infty(0,T;L^1(\Omega\times\mathcal{S}^2\times(0,\infty))),$$

and if ϱ, $\mathbf{u}$, ϑ, I *satisfy their weak formulation, together with the transport equation .*

Full system - the Navier - Stokes - Fourier system

- Global existence of weak solution E. Feireisl et al.
- Singular limits of full system for the Navier type of boundary conditions-E. Feireisl, A. Novotný
- Concept of weak- strong uniqueness- E. Feireisl, A. Novotný, Y. Sun, B. J. Jin

Theorem(stability)

(B. Ducomet, E.Feireisl,Š. N.) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s,the transport coefficients μ, λ, κ, σ_a, and σ_s satisfy the hypothesis. Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to our problem in the sense of Definition of weak solution such that

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \text{ in } L^{5/3}(\Omega),$$

$$\int_\Omega \left(\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) + E_{R,\varepsilon} \right)(0,\cdot) \, \mathrm{d}x$$

$$\equiv \int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}} |(\varrho \mathbf{u})_{0,\varepsilon}|^2 + (\varrho e)_{0,\varepsilon} + E_{R,0,\varepsilon} \right) \, \mathrm{d}x \leq E_0,$$

$$\int_\Omega [\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) + s^R(I_\varepsilon)](0,\cdot) \, \mathrm{d}x \equiv \int_\Omega (\varrho s + s^R)_{0,\varepsilon} \, \mathrm{d}x \geq S_0,$$

and

$$0 \leq I_\varepsilon(0,\cdot) \equiv I_{0,\varepsilon}(\cdot) \leq I_0,\ |I_{0,\varepsilon}(\cdot,\nu)| \leq h(\nu) \text{ for a certain } h \in L^1(0,\infty).$$

Then

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T];L^{5/3}(\Omega)),$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)),$$

and

$$I_\varepsilon \to I \text{ weakly-(*) in } L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),$$

at least for suitable subsequences, where $\{\varrho,\mathbf{u},\vartheta,I\}$ is a weak solution of our problem.

Simplified model $\mathbf{S}_F = 0$

The entropy of a photon gas

$$s^R = -\frac{2k}{c^3}\int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1)\log(n+1)\right] d\omega d\nu,$$

$n = n(I) = \frac{c^2 I}{2h\nu^3}$ is the occupation number

The radiative entropy flux

$$\mathbf{q}^R = -\frac{2k}{c^2}\int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1)\log(n+1)\right] \omega \; d\omega d\nu,$$

Entropy

$$\partial_t s^R + \mathrm{div}_x \mathbf{q}^R = -\frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \log \frac{n}{n+1} \; S \; d\omega d\nu =: \varsigma^R.$$

"Total Entropy"

$$\partial_t \left(\varrho s + s^R\right) + \mathrm{div}_x \left(\varrho s \mathbf{u} + \mathbf{q}^R\right) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) = \varsigma + \varsigma^R.$$

$$\varsigma^R =:$$

$$\frac{S_E}{\vartheta} - \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1}\right] \sigma_a (B - I)\ d\omega d\nu$$

$$- \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1}\right] \sigma_s (\tilde{I} - I)\ d\omega d\nu,$$

$$\varsigma =: \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right) - \frac{S_E}{\vartheta},$$

$$L_{ref}\,,\;\; T_{ref}\,,\;\; U_{ref}\,, \rho_{ref}\,,\;\; \vartheta_{ref}\,,\;\; p_{ref}\,,\;\; e_{ref}\,,\;\; \mu_{ref}\,,\;\; \kappa_{ref}\,,$$

the reference hydrodynamical quantities (length, time, velocity, density, temperature, pressure, energy, viscosity, conductivity)

$$I_{ref}\,,\;\; \nu_{ref}\,,\;\; \sigma_{a,ref}\,,\;\; \sigma_{s,ref}\,,$$

the reference radiative quantities (radiative intensity, frequency, absorption and scattering coefficients).

$$Sr := \frac{L_{ref}}{T_{ref}\, U_{ref}}, \;\; Ma = \frac{U_{ref}}{\sqrt{\rho_{ref}\, p_{ref}}}, \;\; Re = \frac{U_{ref}\, \rho_{ref}\, L_{ref}}{\mu_{ref}}, \;\; Pe = \frac{U_{ref}\, p_{ref}\, L_{ref}}{\vartheta_{ref}\, \kappa_{ref}},$$

the Strouhal, Mach, Reynolds, Péclet (dimensionless) numbers corresponding to hydrodynamics, and by

$$\mathcal{C} = \frac{c}{U_{ref}}, \;\; \mathcal{L} = L_{ref}\sigma_{a,ref}\,, \;\; \mathcal{L}_s = \frac{\sigma_{s,ref}}{\sigma_{a,ref}}, \;\; \mathcal{P} = \frac{2k_B^4 \vartheta_{ref}^4}{h^3 c^3 \;\; \rho_{ref}\, e_{ref}},$$

various dimensionless numbers corresponding to radiation.

Diffusion limit

Equilibrium diffusion regime

$\mathcal{P} = O(\varepsilon)$- a small amount of radiation is present

$\mathcal{C} = O(\varepsilon^{-1})$ -the flow is strongly under-relativistic

$Ma = Sr = Pe = Re = 1,\ \mathcal{P} = \varepsilon,\ \mathcal{C} = \varepsilon^{-1},\ \mathcal{L}_s = \varepsilon^2$ and $\mathcal{L} = \varepsilon^{-1}$,

$$\varepsilon\ \partial_t I + \omega \cdot \nabla_x I = \frac{1}{\varepsilon}\sigma_a (B - I) + \varepsilon\sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I \ \mathrm{d}\omega - I \right),$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) - \ \mathrm{div}_x \mathbb{T} = 0.$$

Entropy inequality

$$\partial_t\left(\varrho s + \varepsilon s_R\right) + \mathrm{div}_x\left(\varrho \mathbf{u} s + \mathbf{q}_R\right) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

$$+\frac{1}{\varepsilon}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(I-B)\ d\omega d\nu$$

$$+\varepsilon\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(I-\tilde{I})\ d\omega d\nu,$$

$$\frac{d}{dt}\int_\Omega\left(\frac{1}{2}\ \varrho|\mathbf{u}|^2 + \varrho e + E_R\right)\ dx + \frac{1}{\varepsilon}\int_0^\infty\int_{\Gamma_+}\omega\cdot\mathbf{n}\ I\ d\Gamma_+ d\nu = 0.$$

The " non-equilibrium diffusion regime"

$Ma = Sr = Pe = Re = 1, \ \mathcal{P} = \varepsilon, \ \mathcal{C} = \varepsilon^{-1}, \ \mathcal{L} = \varepsilon^2$ and $\mathcal{L}_s = \varepsilon^{-1}$.

$$\varepsilon \ \partial_t I + \omega \cdot \nabla_x I = \varepsilon \sigma_a (B - I) + \frac{1}{\varepsilon} \sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I \ \mathrm{d}\omega - I \right),$$

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0,$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) - \ \mathrm{div}_x \mathbb{T} = 0.$$

$$\partial_t \left(\varrho s + \varepsilon s_R \right) + \mathrm{div}_x \left(\varrho \mathbf{u} s + \mathbf{q}_R \right) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$+ \varepsilon \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (I - B) \; d\omega d\nu$$

$$+ \frac{1}{\varepsilon} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (I - \tilde{I}) \; d\omega d\nu.$$

The limit system–equilibrium system

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

The momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T}(\varrho, \vartheta),$$

The energy equation

$$\partial_t \left(\varrho \mathcal{E}(\varrho, \vartheta)\right) + \mathrm{div}_x \left(\varrho e(\varrho, \vartheta)\mathbf{u}\right) + \mathrm{div}_x \left(\mathcal{K}(\varrho, \vartheta)\nabla_x \vartheta\right)$$
$$= \mathbb{S}(\varrho, \vartheta) : \nabla_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u},$$

Radiative transfer

$$I = B(\nu, \vartheta).$$

Boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0, \;\; \nabla\vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

Initial condition

$$(\varrho(x,t), \;\mathbf{u}(x,t), \;\vartheta(x,t))|_{t=0} = \left(\varrho^0(x), \;\mathbf{u}^0(x), \;\vartheta^0(x)\right),$$

The compatibility conditions

$$\mathbf{u}^0(x)|_{\partial\Omega} = 0, \;\; \nabla\vartheta^0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

$$\mathcal{E}(\varrho,\vartheta) = e(\varrho,\vartheta) + \frac{B(\vartheta)}{\varrho}, \text{ and } \mathcal{K}(\vartheta) = \kappa(\vartheta) - \frac{1}{3\sigma_a(\vartheta)}\ \partial_\vartheta B(\vartheta).$$

- existence of a global solution for the small data - Matsumura and Nishida

$(\overline{\varrho}, 0, \overline{\vartheta})$ be a given constant state with $\overline{\varrho} > 0$ and $\overline{\vartheta} > 0$.

$e_0 := \|\varrho_0 - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\mathbf{u}_0\|_{H^1(\Omega)} + \|\vartheta_0 - \overline{\vartheta}\|_{H^1(\Omega)} + \|\mathbb{T}_0\|_{L^2(\Omega)} + \|\mathbb{V}_0\|_{L^4(\Omega)}$,

where $\mathbb{V}_0$ is the initial vorticity $\mathbb{V}_{ij} = \partial_j u_i - \partial_i u_j)$,

$$E_0 := e_0 + \|\nabla_x \varrho_0\|_{L^2(\Omega)} + \|\nabla_x \varrho_0\|_{L^\alpha(\Omega)} + \|\nabla_x \mathbb{T}_0\|_{L^2(\Omega)},$$

for an arbitrary fixed α such that $3 < \alpha < 6$.

The non-equilibrium diffusion regime, Navier-Stokes-Rosseland system

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

The momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T}(\varrho, \vartheta),$$

Energy equation

$$\partial_t \left(\varrho e(\varrho, \vartheta)\right) + \mathrm{div}_x \left(\varrho e(\varrho, \vartheta)\mathbf{u}\right) + \mathrm{div}_x \left(\kappa(\vartheta)\nabla_x \vartheta\right)$$
$$= \mathbb{S}(\varrho, \vartheta) : \nabla_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u} - \sigma_a(\vartheta)[B(\vartheta) - N],$$

Diffusion equation

$$\partial_t N - \frac{1}{3}\,\mathrm{div}_x\left(\frac{1}{\sigma_s(\vartheta)}\nabla_x N\right) = \sigma_a(\vartheta)\,(B(\vartheta) - N)\,.$$

Boundary equations

$$\mathbf{u}|_{\partial\Omega} = 0,\;\; \nabla\vartheta\cdot\mathbf{n}|_{\partial\Omega} = 0,$$

$$N := \int_0^\infty I_0\; d\nu$$

$$N|_{\partial\Omega} = 0.$$

Initial conditions

$$(\varrho(x,t),\;\mathbf{u}(x,t),\;\vartheta(x,t), N(x,t))|_{t=0} = \left(\varrho^0(x),\;\mathbf{u}^0(x),\;\vartheta^0(x), N^0(x)\right),$$

The compatibility conditions

$$\mathbf{u}^2|_{\partial\Omega} = 0, \ \nabla\vartheta^0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \ N^0\big|_{\partial\Omega} = 0.$$

$$N^0(x) = \int_0^\infty \int_{\mathcal{S}^2} I^0(x,\nu,\omega)\ d\omega\ d\nu$$

.

- Strong solution of the limit system for small data

Let $(\overline{\varrho}, 0, \overline{\vartheta}, \overline{N})$ be a given constant state with $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ and $\overline{N} = B(\overline{\vartheta})$.

$$e_0 := \|\varrho^0 - \overline{\varrho}\|_{L^\infty(\Omega)} + \|\mathbf{u}^0\|_{H^1(\Omega)} + \|\vartheta^0 - \overline{\vartheta}\|_{H^1(\Omega)} + \|N^0 - \overline{N}\|_{H^1(\Omega)}$$

$$+\|\mathbb{T}^0\|_{L^2(\Omega)} + \|\mathbb{V}^0\|_{L^4(\Omega)},$$

and

$$E_0 := e_0 + \|\nabla_x \varrho^0\|_{L^2(\Omega)} + \|\nabla_x \varrho^0\|_{L^\alpha(\Omega)} + \|\nabla_x \mathbb{T}^0\|_{L^2(\Omega)},$$

for an arbitrary fixed α such that $3 < \alpha < 6$.

Given three numbers $\overline{\varrho} \in \mathbb{R}_+$, $\overline{\vartheta} \in \mathbb{R}_+$ and $\overline{E} \in \mathbb{R}_+$ we define $\mathcal{O}^H_{ess}$ the set of hydrodynamical essential values essH
$\mathrm{O}^H_{ess} := \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 \;:\; \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho},\; \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta} \right\}$,O^R_{ess} the set of radiative essential values essR
$\mathrm{O}^R_{ess} := \left\{ E^R \in \mathbb{R} \;:\; \frac{\overline{E}}{2} < E^R < 2\overline{E} \right\}, with \mathrm{O}_{ess} := \mathcal{O}^H_{ess} \cup \mathcal{O}^R_{ess}$, and their residual counterparts res
$\mathrm{O}^H_{res} := (\mathbb{R}_+)^2 \backslash \mathcal{O}^H_{ess}, \quad \mathcal{O}^R_{res} := \mathbb{R}_+ \backslash \mathcal{O}^R_{ess}, \quad \mathcal{O}_{res} := (\mathbb{R}_+)^3 \backslash \mathcal{O}_{ess}.$

Theorem(Equilibrium case):

- $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$.
- The thermodynamic functions p, e, s satisfy hypotheses,
- $P \in C^1[0,\infty) \cap C^2(0,\infty)$,
- the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B satisfy hypothesis, $B \in C^1$.

Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system for $(t,x,\omega,\nu) \in [0,T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions and the initial conditions $(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$

such that

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \quad \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \varepsilon \vartheta^{(1)}_{0,\varepsilon},$$

where $(\varrho_0, \mathbf{u}, \vartheta_0) \in H^3(\Omega)$ are smooth functions such that (ϱ_0, ϑ_0) belong to the set $\mathcal{O}^H_{ess}$, where $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$, are two constants and $\int_\Omega \varrho^{(1)}_{0,\varepsilon} \ dx = 0$, $\int_\Omega \vartheta^{(1)}_{0,\varepsilon} \ dx = 0$.

Suppose also that

$$\mathbf{u}_{0,\varepsilon} \longrightarrow \mathbf{u}_0 \quad \textit{strongly in } L^\infty(\Omega; \mathbb{R}^3),$$

$$\varrho_{0,\varepsilon}^{(1)} \longrightarrow \varrho_0^{(1)} \textit{ strongly in } L^2(\Omega),$$

$$\vartheta_{0,\varepsilon}^{(1)} \longrightarrow \vartheta_0^{(1)} \textit{ strongly in } L^2(\Omega),$$

$$I_{0,\varepsilon}^{(1)} \longrightarrow I_0^{(1)} \textit{ strongly in } L^\infty((0,T) \times \Omega \times (0,\infty)).$$

Then up to subsequences

$$\varrho_\varepsilon \longrightarrow \varrho \quad \textit{strongly in } L^\infty(0,T; L^{\frac{5}{3}}(\Omega)),$$

$$\mathbf{u}_\varepsilon \longrightarrow \mathbf{u} \textit{ strongly in } L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3)),$$

$$\vartheta_\varepsilon \longrightarrow \vartheta \quad \textit{strongly in } L^\infty(0,T; L^4(\Omega)),$$

where $(\varrho, \mathbf{u}, \vartheta)$ is the smooth solution of the equilibrium decoupled system on $[0,T] \times \Omega$ and $I(t,x,\nu,\omega) = B(\nu, \vartheta(t,x))$, with initial data $(\varrho_0, \mathbf{u}_0, \vartheta_0)$.

Theorem (Non-equilibrium):

- $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$.
- Assume that the thermodynamic functions p, e, s, $P \in C^1[0,\infty) \cap C^2(0,\infty)$,
- the transport coefficients μ, λ, κ, σ_a, σ_s satisfy hypothesis together with $B \in C^1$.

Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system for $(t, x, \omega, \nu) \in [0, T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions and the initial conditions $(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ such that

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \varepsilon\varrho^{(1)}_{0,\varepsilon}, \quad \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \varepsilon\vartheta^{(1)}_{0,\varepsilon},$$

$$I_\varepsilon(0,\cdot) = I_0 + \varepsilon I^{(1)}_{0,\varepsilon},$$

where the functions $(\varrho_0, \mathbf{u}, \vartheta_0)$ and $x \longrightarrow I_0(x,\omega,\nu)$ belong to $H^3(\Omega)$ and are such that $(\underline{\varrho_0}, \vartheta_0, E_R(I_0))$ belong to the set $\mathcal{O}_{ess}$, where $\overline{\varrho} > 0,\ \overline{\varrho} > 0, \overline{E_R} > 0$ are three constants and $\int_\Omega \varrho^{(1)}_{0,\varepsilon}\ dx = 0,\ \int_\Omega \vartheta^{(1)}_{0,\varepsilon}\ dx = 0,\ \int_\Omega I^{(1)}_{0,\varepsilon}\ dx = 0.$

Suppose also that

$$\mathbf{u}_{0,\varepsilon} \longrightarrow \mathbf{u}_0 \quad \textit{strongly in } L^\infty(\Omega;\mathbb{R}^3),$$

$$\varrho_{0,\varepsilon}^{(1)} \longrightarrow \varrho_0^{(1)} \textit{ strongly in } L^2(\Omega),$$

$$\vartheta_{0,\varepsilon}^{(1)} \longrightarrow \vartheta_0^{(1)} \textit{ strongly in } L^2(\Omega),$$

$$I_{0,\varepsilon}^{(1)} \longrightarrow I_0^{(1)} \textit{ strongly in } L^\infty((0,T)\times\Omega\times(0,\infty)).$$

Then up to subsequences

$$\varrho_\varepsilon \longrightarrow \varrho \quad \textit{strongly in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

$$\mathbf{u}_\varepsilon \longrightarrow \mathbf{u} \textit{ strongly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \longrightarrow \vartheta \quad \textit{strongly in } L^\infty(0,T;L^4(\Omega)),$$

and

$$N_\varepsilon \to N \quad \textit{strongly in } L^\infty((0,T)\times\Omega),$$

where $N_\varepsilon = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon \, d\omega \quad d\nu$ and $(\varrho, \mathbf{u}, \vartheta, N)$ is the smooth solution of the Navier-Stokes-Rosseland system on $[0,T]\times\Omega$ with initial data $(\varrho_0, \mathbf{u}_0, \vartheta_0, N_0)$.

ballistic free energy

$$H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta\varrho s(\varrho,\vartheta),$$

radiative ballistic free energy

$$H^R_\Theta(I) = E^R(I) - \Theta s^R(I).$$

relative entropy

$$\mathcal{E}(\varrho,\vartheta|r,\Theta) := H_\Theta(\varrho,\vartheta) - \partial_\varrho H_\Theta(r,\Theta)(\varrho-\Theta) - H_\Theta(r,\Theta).$$

$$\int_\Omega \left(\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + \varepsilon H^R(I_\varepsilon) \right)(\tau, \cdot)\ dx$$

$$+ \int_0^\tau \int_{\Gamma_+} \omega \cdot \mathbf{n}_x I_\varepsilon(t, x, \omega, \nu)\ d\Gamma\ d\nu\ dt$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \mathbf{u}_\varepsilon - \frac{\mathbf{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right)\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon) + 1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon) + 1} \right] \sigma_{a_\varepsilon}^{(j)} (B_\varepsilon - I_\varepsilon)\ d\omega\ d\nu\ dx\ dt$$

$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon) + 1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon) + 1} \right] \sigma_{s_\varepsilon}^{(j)} (\tilde{I}_\varepsilon - I_\varepsilon)\ d\omega\ d\nu\ dx\ dt, \leq$$

$$\int_\Omega \frac{1}{2} \left(\varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \mathbf{U}(0, \cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | r(0, \cdot), \Theta(0, \cdot)\right) + \varepsilon H^R(I_{0,\varepsilon}) \right)\ dx + \mathcal{R}$$

$$\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{u}|^2+\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|\varrho,\vartheta\right)\right)\leq$$

$$\{\int_\Omega \Bigg(\frac{1}{2}\left(\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}-\mathbf{u}(0,\cdot)|^2+\mathcal{E}\left(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}|\varrho(0,\cdot),\theta(0,\cdot)\right)+\right.$$

$$\left.\varepsilon H^R(I_{0,\varepsilon})\right)\Bigg)dx+e_0\}e^{K_3 t}$$

Suppose that $e_0 \leq C\varepsilon^2$ and the initial data of the primitive system and any of the target systems are close in the following sense

$$\|\varrho_{0,\varepsilon} - \varrho_0\|_{L^2(\Omega)} \leq C\varepsilon, \|\vartheta_{0,\varepsilon} - \vartheta_0\|_{L^2(\Omega)} \leq C\varepsilon, \|\sqrt{\varrho_{0,\varepsilon}}\,(\mathbf{u}_{0,\varepsilon} - \mathbf{u})\|_{L^2(\Omega;\mathbb{R}^3)} \leq C\varepsilon.$$

Semi-relativistic model

Definition of function B

$$B(\nu,\vec{\omega},\mathbf{u},\vartheta)=\frac{2h}{c^2}\frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}\left(1-\alpha\frac{\vec{\omega}\cdot\vec{u}}{c}\right)}-1},$$

$0\leq\alpha(\vartheta)\leq 1$ **Remark:** If $\frac{|\mathbf{u}|}{c}<<1$ one recovers the standard equilibrium Planck's function $B(\nu,\vartheta)=\frac{2h}{c^2}\frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}}-1}$.

Berthon, Buet, Coulombel, Depres, Dubois, Goudon, Morel, Turpault

$M1$ Levermore model

$$\alpha=\frac{\sigma_a+\sigma_s}{\sigma_a+2\sigma_s},$$

$\sigma_a(\vartheta, \mathbf{u}) = \chi(|\mathbf{u}|)\tilde{\sigma}_a(\vartheta) \geq 0$ and $\sigma_s(\vartheta) \geq 0$

$$\chi(s) = \begin{cases} 1 & \text{if} \quad s \leq c, \\ 0 & \text{if} \quad s \geq c + \beta, \end{cases}$$

for an arbitrary $\beta > 0$.

The role of this cut-off is to deal with the singularity of B

In the "over-relativistic" regime ($|\mathbf{u}| \geq c$) we decide to decouple matter and radiation.

Entropy inequality

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) \geq$$

$$\frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x \vartheta}{\vartheta}\right) - \frac{S_E}{\vartheta} + \frac{\mathbf{S}_F\cdot\mathbf{u}}{\vartheta}$$

$$=: \varsigma,$$

where the first term of the right hand side $\varsigma_m := \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x \vartheta}{\vartheta}\right)$ is the (positive) matter entropy production.

The formula for the entropy of a photon gas

$$s^R = -\frac{2k}{c^3} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] d\omega d\nu,$$

$n = n(I) = \frac{c^2 I}{2h\alpha^3 \nu^3}$ is the occupation number.

The radiative entropy flux

$$\mathbf{q}^R = -\frac{2k}{c^2} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] \omega \; d\omega d\nu,$$

The radiative transfer equation

$$\partial_t s^R + \mathrm{div}_x \mathbf{q}^R = -\frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \log \frac{n}{n+1} \; S \; d\omega d\nu =: \varsigma^R.$$

The equilibrium-diffusion regime- Limit system

$$Ma = Sr = Pe = Re = \mathcal{P} = 1,$$

$$\mathcal{C} = \varepsilon^{-1},\ \mathcal{L}_s = \varepsilon^2 \text{ and } \mathcal{L} = \varepsilon^{-1},$$

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \mathbf{p} = \mathrm{div}_x \mathbb{S},$$

Energy equation

$$\partial_t \left(\tfrac{1}{2}\ \varrho |\mathbf{u}_0|^2 + \varrho \mathbf{e}\right) + \mathrm{div}_x \left(\left(\tfrac{1}{2}\ \varrho |\mathbf{u}|^2 + \varrho \mathbf{e} + \mathbf{p}\right) \mathbf{u} + \vec{\mathbf{q}} - \mathbb{S}\mathbf{u}\right) = 0,$$

Entropy equation

$$\partial_t(\varrho \mathbf{s}) + \mathrm{div}_x(\varrho \mathbf{s}\mathbf{u}) + \mathrm{div}_x\left(\frac{\vec{\mathbf{q}}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\vec{\mathbf{q}} \cdot \nabla_x \vartheta}{\vartheta}\right),$$

Radiation

$$I = B(\nu, \vartheta),$$

where $\mathbf{p}(\varrho,\vartheta) = p(\varrho,\vartheta) + \frac{a}{3}\vartheta^4$, $\mathbf{e}(\varrho,\vartheta) = e(\varrho,\vartheta) + \frac{a}{\varrho}\vartheta^4$, $\mathbf{k}(\vartheta_0) = \kappa(\vartheta) + \frac{4a}{3\sigma_a}\vartheta^3$, $\vec{\mathbf{q}} = -\mathbf{k}(\vartheta)\nabla_x\vartheta$ and $\varrho\mathbf{s} = \varrho s + \frac{4}{3}a\vartheta^3$.

Boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \nabla\vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

Initial conditions

$$(\varrho(x,t), \ \mathbf{u}(x,t), \ \vartheta(x,t))|_{t=0} = \left(\varrho^0(x), \ \mathbf{u}^0(x), \ \vartheta^0(x)\right),$$

for any $x \in \Omega$,

Compatibility conditions

$$\mathbf{u}^0(x)|_{\partial\Omega} = 0, \ \nabla\vartheta^0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

- existence of unique solution for strong solution (**local** (small time) or **global** for (small data))
- rigorous proof of singular limit using relative entropy inequality

The non-equilibrium diffusion regime - limit system

$$Ma = Sr = Pe = Re = \mathcal{P} = 1,$$
$$\mathcal{C} = \varepsilon^{-1},\ \mathcal{L} = \varepsilon^2 \text{ and } \mathcal{L}_s = \varepsilon^{-1}.$$

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \mathbf{p} = \mathrm{div}_x \mathbb{S},$$

Total Energy

$$\partial_t \left(\frac{1}{2}\ \varrho|\mathbf{u}_0|^2 + \varrho \mathbf{e} \right) + \mathrm{div}_x \left(\left(\frac{1}{2}\ \varrho|\mathbf{u}|^2 + \varrho \mathbf{e} + \mathbf{p} \right) \mathbf{u} + \vec{\mathbf{q}} - \mathbb{S}\mathbf{u} \right) = 0,$$

Entropy equation

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) =$$

$$\frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) + \frac{1}{3} \frac{\nabla_x N \cdot \mathbf{u}}{\vartheta} - \frac{\sigma_a(\vartheta)}{\vartheta} \left(a\vartheta^4 - N \right),$$

Diffusion equation

$$\partial_t N - \frac{1}{3}\,\mathrm{div}_x\left(\frac{1}{\sigma_s(\vartheta)}\nabla_x N\right) = \sigma_a(\vartheta)\left(a\vartheta^4 - N\right),$$

where $\mathbf{p} = p + \frac{1}{3}N$, $\mathbf{e} = e + \frac{N}{\varrho}$ and $\vec{\mathbf{q}} = \kappa\nabla_x\vartheta + \frac{1}{3\sigma_s}\nabla_x N$

Boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0,\ \ \nabla\vartheta\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \ N|_{\partial\Omega} = 0,$$

Initial conditions

$$\left(\varrho(x,t),\ \mathbf{u}(x,t),\ \vartheta(x,t), N(x,t)\right)|_{t=0} = \left(\varrho^0(x),\ \mathbf{u}^0(x),\ \vartheta^0(x), N^0(x)\right),$$

for any $x\in\Omega$, with $N^0(x) = \int_0^\infty \int_{\mathcal{S}^2} I^0(x,\nu,\omega)\ d\omega\ d\nu$

Compatibility conditions

$$\mathbf{u}^0|_{\partial\Omega} = 0, \ \ \nabla\vartheta^0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ N^0\big|_{\partial\Omega} = 0.$$

$$N = a\theta_r^4.$$

- existence of unique solution for strong solution (**local** (small time) or **global** for (small data))
- rigorous proof of singular limit using relative entropy inequality

Crucial Lemma

Let $(\varrho^{(e)}, \mathbf{u}^{(e)}, \theta^{(e)})$ be the solution of problem full N-S-F with radiation satisfying the conditions of Theorem 1 (equilibrium case) and let $(\varrho^{(ne)}, \mathbf{u}^{(ne)}, \theta^{(ne)}, \theta_r^{(ne)})$ be the solution of problem N-S-F with radiation satisfying the conditions of Theorem 2 (non equilibrium case) and choose $(r, \mathbf{U}, \Theta) = (\varrho^{(e)}, \mathbf{u}^{(e)}, \theta^{(e)})$ in the equilibrium case or $(r, \mathbf{U}, \Theta, \Theta_r) = (\varrho^{(ne)}, \mathbf{u}^{(ne)}, \theta^{(ne)}, \theta_r^{(ne)})$ in the non equilibrium case.

Relative entropy inequality

$$\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + H^R(I_\varepsilon) \right) (t, \cdot)\ dx$$

$$\leq \frac{1}{\varepsilon} \left[\mathcal{C} e_0 + \int_\Omega \left(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \mathbf{U}(0,\cdot)|^2 + \right.\right.$$

$$\left.\left. \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | r(0,\cdot), \Theta(0,\cdot)\right) + H^R(I_{0,\varepsilon}) \right) dx e^{\frac{\mathcal{C}'}{\varepsilon} t},$$

where $\mathcal{C}$ and $\mathcal{C}'$ are positive constant depending on $(r, \mathbf{U}, \Theta, \Theta_r)$ and e_0 is the same as in Theorems 1 and 2.

NSFR with Electromagnetic field

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \quad \text{in } (0,T) \times \Omega, \tag{1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p + \varrho \vec{\chi} \times \mathbf{u}$$
$$= \mathrm{div}_x \mathbb{S} + \varrho \nabla \Psi - \varrho \nabla_x |\vec{\chi} \times \vec{x}|^2 + \vec{j} \times \vec{B} \qquad \text{in } (0,T) \times \Omega, \tag{2}$$

$$\partial_t (\varrho e) + \mathrm{div}_x (\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u} + \vec{j} \cdot \vec{E} - S_E \quad \text{in } (0,T) \times \Omega, \tag{3}$$

$$\frac{1}{c} \partial_t I + \omega \cdot \nabla_x I = S \quad \text{in } (0,T) \times \Omega \times (0,\infty) \times \mathcal{S}^2. \tag{4}$$

$$\partial_t \vec{B} + \mathrm{curl}_x(\vec{B} \times \mathbf{u}) + \mathrm{curl}_x(\lambda \ \mathrm{curl}_x \vec{B}) = 0 \qquad \text{in } (0,T) \times \Omega. \tag{5}$$

$$-\Delta \Psi = 4\pi G(\eta \varrho + g) \quad \text{in } (0,T) \times \Omega_\epsilon. \tag{6}$$

electric current - $\vec{j}$ electric field $\vec{E}$
Ohm's law

$$\vec{j} = \sigma(\vec{E} + \mathbf{u} \times \vec{B}),$$

and *Ampère's law*

$$\zeta \vec{j} = \mathrm{curl}_x \vec{B},$$

where $\zeta > 0$ is the (constant) magnetic permeability,
$\lambda = \lambda(\vartheta) > 0$ is the magnetic diffusivity of the fluid.
We also assume that the system is globally rotating at uniform velocity χ around the vertical direction $\vec{e}_3$

$$\Psi(t,x) = G \int_\Omega K(x-y)(\eta\varrho(t,y) + g(y))\, dy,$$

where $K(x) = \frac{1}{|x|}$, and the parameter η may take the values 0 or 1: for $\eta = 1$ selfgravitation is present and for $\eta = 0$ gravitation only acts as an external field

Boundary conditions:

$$\mathbf{u}|_{\partial\Omega} = 0,\ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \vec{B}\cdot\vec{n}|_{\partial\Omega} = 0,\ \vec{E}\times\vec{n}|_{\partial\Omega} = 0, \tag{7}$$

$$I(t,x,\nu,\omega) = 0 \text{ for } x \in \partial\Omega,\ \omega\cdot\mathbf{n} \le 0, \tag{8}$$

where $\mathbf{n}$ denotes the outer normal vector to $\partial\Omega$.

Primitive system

$$\varepsilon\partial_t I + \omega\cdot\nabla_x I = \sigma_a\left(B - I\right) + \sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I \,\mathrm{d}\omega - I\right), \tag{9}$$

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{10}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho,\vartheta) + \varrho\vec{\chi}\times\mathbf{u} = \mathrm{div}_x\mathbb{S} + \frac{1}{\varepsilon}\varrho\nabla\Psi$$
$$-\varrho\nabla_x|\vec{\chi}\times\vec{x}|^2 + \frac{1}{\varepsilon^2}\vec{j}\times\vec{B}$$

$$\partial_t\left(\varrho e + \varepsilon E^R\right) + \mathrm{div}_x\left(\varrho e\mathbf{u} + \mathbf{F}^R\right) + \mathrm{div}_x\mathbf{q} = \varepsilon^2\mathbb{S}:\nabla_x\mathbf{u} - p\,\mathrm{div}_x\mathbf{u} + \vec{j}\cdot\vec{E} \tag{12}$$

$$\partial_t \left(\varrho s + \varepsilon s^R\right) + \mathrm{div}_x \left(\varrho s \mathbf{u} + \mathbf{q}^R\right) + \ \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) \geq \varsigma_\varepsilon, \tag{13}$$

with

$$\varsigma_\varepsilon = \frac{1}{\vartheta}\left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta} + \frac{\lambda}{\zeta}|\mathrm{curl}_x \vec{B}|^2\right)$$

$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right] \sigma_a(I - B)\ d\omega d\nu$$

$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right] \sigma_s(I - \tilde{I})\ d\omega d\nu,$$

$$\partial_t \vec{B} + \mathrm{curl}_x(\vec{B} \times \mathbf{u}) + \mathrm{curl}_x(\lambda \ \mathrm{curl}_x \vec{B}) = 0, \tag{14}$$

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \ \varepsilon^2 \ \varrho |\mathbf{u}|^2 + \varrho e + \varepsilon E^R + \frac{1}{2\zeta} |\vec{B}|^2 - \frac{1}{2} \varepsilon \varrho \Psi + \frac{1}{2} \varrho |\vec{\chi} \times \vec{x}|^2 \right) \ dx$$

$$+ \int_0^\infty \int_{\Gamma_+} \omega \cdot \mathbf{n} I \ d\Gamma_+ d\nu = 0 \tag{15}$$

where $\Gamma_+ = \{(x, \omega) \in \partial\Omega \times \mathcal{S}^2 \ : \ \omega \cdot \mathbf{n}_x > 0\}$

We consider the pressure in the form

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ \ a > 0, \tag{16}$$

where $P : [0, \infty) \to [0, \infty)$ is a given function with the following properties:

$$P \in C^1[0, \infty), \ \ P(0) = 0, \ \ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{17}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{18}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{19}$$

the specific internal energy e is

$$e(\varrho, \vartheta) = \frac{3}{2}\vartheta \left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{20}$$

and the associated specific entropy reads

$$s(\varrho, \vartheta) = M\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{21}$$

with

$$M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0.$$

$$0 < c_1(1+\vartheta) \le \mu(\vartheta),\ \ \mu'(\vartheta) < c_2,\ \ 0 \le \eta(\vartheta) \le c(1+\vartheta), \tag{22}$$

$$0 < c_1(1+\vartheta^3) \le \kappa(\vartheta),\ \ \lambda(\vartheta) \le c_2(1+\vartheta^3) \tag{23}$$

for any $\vartheta \ge 0$. Moreover we assume that σ_a, σ_s, B are continuous functions of ν, ϑ such that

$$0 \le \sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), |\partial_\vartheta \sigma_a(\nu,\vartheta)|, |\partial_\vartheta \sigma_s(\nu,\vartheta)| \le c_1, \tag{24}$$

$$0 \le \sigma_a(\nu,\vartheta)B(\nu,\vartheta), |\partial_\vartheta \{\sigma_a(\nu,\vartheta)B(\nu,\vartheta)\}| \le c_2, \tag{25}$$

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), \sigma_a(\nu,\vartheta)B(\nu,\vartheta) \le h(\nu),\ \ h \in L^1(0,\infty). \tag{26}$$

for all $\nu \ge 0$, $\vartheta \ge 0$, where $c_{1,2,3}$ are positive constants.

Target system

$$\mathrm{div}_x \mathbf{U} = 0, \tag{27}$$

$$\overline{\varrho}(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U})) + \nabla_x \Pi = \mathrm{div}_x(2\overline{\mu}\, \mathbb{D}(\vec{U})) + \frac{1}{\zeta}\mathrm{curl}_x \vec{B} \times \vec{B} + \vec{F} \tag{28}$$

$$\partial_t \vec{B} + \mathrm{curl}_x(\vec{B} \times \vec{U}) + \mathrm{curl}_x(\overline{\lambda}\ \mathrm{curl}_x \vec{B}) = 0, \tag{29}$$

$$\mathrm{div}_x \mathbf{B} = 0, \tag{30}$$

$$\overline{\varrho}\ \overline{c_P}(\partial_t \Theta + \mathrm{div}_x(\Theta \mathbf{U})) - \mathrm{div}_x(\overline{\kappa} \nabla \Theta) = G, \tag{31}$$

$$\omega \cdot \nabla_x I_0 = \sigma_a (B - I_0) + \sigma_s \left(\tilde{I}_0 - I_0\right), \tag{32}$$

$$\omega \cdot \nabla_x I_1 = \Big(\sigma_a \partial_\vartheta B + \partial_\vartheta \sigma_a (B - I_0) + \partial_\vartheta \sigma_s (\tilde{I}_0 - I_0)\Big)\Theta - \sigma_a I_1 + \sigma_s \left(\tilde{I}_1 - I_1\right), \tag{33}$$

We finally consider the boundary conditions

$$\mathbf{U}|_{\partial\Omega} = 0,\ \nabla\Theta \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \vec{B} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \mathrm{curl}_x \vec{B} \times \mathbf{n}|_{\partial\Omega} = 0 \tag{34}$$

for (27)-(31) and

$$I_0(x,\nu,\omega) = 0 \text{ for } x \in \partial\Omega,\ \omega \cdot \mathbf{n} \le 0 \tag{35}$$

$$I_1(x,\nu,\omega) = 0 \text{ for } x \in \partial\Omega,\ \omega \cdot \mathbf{n} \le 0 \tag{36}$$

for (32) and (33), and the initial conditions

$$\mathbf{U}|_{t=0} = \mathbf{U}_0,\quad \Theta|_{t=0} = \Theta_0,\quad \vec{B}|_{t=0} = \vec{B}_0,\quad I_0|_{t=0} = I_{0,0},\quad I_1|_{t=0} = I_{1,0}. \tag{37}$$

Theorem:

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (16 - 21) with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, and that the transport coefficients μ, η, κ, λ, σ_a, σ_s and the equilibrium function B comply with (22 - 26).
Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon, \mathbf{B}_\varepsilon, I_\varepsilon)$ be a weak solution of the scaled system (1 - 6) for $(t, x, \omega, \nu) \in [0, T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (7 - 8) and initial conditions $(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \mathbf{B}_{0,\varepsilon}, I_{0,\varepsilon})$ given by

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon \varrho_{0,\varepsilon}^{(1)},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \vartheta_\varepsilon(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta_{0,\varepsilon}^{(1)},\ I_\varepsilon(0,\cdot) = \overline{I} + \varepsilon I_{0,\varepsilon}^{(1)},$$

$$\mathbf{B}_\varepsilon(0,\cdot) = \varepsilon \mathbf{B}_{0,\varepsilon}^{(1)},$$

where $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$, $\overline{I} > 0$ are constants and

$$\int_\Omega \varrho_{0,\varepsilon}^{(1)}\, dx = 0,\ \int_\Omega \vartheta_{0,\varepsilon}^{(1)}\, dx = 0,\ \int_\Omega I_{0,\varepsilon}^{(1)}\, dx = 0,\ \int_\Omega \vec{B}_{0,\varepsilon}^{(1)}\, dx = 0 \ \text{ for all } \varepsilon > 0.$$

Assume that

$$\begin{cases} \varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \quad \textit{weakly} - (*) \textit{ in } L^\infty(\Omega), \\ \mathbf{u}_{0,\varepsilon}^{(1)} \to \mathbf{U}_0 \quad \textit{weakly} - (*) \textit{ in } L^\infty(\Omega;\mathbb{R}^3), \\ \vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \quad \textit{weakly} - (*) \textit{ in } L^\infty(\Omega), \\ I_{0,\varepsilon}^{(1)} \to I_0^{(1)} \quad \textit{weakly} - (*) \textit{ in } L^\infty(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+), \\ \mathbf{B}_{0,\varepsilon}^{(1)} \to \mathbf{B}_0^{(1)} \quad \textit{weakly} - (*) \textit{ in } L^\infty(\Omega;\mathbb{R}^3), \end{cases}$$

Then

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t) - \overline{\varrho}\|_{L^{\frac{4}{3}}(\Omega)} \leq C\varepsilon, \tag{38}$$

and up to subsequences

$$\mathbf{u}_\varepsilon \to \mathbf{U} \ weakly - (*) \ in \ L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{39}$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} = \vartheta^{(1)} \to \Theta \quad weakly - (*) \ in \ L^2(0,T;W^{1,2}(\Omega)), \tag{40}$$

$$I_\varepsilon \to I_0 \quad weakly - (*) \ in \ L^2(0,T;L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)), \tag{41}$$

$$\frac{\mathbf{B}_\varepsilon}{\varepsilon} = \mathbf{B}^{(1)} \to \mathbf{B} \ weakly - (*) \ in \ L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{42}$$

and

$$\frac{I_\varepsilon - \overline{I}}{\varepsilon} = I^{(1)} \to I_1 \quad weakly - (*) \ in \ L^2(0,T;L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)), \tag{43}$$

where $(\mathbf{U}, \Theta, \mathbf{B}, I_0, I_1)$ solves the system (27)-(33).

COMPRESSIBLE NAVIER-STOKES SYSTEM:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{44}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \nu \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{45}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{46}$$

the no-slip boundary condition

$$\mathbf{u}|_{\partial \Omega_M} = 0, \tag{47}$$

$\Omega_M \subset R^3$ is a smooth, bounded, simply connected domain.

Kelliher, Lopes, and Nussenzveig-Lopes 2009

- the inviscid limit of the *incompressible* Navier-Stokes system on a family of domains $\Omega_M = M\Omega$, $M \to \infty$, We consider a family of domains $\{\Omega_M\}_{M>0}$ enjoying the following properties:

- $\Omega_M \subset R^3$ are simply connected, bounded C^2 domains, uniformly for $M \to \infty$;
- there exists $\omega > 0$ such that

$$\left\{ x \in R^3 \;\middle|\; |x| < \omega M \right\} \subset \Omega_M; \tag{48}$$

- there exists $\beta > 0$ such that

$$|\partial\Omega_M|_2 \leq \beta M^2, \tag{49}$$

where $|\cdot|_2$ denotes the standard two-dimensional Hausdorff measure.

Our goal is to identify the triple singular limit, where

$$\varepsilon \to 0, \ \nu \to 0, \ \text{while } M \to \infty.$$

$$p \in C[0,\infty) \cap C^3(0,\infty),\ p(0)=0,\ p'(\varrho)>0 \text{ for } \varrho>0,\ \lim_{\varrho\to\infty}\frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty, \tag{50}$$

where

$$\gamma > \frac{3}{2}. \tag{51}$$

We consider the *ill-prepared initial data* in the form

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = 1 + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \tag{52}$$

$$\varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_{0} \text{ in } L^2(R^3), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(R^3;R^3) \text{ as } \varepsilon \to 0. \tag{53}$$

We expect that

$$\varrho \to 1, \ \mathbf{u} \to \mathbf{v},$$

where $\mathbf{v}$ is a solution of the incompressible Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \ \mathrm{div}_x \mathbf{v} = 0 \text{ in } R^3. \tag{54}$$

The principal difficulties of a rigorous proof of such a scenario are:

- The *target* Euler system is defined on R^3 while the *primitive* system (44 - 53) on Ω_M, the solution $\mathbf{v}$ is not an admissible test function in the relative entropy inequality.
- The same problem occurs with the solutions of the associated acoustic system.

The class of *finite energy weak solutions* of the compressible Navier-Stokes system (44-47) satisfying, besides the standard weak formulation of the equations (44 - 46), the *energy inequality*

$$\int_{\Omega_M} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} H(\varrho) \right] (\tau, \cdot) + \nu \int_0^\tau \int_{\Omega_M} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}t \quad (55)$$

$$\leq \int_{\Omega_M} \left[\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} H(\varrho_{0,\varepsilon}) \right] \text{ for a.a. } \tau > 0,$$

where we have set

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z.$$

Solutions of the target system

$\mathbf{u}_0 \in C^m(R^3; R^3)$ for a certain $m > 4$, $\mathrm{supp}[\mathbf{u}_0]$ compact in R^3.

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0], \tag{56}$$

where $\mathbf{H}$ denotes the standard Helmholtz projection onto the space of solenoidal functions, possesses a smooth solution

$$\mathbf{v} \in C^k([0, T_{\max}); W^{m-k,2}(R^3; R^3)), k = 1, .., m-1 \tag{57}$$

defined on a maximal time interval $[0, T_{\max})$, $T_{\max} > 0$.

Acoustic system

Lighthill's acoustic analogy

$$\varepsilon\partial_t \frac{\varrho - 1}{\varepsilon} + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\varepsilon\partial_t(\varrho\mathbf{u}) + p'(1)\nabla_x \frac{\varrho - 1}{\varepsilon} =$$

$$\varepsilon\left[\nu \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) - \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) - \nabla_x\left(p(\varrho) - p'(1)\frac{\varrho - 1}{\varepsilon} - p(1)\right)\right],$$

The *acoustic system*

$$\varepsilon\partial_t s + \Delta\Psi = 0,\ \ \varepsilon\partial_t \nabla_x \Psi + a\nabla_x s = 0,\ \ a = p'(1) > 0, \tag{58}$$

the initial data

$$s(0,\cdot) = \varrho_0^{(1)},\ \ \nabla_x\Psi(0,\cdot) = \nabla_x\Psi_0 = \mathbf{u}_0 - \mathbf{H}[\mathbf{u}_0]. \tag{59}$$

Main results

Theorem:
Let the pressure p satisfy the hypotheses (50), (51). Let $\{\Omega_M\}_{M>0}$ be a family of uniformly C^2-domains in R^3 such that (48), (49) hold for $M = M(\varepsilon)$,

$$\varepsilon M(\varepsilon) \to \infty \text{ as } \varepsilon \to 0. \tag{60}$$

Let the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$ for the compressible Navier-Stokes system (44 - 47) be of the form

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = 1+\varepsilon\varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \|\varrho^{(1)}_{0,\varepsilon}\|_{L^2\cap L^\infty(R^3)}+\|\mathbf{u}_{0,\varepsilon}\|_{L^2(R^3;R^3)} \leq D. \tag{61}$$

In addition, suppose we are given functions $\mathbf{u}_0$, $\varrho^{(1)}_0$ such that

$$\mathbf{u}_0 \in C^m(R^3;R^3),\ \varrho^{(1)}_0 \in C^m(R^3),\ \|\mathbf{u}_0\|_{C^m(R^3;R^3)}+\|\varrho^{(1)}_0\|_{C^m(R^3)} \leq D,\ m > 4, \tag{62}$$

$$\mathrm{supp}[\mathbf{u}_0],\ \mathrm{supp}[\varrho^{(1)}_0] \text{ compact in } R^3. \tag{63}$$

Let $T_{\max} > 0$ be the life-span of the smooth solution $\mathbf{v}$ of the Euler system (54), endowed with the initial datum $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$, and let $0 < T < T_{\max}$. Let $[s, \Psi]$ be the solution of the acoustic system (58), with the initial data (59). Then
there exists $\varepsilon_0 > 0$ such that

$$\left\| \sqrt{\varrho}\Big(\mathbf{u} - \nabla_x \Psi - \mathbf{v} \Big)(\tau, \cdot) \right\|_{L^2(\Omega_M;R^3)} + \left\| \left(\frac{\varrho - 1}{\varepsilon} \right)(\tau, \cdot) - s(\tau, \cdot) \right\|_{L^2 + L^\gamma(\Omega_M)} \tag{64}$$

$$\leq c(D, T, \alpha) \left[\| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \|_{L^2(\Omega_M;R^3)} + \left\| \varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)} \right\|_{L^2(\Omega_M)} + \left(\nu + \varepsilon^\alpha + \frac{1}{\varepsilon M(\varepsilon)} \right)^{1/2} \right],$$

$$\tau \in [0, T],\ 0 < \alpha < 1, \text{ and } 0 < \varepsilon \leq \varepsilon_0,$$

for any weak solution $[\varrho, \mathbf{u}]$ of the compressible Navier-Stokes system (44 - 47).

Corollary

In addition to the hypotheses of Theorem 1, assume that

$$\varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \text{ in } L^2(R^3),\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(R^3;R^3) \text{ as } \varepsilon \to 0,$$

and

$$\nu = \nu(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Then

$$\mathrm{ess}\sup_{t\in(0,T)} \|\varrho - 1\|_{L^2+L^\gamma(K)} \to 0 \text{ as } \varepsilon \to 0,$$

$$\mathrm{ess}\sup_{t\in(\delta,T)} \left\| \sqrt{\varrho}\Big(\mathbf{u} - \mathbf{v}\Big)(t,\cdot) \right\|_{L^2(K;R^3)} \to 0 \text{ as } \varepsilon \to 0 \qquad (65)$$

for any $0 < \delta < T$ and any compact $K \subset R^3$.

- Reduction of dimension
  A straight layer $\Omega_\epsilon = \omega \times (0, \epsilon)$, where ω is a 2-D domain. $\epsilon \longrightarrow 0$ in N-S and N-S-F case
- Singular limit on expanding domain with rotation
- Introducing relative entropy inequality to numerical numerical analysis T.Karper,A.Novotny, E.Feireisl
- Coupling N-S-F with magnetic field
- low Mach number limit in domain dependent on time
- singular limit in fluid-structure interaction