

CONTINUITY PROPERTIES OF PRANDTL-ISHLINSKII OPERATORS IN THE SPACE OF REGULATED FUNCTIONS

WEI LIU*

College of Science, Hohai University, Nanjing 210098, China

PAVEL KREJČÍ

Institute of Mathematics, Czech Academy of Sciences
Žitná 25, CZ-11567 Praha 1, Czech Republic

GUOJU YE

College of Science, Hohai University, Nanjing 210098, China

(Communicated by Yuan Lou)

ABSTRACT. It is well known that the Prandtl-Ishlinskii hysteresis operator is locally Lipschitz continuous in the space of continuous functions provided its primary response curve is convex or concave. This property can easily be extended to any absolutely continuous primary response curve with derivative of locally bounded variation. Under the same condition, the Prandtl-Ishlinskii operator in the Kurzweil integral setting is locally Lipschitz continuous also in the space of regulated functions. This paper shows that the Prandtl-Ishlinskii operator is still continuous if the primary response curve is only monotone and continuous, and that it may not even be locally Hölder continuous for continuously differentiable primary response curves.

1. Introduction. The Prandtl-Ishlinskii operator was introduced in the classical monograph [8] by Krasnosel'skii and Pokrovskii (under the name “Ishlinskii operator”) as a finite or infinite linear combination of the so-called play operators defined as solution operators of rate independent variational inequalities and describing, following Prandtl in [17] and Ishlinskii in [7], elementary uniaxial parallel constitutive models of elastoplasticity. More about mathematical questions related to the Prandtl-Ishlinskii operator can be found in the monographs [2, 9]. Engineers appreciate the fact that it admits an explicit inverse, which is a feature that allows for constructing simple and robust algorithms for inverse control of technical processes with the goal to eliminate the undesired influence of hysteresis, see [1, 15, 18].

The classical theory deals with input functions which are either continuous or piecewise constant. More recent applications in modeling economic processes, where

2010 *Mathematics Subject Classification.* Primary: 34C55; Secondary: 58C07, 49J40.

Key words and phrases. Hysteresis, Prandtl-Ishlinskii operator, regulated function, Kurzweil integral, continuity.

This work was supported by the Program of High-end Foreign Experts of the SAFEA (No. GDW20163200216). The work of the second author was partially supported by the GAČR Grant 15-12227S and RVO: 67985840.

The hospitality of the Hohai University in Nanjing during the second author's stay in October-November 2016 is gratefully acknowledged.

* Corresponding author: Wei Liu.

jump discontinuities may spontaneously occur during the evolution, as for example in [3, 4, 5, 6, 11, 12], require to consider a more general class of inputs. A good candidate seems to be the space $G(0, T)$ of *regulated functions*, that is, functions $f : [0, T] \rightarrow \mathbb{R}$ that admit both one-sided limits $f(t-)$, $f(t+)$ with the convention $f(0-) = f(0)$, $f(T+) = f(T)$. The Kurzweil integral introduced in [16] offers an analytical tool for this study and a corresponding theory of rate independent integral variational inequalities with regulated inputs has been developed in [14]. A further extension to an even larger subspace of L^∞ has been done in [13].

The shape of the Prandtl-Ishlinskii hysteresis loops and the analytical properties of the operator are determined by one single nondecreasing function called the *generating function* or the *primary response curve*. It is well known that the Prandtl-Ishlinskii hysteresis operator is locally Lipschitz continuous in the space of continuous functions provided its primary response curve is convex or concave. This property can easily be extended to any absolutely continuous primary response curve with derivative of locally bounded variation. Under the same condition, the Prandtl-Ishlinskii operator in the Kurzweil integral setting is locally Lipschitz continuous also in the space of regulated functions. Indeed, no continuity can be expected if the primary response curve is discontinuous. The main result of this paper consists in proving that the necessary condition for continuity, namely the continuity of the primary response curve, is also sufficient for right continuous regulated inputs. We also present examples showing that even continuous differentiability of the primary response curve may not be sufficient for local Hölder continuity of the Prandtl-Ishlinskii operator.

The structure of the paper is as follows. In Section 2 we recall some basic notions of the Kurzweil integral variational inequalities. Section 3 is devoted to the proof of the main continuity results, and Section 4 contains some counterexamples.

2. The play operator in the space of regulated functions. The play operator was defined in [8] first for continuous piecewise monotone inputs and it was shown that it can be extended to a Lipschitz continuous operator in the space $C[0, T]$ of continuous functions on $[0, T]$. Brokate and Sprekels proved in [2, Theorem 2.7.7] that the play operator is the main building block for all hysteresis operators satisfying the Madelung memory rules (nowadays called *return point memory operators*). Roughly speaking, their result says that every return point memory operator can be represented by a functional on the space of memory curves generated by the system of play operators. Note that linear functionals correspond exactly to Prandtl-Ishlinskii operators which constitute the main topic of this paper.

Here, we follow the formalism of [10] and restrict ourselves to the space $G_R(0, T)$ of right continuous regulated functions. Then the play operator with threshold $r > 0$ is defined as the mapping \mathfrak{p}_r which with a given input function $u \in G_R(0, T)$ and with an initial condition λ_{-1} associates the solution $\xi_r \in G_R(0, T)$ of the Kurzweil integral variational inequality

$$\left. \begin{aligned} |u(t) - \xi_r(t)| &\leq r, \quad \forall t \in [0, T], \\ \xi_r(0) &= \min\{u(0) + r, \max\{\lambda_{-1}(r), u(0) - r\}\}, \\ \int_0^T (u(t) - \xi_r(t) - z(t)) \, d\xi_r(t) &\geq 0, \quad \forall z \in G(0, T), \quad |z(t)| \leq r \quad \forall t \in [0, T]. \end{aligned} \right\} \quad (1)$$

The definition is meaningful provided $\lambda_{-1} : [0, \infty) \rightarrow \mathbb{R}$, is chosen to be Lipschitz continuous with $|\lambda'_{-1}(r)| \leq 1$ a. e., and there exists a constant $K > 0$ such that $\lambda_{-1}(r) = 0$ for $r \geq K$. Here, we restrict ourselves to the canonical initial condition

$$\lambda_{-1}(r) = 0 \quad \forall r \geq 0 \quad (2)$$

and write simply $\xi_r(t) = \mathbf{p}_r[u](t)$ for $u \in G_R(0, T)$ and $t \in [0, T]$. If u is a right continuous step function of the form

$$u(t) = \sum_{j=1}^m u_{j-1} \chi_{[t_{j-1}, t_j)}(t) + u_m \chi_{\{T\}}(t)$$

corresponding to a division $0 = t_0 < t_1 < \dots < t_m = T$ of the interval $[0, T]$, where χ_A for $A \subset [0, T]$ denotes the characteristic function of the set A , that is, $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$, then ξ_r has the same form

$$\xi_r(t) = \sum_{j=1}^m \xi_{j-1}^r \chi_{[t_{j-1}, t_j)}(t) + \xi_m^r \chi_{\{T\}}(t)$$

with

$$\xi_j^r = \min\{u_j + r, \max\{\xi_{j-1}^r, u_j - r\}\} \quad \text{for } j = 0, \dots, m, \quad \xi_{-1}^r = \lambda_{-1}(r), \quad (3)$$

see [14].

As a special case of [12, Lemma 3.2] we have the following comparison result.

Lemma 2.1. *Let $u, v \in G_R(0, T)$ be given, and let $\xi_r = \mathbf{p}_r[u]$, $\eta_r = \mathbf{p}_r[v]$. Assume that $u(t) \geq v(t)$ for all t in an interval $[a, b] \subset [0, T]$, and that $\xi_r(a) \geq \eta_r(a)$. Then $\xi_r(t) \geq \eta_r(t)$ for all $t \in [a, b]$.*

We sketch here the proof of the following property of the play operator with right continuous regulated inputs which is known for step functions from [2] and for continuous functions from [9, Chapter II].

Lemma 2.2. *Let $u \in G_R(0, T)$, $t \in [0, T]$, and $\underline{t} \in [0, t]$ be given, and let us denote $\xi_r = \mathbf{p}_r[u]$, $\underline{\lambda}(r) := \min\{\xi_r(\underline{t}-), \xi_r(\underline{t})\}$. Set*

$$\bar{t} = \max\{\tau \in [\underline{t}, t] : \bar{u} := \max\{u(\tau-), u(\tau)\} = \sup\{u(s); s \in [\underline{t}, t]\}\}.$$

Assume that

$$u(\tau) \geq \underline{u} := \min\{u(\underline{t}-), u(\underline{t})\} \quad \forall \tau \in [\underline{t}, t].$$

Then $\bar{\lambda}(r) := \max\{\xi_r(\bar{t}-), \xi_r(\bar{t})\} = \max\{\bar{u} - r, \underline{\lambda}(r)\}$.

Proof. We define

$$\begin{aligned} \underline{v}(t) &= \begin{cases} \underline{u} & \text{for } t \in [\underline{t}, \bar{t}), \\ \bar{u} & \text{for } t = \bar{t}, \end{cases} & \bar{v}(t) &= \bar{u} \quad \text{for } t \in [\underline{t}, \bar{t}], \\ \mathbf{p}_r[\underline{v}](\underline{t}) &= \underline{\lambda}(r), & \mathbf{p}_r[\bar{v}](\underline{t}) &= \max\{\bar{u} - r, \underline{\lambda}(r)\}. \end{aligned}$$

By (3) we have $\mathbf{p}_r[\underline{v}](\bar{t}) = \mathbf{p}_r[\bar{v}](\bar{t}) = \max\{\bar{u} - r, \underline{\lambda}(r)\}$, and since $\underline{u} \leq u \leq \bar{u}$ in $[\underline{t}, \bar{t}]$, the assertion follows from Lemma 2.1. \square

The same argument, where we interchange the local maxima and minima, yields the following reverse statement.

Lemma 2.3. *Let $u \in G_R(0, T)$, $t \in [0, T]$, and $\bar{t} \in [0, t]$ be given, and let us denote $\xi_r = \mathbf{p}_r[u]$, $\bar{\lambda}(r) := \max\{\xi_r(\bar{t}-), \xi_r(\bar{t})\}$. Set*

$$\underline{t} = \max\{\tau \in [\bar{t}, t] : \underline{u} := \min\{u(\tau-), u(\tau)\} = \inf\{u(s); s \in [\bar{t}, t]\}\}.$$

Assume that

$$u(\tau) \leq \bar{u} := \max\{u(\bar{t}-), u(\bar{t})\} \quad \forall \tau \in [\bar{t}, t].$$

Then $\underline{\lambda}(r) := \min\{\xi_r(\underline{t}-), \xi_r(\underline{t})\} = \min\{\underline{u} + r, \bar{\lambda}(r)\}$.

By induction following the argument of [9, Proposition II.2.5] we thus construct at each time t a decreasing sequence (finite or infinite) $\{u_{2i-1}\}$ of local suprema and an increasing sequence $\{u_{2i}\}$ of local infima of the function u such that

$$\mathbf{p}_r[u](t) = \begin{cases} 0 & \text{for } r > \bar{\sigma} := \sup_{\tau \in [0,t]} |u(\tau)|, \\ u_j + (-1)^j r & \text{for } r \in (\sigma_{j+1}, \sigma_j] \end{cases} \quad (4)$$

with

$$\sigma_{j+1} = \frac{(-1)^j}{2}(u_{j+1} - u_j), \quad (5)$$

and

$$\begin{aligned} u_{2i} &= \min\{u(t_{2i}), u(t_{2i}-)\}, \\ u_{2i-1} &= \max\{u(t_{2i-1}), u(t_{2i-1}-)\}, \quad i = 1, 2, \dots, \quad t_1 > t_2 > \dots \geq 0. \end{aligned} \quad (6)$$

3. A continuity theorem. For a nondecreasing right continuous function $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\psi(0) = 0$ called the generating function or the primary response curve, we define the Prandtl-Ishlinskii operator generated by ψ by the Kurzweil integral formula

$$\mathcal{F}[u](t) = - \int_0^\infty \frac{\partial^-}{\partial r} \mathbf{p}_r[u](t) \, d\psi(r), \quad (7)$$

where $\frac{\partial^-}{\partial r}$ denotes the left derivative. It was shown in [12] that if $|u|_{[0,T]} \leq K$ and

$$\lambda_{-1}(r) = \min\{0, -K + r\}, \quad (8)$$

then $\mathcal{F}[u] \in G_R(0, T)$. Here, we prove the following continuity result.

Theorem 3.1. *Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a nondecreasing continuous function, $\psi(0) = 0$, and let $\lambda_{-1}(r) = 0$. Then the operator $\mathcal{F} : G_R(0, T) \rightarrow G_R(0, T)$ given by (7) is continuous.*

Indeed, no continuity can be expected if ψ is discontinuous. Before passing to the proof of Theorem 3.1, we state and prove an easy Lemma.

Lemma 3.2. *Under the hypotheses of Theorem 3.1, the function $w(t) = \mathcal{F}[u](t)$ given by (7) belongs to $G_R(0, T)$ for every $u \in G_R(0, T)$, and belongs to $C[0, T]$ for every $u \in C[0, T]$.*

Proof of Lemma 3.2. Let $u \in G_R(0, T)$ be given, and set $K := |u|_{[0,T]}$. Since ψ is nondecreasing and continuous, there exists a sequence $\{\psi_n\}$ of nondecreasing C^2 -functions such that $\psi_n(0) = 0$ and $\sup_{r \in [0,K]} |\psi_n(r) - \psi(r)| \rightarrow 0$ as $n \rightarrow \infty$. For $n \in \mathbb{N}$ and $t \in [0, T]$ set

$$w_n(t) = \mathcal{F}_n[u](t) := - \int_0^\infty \frac{\partial^-}{\partial r} \mathbf{p}_r[u](t) \, d\psi_n(r) = \psi'_n(0)u(t) + \int_0^K \mathbf{p}_r[u](t)\psi''_n(r) \, dr.$$

We first check that all w_n belong to $G_R(0, T)$. Indeed, let $t \in (0, T]$ and $0 < \beta < \alpha < t$ be arbitrary. We have

$$\begin{aligned} w_n(t - \beta) - w_n(t - \alpha) &= \psi'_n(0)(u(t - \beta) - u(t - \alpha)) \\ &\quad + \int_0^K (\mathbf{p}_r[u](t - \beta) - \mathbf{p}_r[u](t - \alpha))\psi''_n(r) \, dr, \end{aligned} \quad (9)$$

hence, using the Lipschitz continuity result of the play operator in [10, Theorem 2.1], one has

$$|w_n(t - \beta) - w_n(t - \alpha)| \leq \sup_{\tau \in [t-\alpha, t-\beta]} |u(\tau) - u(t - \alpha)| \left(\psi'_n(0) + \int_0^K |\psi''_n(r)| \, dr \right),$$

and we conclude that $w_n(t-)$ exists for all $t \in (0, T]$. Similarly, for $t \in [0, T)$ and $0 < \alpha < T - t$ we have

$$w_n(t + \alpha) - w_n(t) = \psi'_n(0)(u(t + \alpha) - u(t)) + \int_0^K (\mathfrak{p}_r[u](t + \alpha) - \mathfrak{p}_r[u](t))\psi''_n(r) \, dr,$$

hence

$$|w_n(t + \alpha) - w_n(t)| \leq \sup_{\tau \in [t, t+\alpha]} |u(\tau) - u(t)| \left(\psi'_n(0) + \int_0^K |\psi''_n(r)| \, dr \right),$$

and we see that $w_n(t+)$ exists and equals $w_n(t)$ for every n .

To complete the proof, it suffices to check that w_n converge uniformly to w . In other words, we have to prove the following implication.

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall t \in [0, T] : n > n_0 \Rightarrow |w_n(t) - w(t)| < \varepsilon. \tag{10}$$

Let $\varepsilon > 0$ be given. We fix $r_0 > 0$ such that $\psi(r_0) < \varepsilon/4$ and an integer $n_1 \in \mathbb{N}$ such that $\sup_{r \in [0, K]} |\psi_n(r) - \psi(r)| < \varepsilon/4$ for $n > n_1$. In particular, for every $t \in [0, T]$ we have

$$\left| \int_0^{r_0} \frac{\partial^-}{\partial r} \mathfrak{p}_r[u](t) \, d(\psi_n - \psi)(r) \right| \leq \text{Var}_{[0, r_0]}(\psi_n - \psi) \leq \psi_n(r_0) + \psi(r_0) < \frac{3}{4}\varepsilon. \tag{11}$$

On $[r_0, K]$, the function $\lambda'(r) = \frac{\partial^-}{\partial r} \mathfrak{p}_r[u](t)$ is a step function for each $t \in [0, T]$. More specifically, by (4), there exists a sequence $r_0 < r_1 < \dots < r_m \leq K$ such that $\lambda'(r) = 0$ for $r > r_m$ and $\lambda'(r) = \pm 1$ in $(r_{j-1}, r_j]$. Since u is regulated, the number m is bounded above by some M independent of t . Hence, $\text{Var}_{[r_0, K]} \lambda' \leq 2M + 2$, so that

$$\begin{aligned} \left| \int_{r_0}^K \frac{\partial^-}{\partial r} \mathfrak{p}_r[u](t) \, d(\psi_n - \psi)(r) \right| &\leq \left(|\lambda'(r_0)| + \text{Var}_{[r_0, K]} \lambda' \right) \sup_{r \in [r_0, K]} |\psi_n(r) - \psi(r)| \\ &\leq (2M + 3) \sup_{r \in [r_0, K]} |\psi_n(r) - \psi(r)|. \end{aligned} \tag{12}$$

It remains to choose $n_0 \geq n_1$ such that $\sup_{r \in [0, K]} |\psi_n(r) - \psi(r)| < \varepsilon/(8M + 12)$ for $n > n_0$ and combine (12) with (11) to obtain the assertion. If $u \in C[0, T]$, then we similarly prove that w_n belong to $C[0, T]$ for all $n \in \mathbb{N}$, and the uniform convergence argument completes the proof. \square

Proof of Theorem 3.1. Let $u \in G_R(0, T)$ and $\varepsilon > 0$ be given. We want to prove the following statement:

$$\exists \delta_0 > 0 \forall \hat{u} \in G_R(0, T) : |u - \hat{u}|_{[0, T]} < \delta_0 \forall t_0 \in [0, T] : |\mathcal{F}[u](t_0) - \mathcal{F}[\hat{u}](t_0)| < \varepsilon. \tag{13}$$

We proceed as follows. We fix $t_0 \in [0, T]$ and show that it is possible to choose $\delta_0 \in (0, 1)$ independent of t_0 satisfying the implication

$$\delta := |u - \hat{u}|_{[0, T]} < \delta_0 \Rightarrow |\mathcal{F}[u](t_0) - \mathcal{F}[\hat{u}](t_0)| < \varepsilon. \tag{14}$$

With this given t_0 , we denote $\lambda(r) = \mathfrak{p}_r[u](t_0)$ and fix $r_0 = r_0(\varepsilon) > 0$ such that

$$\text{Var}_{[0, r_0]} \psi = \psi(r_0) < \frac{\varepsilon}{4}. \tag{15}$$

We fix some $K \geq |u|_{[0,T]} + 1$. According to (4)–(5), there exists a sequence $r_0 < r_1 < \dots < r_m \leq r_{m+1} = K$ such that $\lambda(r) = 0$ for $r \geq r_m$, and either

$$\lambda(r) = u_j + (-1)^j r \quad \text{for } r \in [r_{j-1}, r_j], \quad j = 1, \dots, m, \tag{16}$$

or

$$\lambda(r) = u_j - (-1)^j r \quad \text{for } r \in [r_{j-1}, r_j], \quad j = 1, \dots, m, \tag{17}$$

where $u_j = u(t_j)$ or $u_j = u(t_j-)$ for some $t_0 \geq t_1 > t_j > \dots > t_m \geq 0$. Let us consider only the case (16), case (17) is similar.

We have $\lambda(r_j) = u_j + (-1)^j r_j = u_{j+1} + (-1)^{j+1} r_j$ for $j = 1, \dots, m - 1$, hence $u_{j+1} - u_j = 2(-1)^j r_j$. In particular, $|u_{j+1} - u_j| \geq 2r_0 = 2r_0(\varepsilon)$. Since u is regulated, there exists a number $M = M(\varepsilon)$ depending only on $r_0(\varepsilon)$ (in particular, independent of t_0) such that

$$m \leq M(\varepsilon). \tag{18}$$

Let now $\hat{u} \in G_R(0, T)$ be arbitrary such that $\delta := |u - \hat{u}|_{[0,T]} \leq 1$, and put $\hat{\lambda}(r) = \mathfrak{p}_r[\hat{u}](t_0)$. We have by definition

$$|\mathcal{F}[u](t_0) - \mathcal{F}[\hat{u}](t_0)| = \left| \int_0^\infty (\lambda'(r) - \hat{\lambda}'(r)) \, d\psi(r) \right|, \tag{19}$$

where we set

$$\lambda'(r) = \frac{\partial^-}{\partial r} \mathfrak{p}_r[u](t_0), \quad \hat{\lambda}'(r) = \frac{\partial^-}{\partial r} \mathfrak{p}_r[\hat{u}](t_0). \tag{20}$$

The situation is depicted at Figure 1. We first observe that $|\lambda'(r)| \leq 1, |\hat{\lambda}'(r)| \leq 1$ for every $r > 0$, hence

$$\left| \int_0^{r_0} (\lambda'(r) - \hat{\lambda}'(r)) \, d\psi(r) \right| \leq 2 \operatorname{Var}_{[0,r_0]} \psi < \frac{\varepsilon}{2} \tag{21}$$

by virtue of (15). Similarly, for $j = 1, \dots, m$, we have

$$\left| \int_{r_j-2\delta}^{r_j+2\delta} (\lambda'(r) - \hat{\lambda}'(r)) \, d\psi(r) \right| \leq 2(\psi(r_j + 2\delta) - \psi(r_j - 2\delta)). \tag{22}$$

Let $\eta = \eta(\varepsilon)$ be such that

$$0 < r < \hat{r} \leq K, \quad \hat{r} - r < \eta(\varepsilon) \Rightarrow \psi(\hat{r}) - \psi(r) < \frac{\varepsilon}{d(\varepsilon)} \tag{23}$$

with $d(\varepsilon) \geq 1$ which will be specified later. Then, choosing

$$\delta < \delta_0 := \frac{\eta(\varepsilon)}{4} \tag{24}$$

we have

$$\left| \int_{r_j-2\delta}^{r_j+2\delta} (\lambda'(r) - \hat{\lambda}'(r)) \, d\psi(r) \right| \leq \frac{2\varepsilon}{d(\varepsilon)} \quad \text{for } j = 0, \dots, m. \tag{25}$$

Let $J_\delta := \{j \in \{1, \dots, m\} : r_j - r_{j-1} \geq 2\delta\}$. We have indeed $\lambda(r) = \hat{\lambda}(r) = 0$ for $r \geq K$. Hence, it remains to estimate the integrals

$$\int_{r_{j-1}+\delta}^{r_j-\delta} (\lambda'(r) - \hat{\lambda}'(r)) \, d\psi(r) \quad \text{for } j \in J_\delta. \tag{26}$$

The function λ' is constant in $(r_{j-1} + \delta, r_j - \delta)$, and we may assume that $\lambda'(r) = 1$ there. The case that $\lambda'(r) = -1$ for $r \in (r_{j-1} + \delta, r_j - \delta)$ is similar (see Figure 1). Then,

$$\left| \int_{r_{j-1}+\delta}^{r_j-\delta} (\lambda'(r) - \hat{\lambda}'(r)) \, d\psi(r) \right| = \int_{r_{j-1}+\delta}^{r_j-\delta} (1 - \hat{\lambda}'(r)) \, d\psi(r). \tag{27}$$

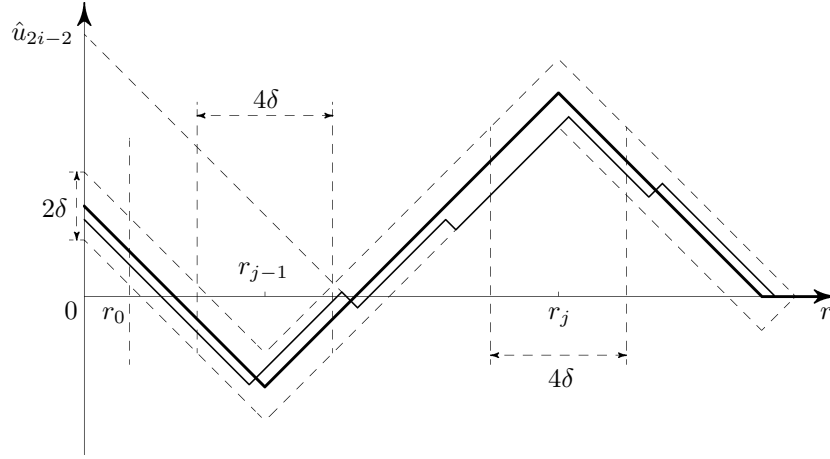


FIGURE 1. The memory curves $\lambda(r)$ (the bold solid line) and $\hat{\lambda}(r)$ (the thin solid line).

There are at most finitely many points in $[r_{j-1} + \delta, r_j - \delta]$ in which $\hat{\lambda}'$ changes sign. Let $r_{j-1} + \delta \leq \hat{r}_0 < \hat{r}_1 < \dots < \hat{r}_{2k-1} \leq r_j - \delta$ be all such points for which $\hat{\lambda}'(r) = 1$ in $(\hat{r}_{2i-1}, \hat{r}_{2i})$, $\hat{\lambda}'(r) = -1$ in $(\hat{r}_{2i-2}, \hat{r}_{2i-1})$, $i = 1, \dots, k$ (see Figure 1). We have for all $r \geq 0$ that

$$|\lambda(r) - \hat{\lambda}(r)| \leq \delta, \tag{28}$$

hence

$$2\delta \geq \int_{r_{j-1}+\delta}^{r_j-\delta} (1 - \hat{\lambda}'(r)) \, dr = 2 \sum_{i=1}^k (\hat{r}_{2i-1} - \hat{r}_{2i-2}) \tag{29}$$

Furthermore,

$$\int_{r_{j-1}+\delta}^{r_j-\delta} (1 - \hat{\lambda}'(r)) \, d\psi(r) = 2 \sum_{i=1}^k (\psi(\hat{r}_{2i-1}) - \psi(\hat{r}_{2i-2})). \tag{30}$$

On the other hand, for every $i = 1, \dots, k$ we have again by (4)–(5) that

$$\hat{\lambda}(r) = \hat{u}_{2i-1} + r \text{ for } r \in [\hat{r}_{2i-1}, \hat{r}_{2i}), \quad \hat{\lambda}(r) = \hat{u}_{2i-2} - r \text{ for } r \in [\hat{r}_{2i-2}, \hat{r}_{2i-1}), \tag{31}$$

where $\hat{u}_l = \hat{u}(\hat{t}_l)$ or $\hat{u}_l = \hat{u}(\hat{t}_l -)$ for $l = 0, \dots, 2k$ and for some sequence $t_0 \geq \hat{t}_1 > \dots \hat{t}_{2k} \geq 0$. In particular, we have

$$\hat{\lambda}(\hat{r}_{2i-1}) = \hat{u}_{2i-1} + \hat{r}_{2i-1} = \hat{u}_{2i-2} - \hat{r}_{2i-1}, \tag{32}$$

hence $\hat{u}_{2i-2} - \hat{u}_{2i-1} = 2\hat{r}_{2i-1} \geq 2(r_j + \delta)$. By virtue of the assumption $|u - \hat{u}|_{[0, T]} = \delta$, the corresponding values of u_{2i-1}, u_{2i-2} satisfy

$$u_{2i-2} - u_{2i-1} \geq 2r_0 \tag{33}$$

and an argument similar to (18) yields

$$k \leq M(\varepsilon). \tag{34}$$

By (29), we have $\hat{r}_{2i-1} - \hat{r}_{2i-2} \leq \delta$, and from (23)–(24) and (30) it follows that

$$\int_{r_{j-1}+\delta}^{r_j-\delta} (1 - \hat{\lambda}'(r)) \, d\psi(r) \leq \frac{2M(\varepsilon)}{d(\varepsilon)} \varepsilon. \tag{35}$$

Putting together (21), (25), and (35), we obtain

$$\begin{aligned} |\mathcal{F}[u](t_0) - \mathcal{F}[\hat{u}](t_0)| &= \left| \int_0^\infty (\lambda'(r) - \hat{\lambda}'(r)) \, d\psi(r) \right| \\ &\leq \left(\frac{1}{2} + \frac{2(M(\varepsilon) + M(\varepsilon)^2)}{d(\varepsilon)} \right) \varepsilon. \end{aligned} \tag{36}$$

To complete the proof, it suffices to choose $d(\varepsilon) = 4(M(\varepsilon) + M(\varepsilon)^2)$. □

4. Counterexamples to Hölder continuity. It is well known that the Prandtl-Ishlinskii operator \mathcal{F} is Lipschitz continuous on bounded sets in $G_R(0, T)$ if ψ is convex or concave. Since every function of bounded variation can be represented as a difference of two nondecreasing functions, the Lipschitz continuity holds whenever ψ' has bounded variation. Since \mathcal{F} is continuous for every continuous function ψ according to Theorem 3.1, one might be tempted to believe that the Lipschitz continuity of ψ suffices for the Lipschitz continuity of \mathcal{F} . We show here that this conjecture is false. In fact, we prove even more, namely that there exists an increasing Lipschitz continuous function ψ such that the operator \mathcal{F} is not even locally α -Hölder continuous for any exponent $\alpha > 0$. In Theorem 4.1 below we show that ψ may even be assumed continuously differentiable and the counterexample still works.

The construction is as follows. We consider a decreasing sequence $\{r_k : k = 0, 1, \dots\}$ of positive numbers, $r_0 \leq 1$, $\lim_{k \rightarrow \infty} r_k = 0$, $r_k - r_{k+1} < r_{k-1} - r_k$ for all $k \geq 1$, set $\rho_k = \frac{1}{2}(r_k + r_{k+1})$ for $k \geq 0$, and define

$$\psi_0(r) = \begin{cases} 0 & \text{for } r \geq r_0, \\ -r_0 + r & \text{for } r \in [\rho_0, r_0), \\ r_{2i-1} - r & \text{for } r \in [\rho_{2i-1}, \rho_{2i-2}), \, i \in \mathbb{N}, \\ -r_{2i} + r & \text{for } r \in [\rho_{2i}, \rho_{2i-1}), \, i \in \mathbb{N}, \\ 0 & \text{for } r = 0, \end{cases} \tag{37}$$

and $\psi(r) = 2r + \psi_0(r)$. Then ψ is indeed increasing and Lipschitz continuous with Lipschitz constant 3. We now construct functions $u, \hat{u} \in G_R(0, T)$ such that the norms $|u|_{[0, T]}$, $|\hat{u}|_{[0, T]}$ can be chosen arbitrarily small, and the difference $|\mathcal{F}[u] - \mathcal{F}[\hat{u}]|_{[0, T]}$ cannot be dominated by any power of $|u - \hat{u}|_{[0, T]}$.

We fix some $n \in \mathbb{N}$, choose an arbitrary sequence $0 = t_0 < t_1 < \dots < T$, $\lim_{k \rightarrow \infty} t_k = T$, and define

$$u(t) = u_k \text{ for } t \in [t_k, t_{k+1}), \, u(T) = 0, \tag{38}$$

where

$$u_k = (-1)^{k+1} r_{2n+k} \text{ for } k = 0, 1, \dots \tag{39}$$

The function u indeed belongs to $G_R(0, T)$. We similarly define

$$\hat{u}(t) = \hat{u}_k \text{ for } t \in [t_k, t_{k+1}), \, \hat{u}(T) = 0, \tag{40}$$

where

$$\hat{u}_k = (-1)^{k+1}r_{2n+k+1} = -u_{k+1} \text{ for } k = 0, 1, \dots \tag{41}$$

Note first that $u_k - \hat{u}_k = (-1)^{k+1}(r_{2n+k} - r_{2n+k+1})$, hence

$$|u - \hat{u}|_{[0, T]} \leq r_{2n} - r_{2n+1}. \tag{42}$$

Set $\lambda(r) = \mathbf{p}_r[u](T)$, $\hat{\lambda}(r) = \mathbf{p}_r[\hat{u}](T)$ for $r \geq 0$, $\lambda'(r) = \frac{\partial^-}{\partial r} \mathbf{p}_r[u](T)$, $\hat{\lambda}'(r) = \frac{\partial^-}{\partial r} \mathbf{p}_r[\hat{u}](T)$ for $r > 0$. Then we have by (4)–(5) that

$$\lambda(r) = \begin{cases} 0 & \text{for } r \geq r_{2n}, \\ u_0 + r & \text{for } r \in [\rho_{2n}, r_{2n}), \\ u_{2i-1} - r & \text{for } r \in [\rho_{2n+2i-1}, \rho_{2n+2i-2}), \\ u_{2i} + r & \text{for } r \in [\rho_{2n+2i}, \rho_{2n+2i-1}), \\ 0 & \text{for } r = 0, \end{cases} \tag{43}$$

$$\hat{\lambda}(r) = \begin{cases} 0 & \text{for } r \geq r_{2n+1}, \\ \hat{u}_0 + r & \text{for } r \in [\rho_{2n+1}, r_{2n+1}), \\ \hat{u}_{2i-1} - r & \text{for } r \in [\rho_{2n+2i}, \rho_{2n+2i-1}), \\ \hat{u}_{2i} + r & \text{for } r \in [\rho_{2n+2i+1}, \rho_{2n+2i}), \\ 0 & \text{for } r = 0, \end{cases} \tag{44}$$

hence $\hat{\lambda}'(r) = -\lambda'(r)$ for $r \in (0, r_{2n+1}]$, $\hat{\lambda}'(r) = 0$ for $r > r_{2n+1}$ (see Figure 2).

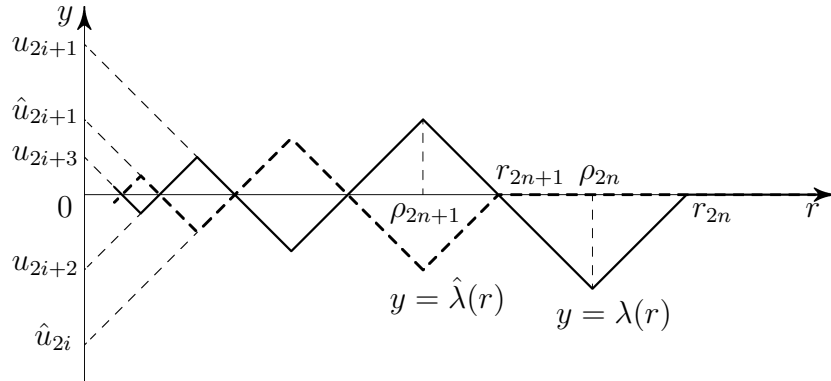


FIGURE 2. The memory curves $\lambda(r)$ (the solid line) and $\hat{\lambda}(r)$ (the dashed line).

Since ψ is Lipschitz continuous, we can rewrite (7) as

$$\mathcal{F}[u](T) = - \int_0^\infty \lambda'(r)\psi'(r) dr, \quad \mathcal{F}[\hat{u}](T) = - \int_0^\infty \hat{\lambda}'(r)\psi'(r) dr,$$

so that

$$\mathcal{F}[\hat{u}](T) - \mathcal{F}[u](T) = 2 \int_0^{r_{2n+1}} \lambda'(r)\psi'(r) dr + \int_{r_{2n+1}}^{r_{2n}} \lambda'(r)\psi'(r) dr. \tag{45}$$

We have by definition $\lambda(r_k) = 0$ for all $k = 0, 1, \dots$, so that the contribution of the linear part of ψ is zero, and we have

$$\mathcal{F}[\hat{u}](T) - \mathcal{F}[u](T) = 2 \int_0^{r_{2n+1}} \lambda'(r)\psi'_0(r) dr + \int_{r_{2n+1}}^{r_{2n}} \lambda'(r)\psi'_0(r) dr. \tag{46}$$

It follows from (37) and (43) that $\psi'_0(r) = \lambda'(r)$ for all $r \in (0, r_{2n}]$, and (46) yields that

$$\mathcal{F}[\hat{u}](T) - \mathcal{F}[u](T) \geq \int_0^{r_{2n}} (\lambda'(r))^2 dr = r_{2n}. \tag{47}$$

Let now $\alpha \in (0, 1]$ be arbitrary. By (47), (42) we have

$$\frac{\mathcal{F}[\hat{u}](T) - \mathcal{F}[u](T)}{|u - \hat{u}|_{[0, T]}^\alpha} \geq \frac{r_{2n}}{(r_{2n} - r_{2n+1})^\alpha}. \tag{48}$$

Choosing for example $r_k = \frac{1}{\log(e+k)}$, we obtain

$$\frac{r_{2n}}{(r_{2n} - r_{2n+1})^\alpha} = \frac{\log^\alpha(e + 2n + 1)}{\log^{1-\alpha}(e + 2n) \log^\alpha(1 + \frac{1}{e+2n})}. \tag{49}$$

We have $\log(1 + \frac{1}{e+2n}) \leq \frac{1}{e+2n}$, which yields

$$\frac{\mathcal{F}[\hat{u}](T) - \mathcal{F}[u](T)}{|u - \hat{u}|_{[0, T]}^\alpha} \geq \frac{(e + 2n)^\alpha \log^\alpha(e + 2n + 1)}{\log^{1-\alpha}(e + 2n)}. \tag{50}$$

We see that for each choice of $\alpha \in (0, 1]$, the right hand side of (50) can become arbitrarily large for large n , which proves that the operator \mathcal{F} is not α -Hölder continuous for any exponent $\alpha > 0$ on the unit ball $B_1(0) = \{u \in G_R(0, T) : |u|_{[0, T]} \leq 1\}$.

We use the above construction to prove the following result.

Theorem 4.1. *There exists a C^1 -function $\psi : [0, \infty) \rightarrow [0, \infty)$, $1 \leq \psi'(r) \leq 3$ for all $r \geq 0$ such that the Prandtl-Ishlinskii operator \mathcal{F} defined by (7) is not α -Hölder continuous on the unit ball $B_1(0) \subset G_R(0, T)$ for any $\alpha \in (0, 1)$.*

Proof. We construct ψ_0, u, \hat{u} as in (37)–(41). For a decreasing sequence $\{\nu_k; k \in \mathbb{N}\}$ in $(0, 1)$, $\lim_{k \rightarrow \infty} \nu_k = 0$ we put

$$\psi_1(r) := \psi_0(r) \sum_{k=1}^\infty \nu_k \chi_{[r_k, r_{k-1})}(r). \tag{51}$$

By definition (37), we have $\psi_0(r_k) = 0$ for every $k = 0, 1, \dots$, hence ψ_1 is continuous (see Figure 3). Moreover, we have

$\psi'_1(r) = (-1)^k \nu_k$ for $r \in (r_k, \rho_{k-1})$, $\psi'_1(r) = (-1)^{k-1} \nu_k$ for $r \in (\rho_{k-1}, r_{k-1})$, hence ψ'_1 is regulated, $|\psi'_1(r)| \leq 1$ for a. e. $r > 0$, and we may put $\psi'_1(0) := \psi'_1(0+) = 0$. The discontinuity points of ψ'_1 are $\{r_k; k = 0, 1, \dots\}$ and $\{\rho_k; k = 0, 1, \dots\}$, and the next step of the proof consists in smoothing the function ψ'_1 in a neighborhood of these points (see Figure 3). We set

$$\eta_{k-1} := \frac{\nu_k}{16} (r_{k-1} - r_k) \text{ for } k \in \mathbb{N} \tag{52}$$

and define

$$\psi'_2(r) := \begin{cases} \psi'_1(r) & \text{for } r \in [0, \infty) \setminus \bigcup_{k=0}^\infty ((\rho_k - \eta_k, \rho_k + \eta_k) \cup (r_k - \eta_k, r_k + \eta_k)), \\ \psi'_1(\rho_k - \eta_k) + \frac{r - (\rho_k - \eta_k)}{2\eta_k} (\psi'_1(\rho_k + \eta_k) - \psi'_1(\rho_k - \eta_k)) & \text{for } r \in (\rho_k - \eta_k, \rho_k + \eta_k), \\ \psi'_1(r_k - \eta_k) + \frac{r - (r_k - \eta_k)}{2\eta_k} (\psi'_1(r_k + \eta_k) - \psi'_1(r_k - \eta_k)) & \text{for } r \in (r_k - \eta_k, r_k + \eta_k). \end{cases}$$

Then ψ'_2 is continuous in $[0, \infty)$, $|\psi'_2(r)| \leq 1$ for all $r \geq 0$ (see Figure 3), and we may set

$$\psi_2(r) := \int_0^r \psi'_2(s) \, ds. \tag{53}$$

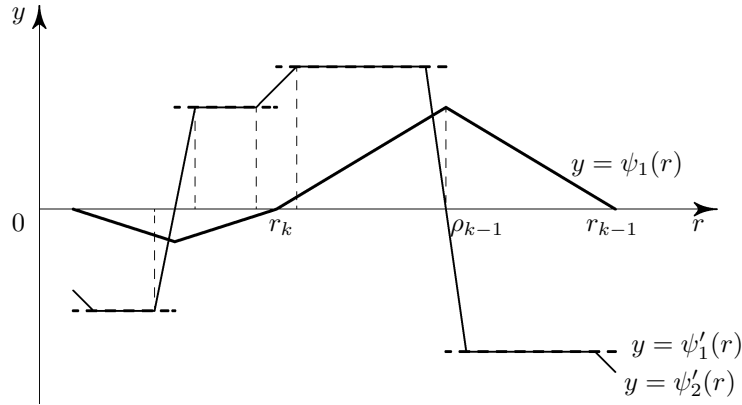


FIGURE 3. The primary response curve ψ_1 (the bold solid line), its derivative ψ'_1 (the bold dashed line), and the piecewise linear regularization ψ'_2 of ψ'_1 (the thin solid line).

With $\lambda, \hat{\lambda}$ as in (43)–(44), we have

$$\psi'_2(r) = \nu_k \lambda'(r) \text{ for } r \in (r_k + \eta_k, \rho_{k-1} - \eta_{k-1}) \cup (\rho_{k-1} + \eta_{k-1}, r_{k-1} - \eta_{k-1}) \tag{54}$$

for all $k \geq 2n + 1$. Let now \mathcal{F} be the Prandtl-Ishlinskii operator generated by the function

$$\psi(r) = 2r + \psi_2(r). \tag{55}$$

In analogy with (46) we have

$$\mathcal{F}[\hat{u}](T) - \mathcal{F}[u](T) = 2 \int_0^{T_{2n+1}} \lambda'(r) \psi'_2(r) \, dr + \int_{T_{2n+1}}^{T_{2n}} \lambda'(r) \psi'_2(r) \, dr. \tag{56}$$

We write

$$\int_{r_k}^{r_{k-1}} = \int_{r_k}^{r_k + \eta_k} + \int_{r_k + \eta_k}^{\rho_{k-1} - \eta_{k-1}} + \int_{\rho_{k-1} - \eta_{k-1}}^{\rho_{k-1} + \eta_{k-1}} + \int_{\rho_{k-1} + \eta_{k-1}}^{r_{k-1} - \eta_{k-1}} + \int_{r_{k-1} - \eta_{k-1}}^{r_{k-1}}.$$

We have (note that the sequence $\{\eta_k\}$ is decreasing)

$$\left| \left(\int_{r_k}^{r_k + \eta_k} + \int_{\rho_{k-1} - \eta_{k-1}}^{\rho_{k-1} + \eta_{k-1}} + \int_{r_{k-1} - \eta_{k-1}}^{r_{k-1}} \right) \lambda'(r) \psi'_2(r) \, dr \right| \leq 4\eta_{k-1},$$

and (54) implies that

$$\int_{r_k + \eta_k}^{\rho_{k-1} - \eta_{k-1}} \lambda'(r) \psi'_2(r) \, dr = \nu_k (\rho_{k-1} - r_k - 2\eta_{k-1}),$$

$$\int_{\rho_{k-1} + \eta_{k-1}}^{r_{k-1} - \eta_{k-1}} \lambda'(r) \psi'_2(r) \, dr = \nu_k (r_{k-1} - \rho_{k-1} - 2\eta_{k-1}),$$

so that by (52) we have for $k \geq 2n + 1$ that

$$\int_{r_k}^{r_{k-1}} \lambda'(r) \psi_2'(r) \, dr \geq \nu_k (r_{k-1} - r_k) - 8\eta_{k-1} = \frac{\nu_k}{2} (r_{k-1} - r_k), \quad (57)$$

and (56) yields

$$\mathcal{F}[\hat{u}](T) - \mathcal{F}[u](T) \geq \frac{1}{2} \sum_{k=2n+1}^{\infty} \nu_k (r_{k-1} - r_k). \quad (58)$$

This yields for $\alpha \in (0, 1]$ that

$$\frac{\mathcal{F}[\hat{u}](T) - \mathcal{F}[u](T)}{|u - \hat{u}|_{[0, T]}^\alpha} \geq \frac{\sum_{k=2n+1}^{\infty} \nu_k (r_{k-1} - r_k)}{2(r_{2n} - r_{2n+1})^\alpha}. \quad (59)$$

It remains to set

$$r_{k-1} - r_k = \frac{c}{(e+k) \log^2(e+k)} \quad (60)$$

for $k \in \mathbb{N}$ with $c > 0$ such that

$$r_0 = c \sum_{k=1}^{\infty} \frac{1}{(e+k) \log^2(e+k)} = 1,$$

and

$$\nu_k = \frac{1}{\log^\beta(e+k)} \quad (61)$$

with any $\beta > 0$. We have

$$\begin{aligned} \sum_{k=2n+1}^{\infty} \nu_k (r_{k-1} - r_k) &= c \sum_{k=2n+1}^{\infty} \frac{1}{(e+k) \log^{2+\beta}(e+k)} \\ &\geq c \int_{2n+1}^{\infty} \frac{dx}{(e+x) \log^{2+\beta}(e+x)} \\ &= \frac{c}{(1+\beta) \log^{1+\beta}(e+2n+1)}. \end{aligned}$$

Inequality (59) then yields

$$\frac{\mathcal{F}[\hat{u}](T) - \mathcal{F}[u](T)}{|u - \hat{u}|_{[0, T]}^\alpha} \geq \frac{c^{1-\alpha}}{2(1+\beta)} (e+2n+1)^\alpha \log^{2\alpha-1-\beta}(e+2n+1) \quad (62)$$

which tends to ∞ when n tends to infinity. This concludes the proof. \square

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Received November 2016; revised April 2017.

E-mail address: liuw626@hhu.edu.cn

E-mail address: krejci@math.cas.cz

E-mail address: yegj@hhu.edu.cn