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**Asymptotic behaviour of the Weyl tensor
in higher dimensions**

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Abstract

We determine the leading order fall-off behaviour of the Weyl tensor in higher dimensional Einstein spacetimes (with and without a cosmological constant) as one approaches infinity along a congruence of null geodesics. The null congruence is assumed to “expand” in all directions near infinity (but it is otherwise generic), which includes in particular asymptotically flat spacetimes. In contrast to the well-known four-dimensional peeling property, the fall-off rate of various Weyl components depends substantially on the chosen boundary conditions, and is also influenced by the presence of a cosmological constant. The leading component is always algebraically special, but in various cases it can be of type N, III or II.

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1 Introduction

The study of isolated systems in general relativity is based on the analysis of asymptotic properties of spacetimes. Under certain assumptions, this enables one to define physical quantities such as mass, angular momentum and energy flux. In particular, properties of gravitational radiation can be determined by considering the spacetime behaviour “far away” along a geodesic null congruence.

In four dimensions, the Weyl tensor decay is described by the well-known peeling property, i.e., components of boost weight (b.w.) w fall off as $1/r^{w+3}$ (where $w = \pm 2, \pm 1, 0$, and the $1/r$ term characterizes radiative fields). This result was obtained by coordinate-based approaches that studied Einstein’s vacuum equations assuming suitable asymptotic “outgoing radiation” conditions, which were formulated in terms either of the metric coefficients [1, 2] or directly of the Weyl tensor [3, 4] (see [5–7] for early results in special cases). From a more geometrical viewpoint, the peeling-off behaviour also naturally follows from Penrose’s conformal definition of asymptotically simple spacetimes (which also allows for a cosmological constant) [8, 9], at least under suitable smoothness conditions on the conformal geometry (see also [10]).

In an n -dimensional spacetime, the definition of asymptotic flatness at null infinity (along with the “news” tensor and Bondi energy-momentum) using a conformal method turns out to be sound only for even n [11] (see also [12]) – linear gravitational perturbation of the metric tensor typically decays as $r^{-(n/2-1)}$ and the unphysical (conformal) metric is thus not smooth at null infinity if n is odd (see [13] for further results for even n). In [14], linear (vacuum) perturbations of Minkowski spacetime were studied in terms of the Weyl tensor, which was found to decay as $r^{-(n/2-1)}$, thus again non-smoothly in odd dimensions.¹ Ref. [14] also pointed out a qualitative difference between $n = 4$ and $n > 4$ in the decay properties of various Weyl components at null infinity and related this to a possible new peeling behaviour when $n > 4$. This expectation was indeed confirmed in the full theory in [15] by studying the Bondi-like metric defined in [16, 17] (also mentioned in [11, 12]) and thus an expansion of the Weyl tensor along the generators of a family of outgoing null hypersurfaces. Not only was the $r^{-(n/2-1)}$ result of [14] recovered at the leading order but at higher orders a new structure of the r -dependence of various Weyl components was also obtained [15]. For odd n , an extra condition on the asymptotic metric coefficients was needed in [15] (see also [16]), in relation to the simultaneous appearance of integer and semi-integer powers in the expansions. (Note that the analysis of [15] includes not only vacuum spacetimes but also possible matter fields which decay “fast enough” at infinity, cf. [15] for details.)

The present contribution studies the asymptotic behaviour of the Weyl tensor in higher dimensional *Einstein spacetimes* ($R_{ab} = \frac{R}{n}g_{ab}$) under more general boundary conditions, for which a different method seems to be more suitable. The basic idea is still to evaluate the Weyl components in a frame parallelly transported along a congruence of “outgoing” null geodesics, affinely parametrized by r (the congruence is rather “generic” and not assumed to be hypersurface orthogonal – its precise properties will be specified

¹In the present paper we discuss the *physical* Weyl tensor only, so here we have accordingly rephrased the results of [14] (where the *unphysical* Weyl tensor of the conformal spacetime was instead considered).

in section 2.1 below). However, on the lines of the classic 4D work [3], we do not make assumptions on the spacetime metric but work directly with the Weyl tensor, in the framework of the higher dimensional Newman-Penrose (NP) formalism [18–22] (we follow the notation of the review [22] and we do not repeat here the definitions of all the symbols). This permits a unified study for both even and odd dimensions and with little extra effort it also allows for a possible cosmological constant. In the case of asymptotically flat spacetimes the Bianchi equations naturally give the “ $r^{-(n/2-1)}$ -result” for the leading Weyl components (see (2) below), as previously obtained with the methods of [14, 15]. In addition to this special case, a complete pattern of possible fall-off behaviours both with (sections 3.1.4, 3.2, 3.3) and without (sections 4.1.4, 4.2, 4.3) a cosmological constant is presented. The precise fall-off for a specific spacetime will be determined by a choice of “boundary condition” at null infinity. These are naturally specified by first fixing a bound on the decay rate of b.w. +2 Weyl components Ω_{ij} (which we will assume to be faster than $1/r^2$), as in four dimensions. However, while in 4D only the fall-off $\Omega_{ij} = O(r^{-5})$ needs to be assumed (and then the standard peeling result follows [3]),² for $n > 4$ the r -dependence of the remaining Weyl components will still be partially undetermined and various possible choices of boundary conditions for lower b.w. components will lead to different fall-off behaviours. More specifically, how such numerous cases (and subcases) arise can be better understood by observing that the Weyl components containing arbitrary integration “constants” are Ψ_{ijk} (at order $1/r^n$ or $1/r^3$) and, for $n > 5$, Φ_{ijkl} (at order $1/r^2$). This will be worked out in the paper.³

Certain cases of physical interest (including asymptotically (A)dS and asymptotically flat spacetime) arise when we set to zero the terms of order $1/r^3$ in Ψ_{ijk} and $1/r^2$ in Φ_{ijkl} . For $R \neq 0$, then we obtain that necessarily $\Omega_{ij} = O(r^{-1-n})$ (or faster), and the fall-off generically is (see (67))

$$\begin{aligned}
\Omega_{ij} &= O(r^{-1-n}), \\
\Psi_{ijk} &= O(r^{-n}), \\
\Phi_{ijkl} &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}) \quad (R \neq 0), \\
\Psi'_{ijk} &= O(r^{2-n}), \\
\Omega'_{ij} &= O(r^{3-n}),
\end{aligned} \tag{1}$$

where components are ordered by decreasing b.w.. Under the same assumptions, more possibilities arise for a vanishing cosmological constant, depending more substantially on the precise fall-off prescribed for Ω_{ij} . In particular, if Ω_{ij} falls faster than $1/r^{n/2}$ but not faster than $1/r^{n/2+1}$ we have (cf. (94) and the discussion after it)

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) \quad \left(\frac{n}{2} < \nu \leq 1 + \frac{n}{2} \right), \\
\Psi_{ijk} &= O(r^{-\nu}), \\
\Phi_{ijkl} &= O(r^{-n/2}), \quad \Phi = O(r^{-\nu}), \quad \Phi_{ij}^A = O(r^{-\nu}), \quad (R = 0), \\
\Psi'_{ijk} &= O(r^{-n/2}), \\
\Omega'_{ij} &= O(r^{1-n/2}).
\end{aligned} \tag{2}$$

This includes the behaviour found in [15] for asymptotically flat radiative spacetimes. The radiative term vanishes if $\nu > 1 + \frac{n}{2}$, in which case the fall-off is completely different (e.g., it is given by (105) for $\nu > n$, but other cases are also possible, see section 4 for details). On the other hand, if Ω_{ij} falls as $1/r^{n/2}$ or slower, one finds instead the behaviour (99) (with $\nu > 3$). Both (1) and (2) are qualitatively different from the corresponding results (69) and (107) for the 4D case (apart from Φ_{ij}^A , (1) with $n = 4$ would look the same as (69) but see comments in the following sections).

²The Ω_{ij} components of the n -dimensional notation correspond to the NP scalar Ψ_0 in 4D.

³To be precise, by “arbitrary integration constants” we refer to r -independent quantities that generically may still depend on coordinates different than r . Additionally, (some of) these may be “arbitrary” only at the level of the r -integration of the (asymptotic) NP equations – the remaining “transverse” NP equations would in fact play a role of “constraint equations”. This is of course important for a full analysis of the characteristic initial value problem but goes beyond the scope of this paper and will not be discussed in the following (for details in 4D see [4] and, e.g., the review [23]).

More general asymptotia can also be of physical interest and the corresponding fall-off properties are given in the paper. Let us just mention here, for example, that a non-zero term of order $1/r^2$ in Φ_{ijkl} may correspond, e.g., to black holes living in generic Einstein spacetimes (this is manifest in the case of static black holes from the Weyl r -dependence given in [24]). Although here we restrict to Einstein spacetimes, several results can presumably be easily extended to include matter fields that fall-off “sufficiently” fast (cf. [15]). The method employed here can also be similarly applied to more general contexts such as the coupled Einstein-Maxwell equations, which we leave for future work. We further note that previous results concerning the (exact) r -dependence of the Weyl tensor for algebraically special Einstein spacetimes include [24–29].

On the invariance of the results

Chosen a null direction ℓ , the results we will present hold in a “generic” parallelly transported frame. One may thus wonder if the behaviour we find is frame-dependent. Similarly as in four dimensions, the answer follows from transformation properties of various Weyl components under null rotations about ℓ , i.e.,

$$\hat{\ell} = \ell, \quad \hat{\mathbf{n}} = \mathbf{n} + z_i \mathbf{m}_i - \frac{1}{2} z_i z^i \ell, \quad \hat{\mathbf{m}}_i = \mathbf{m}_i - z_i \ell. \quad (3)$$

Two different parallelly transported frames are related by a transformation (3) (apart from trivial spatial rotations) with the parameters z_i being r -independent [20]. Under (3), the change of a Weyl component of a given b.w. w is simply a term linear in components of b.w. *smaller* than w , with coefficients determined by the z_i (see, e.g., eqs. (2.27)–(2.35) of [21]). It thus follows, in particular, that at the leading order (when $r \rightarrow \infty$) a certain Weyl component will be unchanged *if* all Weyl components of lower boost weight decay faster. This is always the case, for instance, for the b.w. -2 components Ω'_{ij} when the leading order term is of type N. Therefore, this observation will apply to several of the results of this paper, most notably to the radiative behaviour (2) (or (94)), in which case the leading Weyl component can be related to the Bondi flux [15]. By Contrast, when higher order terms are not invariant in the sense just discussed, a transformation (3) can be used to pick up preferred frames, which may simplify certain expressions and be useful for particular applications (see, e.g., [30] in the case of algebraically special spacetimes). This freedom will not be used here since we are interested in the asymptotic behaviour in a generic parallelly transported frame.

Assumptions and notation

In this paper, we are interested in determining the *leading* order r -dependence of the Weyl tensor of Einstein spacetimes, while a systematic study of subleading terms and the analysis of asymptotic solutions of the NP equations is left for future work (several results have been already obtained in the case of algebraically special spacetimes [30]). For this reason we will not need to assume that the NP quantities (Weyl tensor, Ricci rotation coefficients, derivative operators) admit a series expansion. However, we will assume that for large r the leading terms of those quantities have a power-like behaviour (so that for our purposes the notation $f = O(r^{-\zeta})$ will effectively mean $f \sim r^{-\zeta}$), where the powers will not be restricted to be integer numbers. We will also assume that if $f = O(r^{-\zeta})$ then $\partial_r f = O(r^{-\zeta-1})$ and $\partial_A f = O(r^{-\zeta})$ (where ∂_A denote a derivative w.r.t. coordinates x^A different from r and that need not be further specified for our purposes). In a few cases it will be useful to consider subleading terms of some expressions (most importantly (10)), and it will be understood that those are also assumed to be power-like.

Although we are not interested in giving the full set of asymptotic field equations, in some cases it will be useful to display relations among the leading terms of certain Weyl components. For a generic frame Weyl component “ f ” we thus define the notation

$$f = \frac{f^{(\zeta)}}{r^\zeta} + o(r^{-\zeta}), \quad (4)$$

where $f^{(\zeta)}$ does not depend on r (so that we will have, e.g., $\Phi_{ij}^S = \Phi_{ij}^{S(n-1)} r^{1-n} + o(r^{1-n})$, or $\Psi_{ijk} =$

$\Psi_{ijk}^{(3)}r^{-3} + o(r^{-3})$, etc.). For the Ricci rotation coefficients we will instead denote r -independent quantities by lowercase latin letters, e.g., $L_{1i} = l_{1i}r^{-1} + o(r^{-1})$, $M_{j1}^i = m_{j1}^i + o(1)$, etc..

Many of the equations will take a more compact form using the rescaled Ricci scalar

$$\tilde{R} = \frac{R}{n(n-1)}. \quad (5)$$

We will be interested in the asymptotic behaviour along a geodesic null congruence with an affine parameter r and tangent vector field ℓ . Calculations will be performed in a frame $(\ell, \mathbf{n}, \mathbf{m}_i)$ (with $i, j, k, \dots = 2, \dots, n-1$) which is parallelly transported along ℓ . The above assumptions imply the vanishing of the following Ricci rotation coefficients (cf. [22] for more details on the notation)

$$\kappa_i = 0 = L_{10}, \quad M_{j0}^i = 0, \quad N_{i0} = 0. \quad (6)$$

Directional derivatives along the frame vectors $(\ell, \mathbf{n}, \mathbf{m}_i)$ will be denoted, respectively, by D , δ_i , and Δ .

Section 2.1 and the first parts of sections 3.1 and 4.1 are devoted to results on the Ricci rotation coefficients, to preliminary analysis of the Weyl tensor and to setting up the method. Readers not interested in those details can jump to the summary of the results for the Weyl tensor in sections 3.1.4, 3.2, 3.3 ($\tilde{R} \neq 0$) and 4.1.4, 4.2, 4.3 ($\tilde{R} = 0$). For comparison, four dimensional results are also reproduced in the various cases and given in (62), (69) ($\tilde{R} \neq 0$) and (98) (100), (107), ($\tilde{R} = 0$).

2 Boundary conditions and Ricci rotation coefficients

In this section, we explain our assumptions on the asymptotic behaviour of ℓ and of the Weyl tensor components of b.w. +2, and use those to fix the leading order behaviour of the Ricci rotation coefficients and derivative operators (both for $\tilde{R} \neq 0$ and $\tilde{R} = 0$). It will also follow that subsequent analysis will need to consider three different choices of boundary conditions on the Weyl components of b.w. +1, which we will do in later sections.

2.1 Sachs equation and optical matrix

In the frame $(\ell, \mathbf{n}, \mathbf{m}_i)$ (see above), the optical matrix of $\ell = \partial_r$ is given by

$$\rho_{ij} = l_{a;b} m_{(i}^a m_{j)}^b. \quad (7)$$

From now on, we assume that ρ_{ij} is *asymptotically non-singular and expanding*, i.e., the leading term of ρ_{ij} (for large r) is a matrix with non-zero determinant and non-zero trace. Roughly speaking, this means that near infinity ℓ expands in all spacelike directions at the same speed, which is compatible, in particular, with asymptotically flat spacetimes (as follows from [15–17] – however we will see in the following that this assumptions holds also in more general spacetimes).

Next, one needs to specify the speed at which the Weyl tensor tends to zero for $r \rightarrow \infty$. In general, we will make only the following rather weak assumption for the fall-off for the b.w. +2 components of the Weyl tensor

$$\Omega_{ij} = O(r^{-\nu}), \quad \nu > 2, \quad (8)$$

although, in most cases of interest, ν will in fact be larger, as we will show (recall that in four dimensions the existence of a smooth null infinity requires $\nu \geq 5$ [3, 8–10]).

With the assumptions listed above the Sachs equation reads $D\rho_{ij} = -\rho_{ik}\rho_{kj} - \Omega_{ij}$ (cf. (11g, [20])), from which one finds⁴

$$\rho_{ij} = \frac{\delta_{ij}}{r} + o(r^{-1}). \quad (9)$$

⁴Another solution is $\rho_{ij} = O(r^{-\nu+1})$ (for $\nu > 2$), which however gives an asymptotically non-expanding optical matrix (since Ω_{ij} is traceless), contrary to our assumptions.

In general, it is easy to see from (11g, [20]) that Ω_{ij} will affect ρ_{ij} at order $O(r^{-\nu+1})$. At all lower orders, the r -dependence of ρ_{ij} is given by negative integer powers of r , which can be fixed recursively as done (to arbitrary order) in [30]. Thus, for example, if $\nu > 3$ (which will indeed occur in several cases discussed in the following) one has

$$\rho_{ij} = \frac{\delta_{ij}}{r} + \frac{b_{ij}}{r^2} + o(r^{-2}) \quad (\nu > 3), \quad (10)$$

where the subleading term contains an arbitrary “integration matrix” b_{ij} independent of r . Note that when ℓ is twistfree then $b_{[ij]} = 0$ (the viceversa is also true if ℓ is a WAND [27]).

2.2 Derivative operators and commutators

Taking r as one of the coordinates we can write

$$D = \partial_r, \quad \Delta = U\partial_r + X^A\partial_A, \quad \delta_i = \omega_i\partial_r + \xi_i^A\partial_A, \quad (11)$$

where $\partial_A = \partial/\partial x^A$ and the x^A represent any set of $(n-1)$ scalar functions such that (r, x^A) is a well-behaved coordinate system. From the commutators [19]

$$\Delta D - D\Delta = L_{11}D + L_{i1}\delta_i, \quad (12)$$

$$\delta_i D - D\delta_i = L_{1i}D + \rho_{ji}\delta_j, \quad (13)$$

we obtain the differential equations (cf. also [30])

$$D\omega_i = -L_{1i} - \rho_{ji}\omega_j, \quad (14)$$

$$D\xi_i^A = -\rho_{ji}\xi_j^A, \quad (15)$$

$$DU = -L_{11} - L_{i1}\omega_i, \quad (16)$$

$$DX^A = -L_{i1}\xi_i^A. \quad (17)$$

Using (9), eq. (15) gives

$$\xi_i^A = O(r^{-1}). \quad (18)$$

Similarly as mentioned above for ρ_{ij} , Ω_{ij} will affect ξ_i^A at order $O(r^{-\nu+1})$.

In order to fix the full r -dependence of the derivative operators we also need to study the behaviour of the Ricci rotation coefficients of b.w. 0 and -1. However, the corresponding differential equations will in turn involve also Weyl components of b.w. +1 and 0, respectively, and thus one has to consider the set of the “ D ”-Ricci identities of b.w. b simultaneously with the “ D ”-Bianchi identities of b.w. $(b+1)$ (for $b=+1,0,-1,-2$).

2.3 Ricci rotation coefficients of b.w. 0 and Weyl components of b.w. +1

We need to study (11b, [20]), (11e, [20]), (11n, [20]) and (B8, [18]), along with (14), (17). One starts by assuming a generic behaviour for large r for each of the “unknowns” (e.g., $L_{1i} = O(r^\alpha)$, where α need not be specified a priori). By combining conditions coming from all the considered equations one can constraint such leading terms. For example, from (11b, [20]) it is easy to see that one can only have either

$$L_{1i} = O(r^{-1}), \quad \Psi_i = o(r^{-2}), \quad (19)$$

or

$$L_{1i} = O(r^\alpha), \quad \Psi_i = O(r^{\alpha-1}) \quad (\alpha \neq -1). \quad (20)$$

Working out similar conditions for other quantities from (11n, [20]), (B8, [18]) and (14) and requiring compatibility of all such conditions one concludes that

$$L_{1i} = O(r^{-1}), \quad M_{jk}^i = O(r^{-1}), \quad \omega_i = O(1), \quad (21)$$

where it is understood that for $r \rightarrow \infty$ all terms can go to zero faster than indicated, in special cases. However, we will consider only the generic case, in which this does not happen. For the Weyl tensor components of positive b.w. there are three possibilities:

$$\text{i) } \Psi_{ijk} = O(r^{-\nu}), \quad \Omega_{ij} = O(r^{-\nu}) \quad (\nu > 2),$$

where $\Psi_{ijk}^{(-\nu)}$ can be expressed in terms of $\Omega_{ij}^{(-\nu)}$ using (B8, [18]) (except when $\nu = 3, n$). For $\nu > 3$, this case sets the boundary condition $\Psi_{ijk}^{(3)} = 0$, and for $\nu > n$ also $\Psi_{ijk}^{(n)} = 0$. It includes both the case when ℓ is a *multiple* WAND (in the formal limit $\nu \rightarrow +\infty$) and asymptotically flat radiative spacetimes in higher dimensions (as we will discuss in the following, cf. [15]).

$$\text{ii) } \Psi_{ijk} = O(r^{-n}), \quad \Omega_{ij} = o(r^{-n}),$$

with $(n-3)\Psi_{ijk}^{(n)} = 2\Psi_{[j}^{(n)}\delta_{k]i}$. This case corresponds to the boundary condition $\Psi_{ijk}^{(3)} = 0$, $\Psi_{ijk}^{(n)} \neq 0$. It is compatible with the four-dimensional results of [3, 8, 9] (where $\nu = 5$) for $n = 4$.

$$\text{iii) } \Psi_{ijk} = O(r^{-3}), \quad \Psi_i = o(r^{-3}), \quad \Omega_{ij} = O(r^{-\nu}) \quad (n > 4, \nu > 3),$$

with $\Psi_i = O(r^{-\nu})$ if $3 < \nu \leq 4$, and (using (10)) $\Psi_i = O(r^{-4})$ if $\nu > 4$ (in both cases the leading term of Ψ_i can be determined by the trace of (B8, [18])). This case corresponds to the boundary condition $\Psi_{ijk}^{(3)} \neq 0$. It is not permitted in 4D since $\Psi_i = 0 \Leftrightarrow \Psi_{ijk} = 0$ there [22] and can not be asymptotically flat, cf. [15].

Only cases (ii) and (iii) are permitted if one assumes that asymptotically Ψ_{ijk} goes to zero more slowly than Ω_{ij} .

Furthermore, from (11e, [20]) we have

$$L_{i1} = O(r^{-1}), \tag{22}$$

which with (17) gives

$$X^A = X^{A0} + O(r^{-1}). \tag{23}$$

When the fall-off condition $\nu > 3$ is assumed, thanks to (10) we can strengthen the above results and those of section 2.2 for the derivative operator as follows (assuming that each quantity has a power-like behaviour also at the subleading order):

$$L_{1i} = \frac{l_{1i}}{r} + O(r^{-2}), \quad L_{i1} = \frac{l_{i1}}{r} + O(r^{-2}), \quad M_{jk}^i = \frac{m_{jk}^i}{r} + O(r^{-2}), \tag{24}$$

$$\xi_i^A = \frac{\xi_i^{A0}}{r} + O(r^{-2}), \quad \omega_i = -l_{1i} + O(r^{-1}) \quad (\nu > 3). \tag{25}$$

This will be useful in the following since many cases of interest have indeed $\nu > 3$. Note that using null rotations (3) one can always choose a parallelly transported frame such that, e.g., $l_{1i} = 0$ or $l_{i1} = 0$. This may be convenient for particular computations but for the sake of generality we will keep our frame unspecified.

At this stage, knowing the r -dependence of the derivative operators at the leading order (eq. (11) with (18), (21), (23) and (33) or (34)) of course means also knowing the leading order terms of the spacetime metric (however, to explicitly connect the metric and the Weyl tensor we would need to study higher order terms). In the following, we will analyze in detail the above case (i) (sections 2.4, 3.1, 4.1). For cases (ii) and (iii), we will only summarize the main results (sections 3.2, 3.3, 4.2 and 4.3) without giving intermediate steps since the method to obtain those is essentially the same as for case (i).

2.4 Ricci rotation coefficients of b.w. -1 and Weyl components of b.w. 0: derivation for case (i)

The next step consists in the study of (11a, [20]), (11j, [20]), (11m, [20]), (B5, [18]), (B12, [18]) and (16), also using the results of section 2.3 above. It is convenient to start from (11j, [20]) and (B12, [18]) (since these do not contain L_{11} , M_{j1}^i and U). Let us first focus on (11j, [20]) and consider the leading order behaviour of the following quantities

$$N_{ij} = O(r^\alpha), \quad \Phi_{ij} = O(r^\beta). \tag{26}$$

By inspecting (11j, [20]) we arrive at the following possibilities:

1. For $\tilde{R} \neq 0$:

- (a) $\alpha = 1, \beta < 0$, with $N_{ij} = -\frac{\tilde{R}}{2}\delta_{ij}r + o(r)$
- (b) $\alpha < 1, \beta = 0$, with $\Phi_{ij} = -\tilde{R}\delta_{ij} + o(1)$
- (c) $\alpha \geq 1, \beta = \alpha - 1$.

2. For $\tilde{R} = 0$:

- (a) $\alpha = -1, \beta < -2$, with $N_{ij} = O(r^{-1})$
- (b) $\alpha \geq 1, \beta = \alpha - 1$
- (c) $\alpha < 1, \alpha \neq -1, \beta = \alpha - 1$.

Let us also define the leading order behaviour of

$$\Phi_{ijkl} = O(r^{\beta_c}). \quad (27)$$

Now, in general, the leading order term of eq. (B12, [18]) can be of order $O(r^{\beta_c-1}), O(r^{\beta-1}), O(r^{\alpha-\nu}), O(r^{-\nu-1})$, depending on the relative value of the parameters $\alpha, \beta_c, \beta, \nu$ (recall that here we are restricting to case (i): $\Psi_{ijk} = O(r^{-\nu}), \Omega_{ij} = O(r^{-\nu})$). It is easy to see that in the above cases (1b), (1c) and (2b) the leading term is either $O(r^{\beta_c-1})$ or $O(r^{\beta-1})$ (with possibly $\beta = \beta_c$). However, studying (B12, [18]) at the leading order reveals that such cases (1b), (1c) and (2b) are in fact forbidden, since they all have $\beta \geq 0$. Additionally, it shows that in case (2c) one has a stronger restriction $\alpha < -1$ (for $n = 4$ eq. (B5, [18]) is also needed). In the permitted cases, we can thus in general conclude

$$N_{ij} = -\frac{\tilde{R}}{2}\delta_{ij}r + o(r) \quad \text{if } \tilde{R} \neq 0, \quad (28)$$

$$N_{ij} = O(r^{-1}) \quad \text{if } \tilde{R} = 0. \quad (29)$$

Note also that in all the permitted cases we have $\beta < 0$. This enables us to use (11a, [20]) to readily arrive at

$$L_{11} = \tilde{R}r + o(r) \quad \text{if } \tilde{R} \neq 0, \quad (30)$$

$$L_{11} = l_{11} + o(1) \quad \text{if } \tilde{R} = 0, \quad (31)$$

while (11m, [20]) gives

$$M_{j1}^i = O(1), \quad (32)$$

and (16) leads to

$$U = -\frac{\tilde{R}}{2}r^2 + o(r^2) \quad \text{if } \tilde{R} \neq 0, \quad (33)$$

$$U = -l_{11}r + o(r) \quad \text{if } \tilde{R} = 0. \quad (34)$$

Thanks to the above discussion we can now study the consequences of (B12, [18]), as well as those of (B5, [18]), more systematically. Clearly, from now on it will be necessary to distinguish case 1. ($\tilde{R} \neq 0$) from case 2. ($\tilde{R} = 0$).

3 Case $\tilde{R} \neq 0$

3.1 Case (i): $\Psi_{ijk} = O(r^{-\nu}), \Omega_{ij} = O(r^{-\nu})$ ($\nu > 2$)

3.1.1 Weyl components of b.w. 0

At the leading order of (B12, [18]), we can have only (some of) the terms $O(r^{\beta_c-1})/O(r^{\beta-1}), O(r^{1-\nu})$. (From now on, it will be understood that Φ_{ij}^S and Φ have the same behaviour as Φ_{ijkl} , except when stated otherwise.)

I. If $1 - \nu > \beta_c - 1$ and $1 - \nu > \beta - 1$, eq. (B12, [18]) shows that necessarily $n = 4$ and (B5, [18]) then gives $\nu = 5$. It also turns out that then $\beta_c = \beta = -4$, so that here we can thus have only

$$\Phi_{ijkl} = O(r^{-4}), \quad \Phi_{ij}^A = O(r^{-4}), \quad \Omega_{ij} = O(r^{-5}) \quad (n = 4). \quad (35)$$

II. In all remaining cases, at least one of the terms $O(r^{\beta_c-1})$, $O(r^{\beta-1})$ must appear at the leading order in (B12, [18]). Combing this with (B5, [18]), after some calculations and depending on the value of ν (and of n) one arrives at the following possible behaviours:

(a) $\beta_c = -2$, $\nu = 4$:

$$\Phi_{ijkl} = O(r^{-2}), \quad \Phi = o(r^{-2}), \quad \Phi_{ij}^A = o(r^{-2}), \quad \Omega_{ij} = O(r^{-4}) \quad (n > 4). \quad (36)$$

with $\Phi_{ij}^{S(2)} = \frac{\tilde{R}}{2}\Omega_{ij}^{(4)}$. Since in 4D $\Phi_{ij}^S \propto \delta_{ij}$, this case is permitted only for $n > 4$.

(b) $\beta_c = -2$, $\nu > 4$: it follows from the last remark that here Φ_{ij}^S becomes subleading. It turns out (by comparing (B5, [18]) with the trace of (B12, [18])) that the range $4 < \nu < 5$ is forbidden and we can identify three possible subcases, i.e.,

$$\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = o(r^{-3}), \quad \Phi_{ij}^A = O(r^{-3}), \quad \Omega_{ij} = O(r^{-5}) \quad (n > 5), \quad (37)$$

$$\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = O(r^{1-n}), \quad \Phi_{ij}^A = o(r^{1-n}), \quad \Omega_{ij} = O(r^{-n-1}) \quad (n > 5), \quad (38)$$

$$\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = O(r^{2-\nu}), \quad \Phi = o(r^{2-\nu}), \quad \Phi_{ij}^A = o(r^{2-\nu}), \quad \Omega_{ij} = O(r^{-\nu}) \quad (n > 5, \nu > 5, \nu \neq n + 1). \quad (39)$$

Here, $n > 5$ since in 4D and 5D one has $\Phi_{ijkl} = 0 \Leftrightarrow \Phi_{ij}^S = 0$ [31]. In (37), the ((anti)symmetric parts of the) trace of (B12, [18]) (using (10)) give $\Phi_{ijkl}^{(2)} b_{(jl)} = -\frac{\tilde{R}}{2}(n-4)\Omega_{ik}^{(5)}$ and $(n-4)\Phi_{ij}^{A(3)} = \Phi_{ikjl}^{(2)} b_{[kl]}$ (while $\Phi_{ijkl}^{(2)} b_{(jl)} = 0 = \Phi_{ikjl}^{(2)} b_{[kl]}$ in (38) and (39)). In (37) and (38), Ω_{ij} can go to zero faster than indicated, while $(3-\nu)\Phi_{ij}^{S(\nu-2)} = \frac{\tilde{R}}{2}\Omega_{ij}^{(\nu)}$ ($\nu-5$) in (39) (as obtained from (B5, [18])). In (38), one finds $(2-n)\Phi_{ij}^{S(n-1)} + \Phi^{(n-1)}\delta_{ij} = \frac{\tilde{R}}{2}(n-4)\Omega_{ij}^{(n+1)}$.

(c) $\beta_c = 1 - n$: there is a difference between $n > 4$ and $n = 4$, i.e.,

$$\text{if } n > 4: \quad \Phi_{ijkl} = O(r^{1-n}), \quad \Phi_{ij}^A = o(r^{1-n}), \quad \Omega_{ij} = O(r^{-n-1}), \quad (40)$$

$$\text{if } n = 4: \quad \Phi_{ijkl} = O(r^{-3}), \quad \Phi_{ij}^A = O(r^{-3}), \quad \Omega_{ij} = O(r^{-5}), \quad (41)$$

with $\Phi_{ijkl}^{(3)} = 2\Phi^{(3)}\delta_{j[k}\delta_{l]i}$ for $n = 4$ and $(n-2)(n-3)\Phi_{ijkl}^{(n-1)} = 4\Phi^{(n-1)}\delta_{j[k}\delta_{l]i} - 2(n-3)\tilde{R}(\Omega_{j[k}^{(n+1)}\delta_{l]i} - \Omega_{i[k}^{(n+1)}\delta_{l]j})$ (which implies $(2-n)\Phi_{ij}^{S(n-1)} + \Phi^{(n-1)}\delta_{ij} = \frac{\tilde{R}}{2}(n-4)\Omega_{ij}^{(n+1)}$) for $n > 4$. Note the different behaviour of the ‘‘magnetic’’ term Φ_{ij}^A . In both cases it is understood that Ω_{ij} can go to zero faster (or even vanish identically – for $n = 4$ if $\nu > 5$ then necessarily $\nu \geq 6$). In (41), both Φ_{ijkl} and Φ_{ij}^A can go to zero faster than indicated. The result of (35) can thus be understood as a subcase of (41).

We have not given explicitly the behaviour of Ψ_{ijk} in all the above cases since it always follows from point (i) of section 2.3. Note that not all values of ν are permitted. In particular, although we started from the weak assumption $\nu > 2$, in the end we always have either $\nu = 4$ or $\nu \geq 5$. Thanks to (10) this enables us to specialize (28) to

$$N_{ij} = -\frac{\tilde{R}}{2}\delta_{ij}r + \frac{\tilde{R}}{2}b_{ij} + o(1). \quad (42)$$

Additionally, since in all permitted cases we have $\Phi = o(r^{-2})$ and $\Phi_{ij}^A = o(r^{-2})$ (or faster), eqs. (30), (33) and (32) can be specialize as

$$L_{11} = \tilde{R}r + l_{11} + O(r^{-1}), \quad (43)$$

$$U = -\frac{\tilde{R}}{2}r^2 - l_{11}r + O(1), \quad (44)$$

$$M_{j1}^i = m_{j1}^i + O(r^{-1}). \quad (45)$$

Using (42) in (B5, [18]), one is now able to refine all the “ o ” symbols in eqs. (36), (38), (39) and (40) (but not in (38)) by appropriate “ O ” symbols (e.g. $\Phi = o(r^{-2})$ in (36) can be replaced by $\Phi = O(r^{-3})$, etc). This will be taken into account explicitly in a summary in section 3.1.4.

3.1.2 Ricci rotation coefficients of b.w. -2 and Weyl components of b.w. -1

Let us analyse (11f, [20]) and (B6, [18]), (B9, [18]) and (B1, [18]) in all the possible cases listed above, where we note that always $\nu \geq 4$ (useful for the next comment). First, let us observe from (B9, [18]) that if Ψ'_{ijk} goes to zero more slowly than Φ_{ij} then necessarily it goes to zero as $O(r^{-2})$ (or faster). On the other hand, if Ψ'_{ijk} does *not* go to zero more slowly than Φ_{ij} , we also conclude $\Psi'_{ijk} = O(r^{-2})$ (or faster) since $\Phi_{ij} = O(r^{-2})$ (or faster) in all permitted cases. Thus, we always have $\Psi'_{ijk} = O(r^{-2})$ (or faster), which enables one to use (11f, [20]) (together with the second of (24) and (42)) to arrive at

$$N_{i1} = \frac{\tilde{R}}{2} l_{i1} r + O(1). \quad (46)$$

Thanks to this result we can now employ (B6, [18]) together with (B9, [18]) and arrive at the following results (where the various points are “numbered” so as to correspond to those of section 3.1.1). From now on, it will be understood that Ψ'_i has the same behaviour as Ψ'_{ijk} , except when stated otherwise.

(a) $\Psi'_{ijk} = O(r^{-2})$,

with $\Psi_i^{(2)} = -\frac{\tilde{R}}{2} \Psi_i^{(4)}$ and $\Psi_{ijk}^{(2)}$ can be expressed in terms of $\Omega_{ij}^{(4)}$ and $\Phi_{ijkl}^{(2)}$ using (B6, [18]) (recall that $\Psi_{ijk}^{(4)}$ and its trace $\Psi_i^{(4)}$ can be expressed in terms of $\Omega_{ij}^{(4)}$, as observed in section 2.3).

(b) For the three subcases we find, respectively,

$$\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{-3}), \quad (47)$$

$$\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{1-n}), \quad (48)$$

$$\Psi'_{ijk} = O(r^{-2}), \quad \Psi'_i = O(r^{2-\nu}), \quad (49)$$

with $\Psi_{ijk}^{(2)} = -\Phi_{isjk}^{(2)} l_{s1}$ and where the behaviour of Ψ'_i has been obtained using (B1, [18]).

(c)

$$\text{if } n > 4 : \quad \Psi'_{ijk} = O(r^{1-n}), \quad (50)$$

$$\text{if } n = 4 : \quad \Psi'_{ijk} = O(r^{-2}). \quad (51)$$

3.1.3 Weyl components of b.w. -2

To conclude, let us study (B4, [18]). It will be also useful to use (B13, [18]), whose trace immediately tells us that the terms containing Ω'_{ij} cannot be leading over all the remaining terms in that equation (when $n > 4$). Bearing this in mind, in the various cases listed above (B4, [18]) leads to:

(a) $\Omega'_{ij} = O(1)$,

with $\Omega_{ij}^{(0)} = \left(\frac{\tilde{R}}{2}\right)^2 \Omega_{ij}^{(4)}$.

(b) In the first case (eq. (37)), we find

$$\Omega'_{ij} = O(r^{-1}) \quad (\text{case (37)}), \quad (52)$$

with $\Omega_{ij}^{(1)} = -\left(\frac{\tilde{R}}{2}\right)^2 \Omega_{ij}^{(5)}$ and for the second and third cases (eqs. (38), (39))

$$\Omega'_{ij} = O(r^{-2}) \quad (\text{cases (38), (39)}). \quad (53)$$

The different behaviour in case (37) stems from (B13, [18]) using the fact that $\Phi_{ijkl}^{(2)} b_{(jl)} \neq 0$ when $\nu = 5$. Comparing (B4, [18]) and (B13, [18]) also reveals that in case (39) necessarily $\nu \geq 6$ (and not just $\nu > 5$), as we will summarize in section 3.1.4, so that $\Omega'_{ij} = \Phi_{isjk}^{(2)} l_{s1} l_{k1} + \left(\frac{\tilde{R}}{2}\right)^2 \Omega_{ij}^{(6)}$. For case (38) one has simply $\Omega'_{ij} = \Phi_{isjk}^{(2)} l_{s1} l_{k1}$.

(c)

$$\text{if } n > 4 : \quad \Omega'_{ij} = O(r^{3-n}), \quad (54)$$

$$\text{if } n = 4 : \quad \Omega'_{ij} = O(r^{-1}), \quad (55)$$

where $\Omega_{ij}^{(n-3)} = \left(\frac{\tilde{R}}{2}\right)^2 \Omega_{ij}^{(n+1)}$ for $n > 4$.

It is clear that if $n > 4$ and ℓ is a WAND (possible in cases (c) and (b) above) the fall-off of Ω'_{ij} will be faster since $\Omega_{ij} = 0$ (in agreement with the results of [30] for multiple WANDs).

3.1.4 Summary of case (i)

In all cases given here we have

$$\begin{aligned} \Omega_{ij} &= O(r^{-\nu}), & (\nu \geq 4) \\ \Psi_{ijk} &= O(r^{-\nu}). \end{aligned} \quad (56)$$

These two equations will not be repeated every time below, where we will give only possible further restrictions on ν . See also sections 3.1.1–3.1.3 for relations among the leading order terms of various boost weight.

(a) Here $n > 4$ and

$$\begin{aligned} \Phi_{ijkl} &= O(r^{-2}), & \Phi &= O(r^{-3}), & \Phi_{ij}^A &= O(r^{-3}) & (n > 4, \nu = 4), \\ \Psi'_{ijk} &= O(r^{-2}), \\ \Omega'_{ij} &= O(1). \end{aligned} \quad (57)$$

The leading term at infinity is of order r^0 and it is of type N. At order $1/r^2$ the type becomes II(ad). This case does not seem of great physical interest since the frame components Ω'_{ij} do not decay near infinity. Here ℓ cannot be a WAND.

(b) Here $n > 5$ and we have three subcases. Generically (case (37)) we have

$$\begin{aligned} \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= o(r^{-3}), & \Phi_{ij}^A &= O(r^{-3}) & (n > 5, \nu \geq 5), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{-3}), \\ \Omega'_{ij} &= O(r^{-1}), \end{aligned} \quad (58)$$

where, however, if $\nu \geq 6$ then $\Phi_{ij}^S = O(r^{-4})$ and $\Omega'_{ij} = O(r^{-2})$. The leading term is thus of type N for $5 \leq \nu < 6$ and of type type II(abd) for $\nu \geq 6$. As a special subcase here ℓ can be a multiple WAND, cf. the results of [30].

If $\Phi_{ijkl}^{(2)} b_{[jl]} = 0$ this becomes either

$$\begin{aligned} \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}) & (n > 5, \nu \geq n + 1), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{1-n}), \\ \Omega'_{ij} &= O(r^{-2}), \end{aligned} \quad (59)$$

which describes, in particular, the fall-off along a multiple WAND in Robinson-Trautman Einstein spacetimes [24] (such as static Einstein black holes) or (if $6 \leq \nu < n + 1$, or $\nu > n + 1$ but with $\Phi_{ij}^{S(n-1)} = 0$)

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{2-\nu}), & \Phi &= O(r^{1-\nu}), & \Phi_{ij}^A &= O(r^{1-\nu}) & (n > 5, \nu \geq 6, \nu \neq n + 1) \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{2-\nu}), \\ \Omega'_{ij} &= O(r^{-2}).\end{aligned}\tag{60}$$

The leading term is of type II(abd) in both the above two cases.

(c) This possibility arises when $\Phi_{ijkl}^{(2)} = 0$ and includes the four-dimensional case. For $n > 4$, we have

$$\begin{aligned}\Phi_{ijkl} &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}) & (n > 4, \nu \geq n + 1), \\ \Psi'_{ijk} &= O(r^{1-n}), \\ \Omega'_{ij} &= O(r^{3-n}).\end{aligned}\tag{61}$$

The leading term at infinity is of order $1/r^{n-3}$ (provided $\Omega_{ij}^{(n+1)} \neq 0$) and it is of type N. At order $1/r^{n-1}$ the type becomes II(cd) (II(bcd) if $\Omega_{ij}^{(n+1)} = 0$). In special cases ℓ can be a multiple WAND. This case thus includes the behaviour of the algebraically special spacetimes along a non-degenerate geodesic multiple WAND, for which however $\Omega'_{ij} = O(r^{1-n})$ [30] (the r -dependence at the leading order has been worked out explicitly also for concrete examples such as Kerr-Schild-(A)dS geometries (with a non-degenerate Kerr-Schild vector) [29], including rotating (A)dS black holes, and for Robinson-Trautman spacetimes with (A)dS asymptotics [24], such as the Schwarzschild-Tangherlini (A)dS black hole).

For $n = 4$, one has instead

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-3}), & \Phi_{ij}^A &= O(r^{-3}) & (n = 4, \nu \geq 5), \\ \Psi'_{ijk} &= O(r^{-2}), \\ \Omega'_{ij} &= O(r^{-1}).\end{aligned}\tag{62}$$

This is a special subcase of the standard 4D peeling (69).

3.2 Case (ii): $\Psi_{ijk} = O(r^{-n})$, $\Omega_{ij} = o(r^{-n})$

The behaviour of the Ricci rotation coefficients and derivative operators is the same as in case (i) and it will not be repeated here (in particular, (28), (30), (32), (33) and (46) still apply).

3.2.1 Case $\beta_c = -2$, $n > 5$

All the following cases can occur only for $n > 5$. In general, one has

$$\begin{aligned}\Omega_{ij} &= o(r^{-n}), \\ \Psi_{ijk} &= O(r^{-n}), \\ \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-4}), & \Phi_{ij}^A &= O(r^{-3}), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{-3}), \\ \Omega'_{ij} &= O(r^{-2}),\end{aligned}\tag{63}$$

with $\Phi_{ijkl}^{(2)} b_{(jl)} = 0$, $(n-4)\Phi_{ij}^{A(3)} = \Phi_{ikjl}^{(2)} b_{[kl]}$ and $\Psi_{ijk}'^{(2)} = -\Phi_{isjk}^{(2)} l_{s1}$. Here ℓ can be a single WAND, in special cases. For $\Psi_{ijk}^{(n)} = 0$ this reduces to (58) (with $\nu > n$).

If $\Phi_{ikjl}^{(2)} b_{[kl]} = 0$ (but $\Phi_{ikjl}^{(2)} \neq 0$) we have the subcase:

$$\begin{aligned}\Omega_{ij} &= O(r^{-1-n}), \\ \Psi_{ijk} &= O(r^{-n}), \\ \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}), \\ \Psi_{ijk}' &= O(r^{-2}), & \Psi_i' &= O(r^{2-n}), \\ \Omega_{ij}' &= O(r^{-2}),\end{aligned}\tag{64}$$

with $\Psi_i'^{(n-2)} = \frac{\tilde{R}}{2}\Psi_i^{(n)}$.

If, additionally, $\Phi_{ij}^{S(n-1)} = 0$ we have, depending on the range of ν , either

$$\begin{aligned}\Omega_{ij} &= O(r^{-\nu}) & (n < \nu < 2+n, \nu \neq n+1), \\ \Psi_{ijk} &= O(r^{-n}), \\ \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{2-\nu}), & \Phi &= o(r^{2-\nu}), & \Phi_{ij}^A &= o(r^{2-\nu}), \\ \Psi_{ijk}' &= O(r^{-2}), & \Psi_i' &= O(r^{2-n}), \\ \Omega_{ij}' &= O(r^{-2}),\end{aligned}\tag{65}$$

where the precise power of r for both Φ and Φ_{ij}^A is given by $\max\{1-\nu, -n\}$ or

$$\begin{aligned}\Omega_{ij} &= O(r^{-2-n}), \\ \Psi_{ijk} &= O(r^{-n}), \\ \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-n}), & \Phi_{ij}^A &= O(r^{-n}), \\ \Psi_{ijk}' &= O(r^{-2}), & \Psi_i' &= O(r^{2-n}), \\ \Omega_{ij}' &= O(r^{-2}).\end{aligned}\tag{66}$$

In all the above cases the leading term is of type II(abd).

3.2.2 Case $\beta_c < -2$, $n > 4$

If $\Phi_{ijkl}^{(2)} = 0$ then (64) reduces to

$$\begin{aligned}\Omega_{ij} &= O(r^{-1-n}), \\ \Psi_{ijk} &= O(r^{-n}), \\ \Phi_{ijkl} &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}), \\ \Psi_{ijk}' &= O(r^{2-n}), \\ \Omega_{ij}' &= O(r^{3-n})\end{aligned}\tag{67}$$

with $\Omega_{ij}'^{(n-3)} = \left(\frac{\tilde{R}}{2}\right)^2 \Omega_{ij}^{(n+1)}$ and $(n-2)(n-3)\Phi_{ijkl}^{(n-1)} = 4\Phi^{(n-1)}\delta_{j[k}\delta_{m]i} - 2(n-3)\tilde{R}(\Omega_{j[k}^{(n+1)}\delta_{m]i} - \Omega_{i[k}^{(n+1)}\delta_{m]j})$.

The leading term is type N. If $\Psi_{ijk}^{(n)} = 0$ this reduces to (61) with $\nu = n+1$. Although the above fall-off looks very similar to the standard 4D peeling (69), an important difference for $n > 4$ is that $\Omega_{ij}'^{(n-3)} \neq 0$ implies that ℓ is not a WAND.

If $\Phi^{(n-1)} = 0 = \Omega_{ij}^{(n+1)}$ this becomes

$$\begin{aligned}
\Omega_{ij} &= O(r^{-2-n}), \\
\Psi_{ijk} &= O(r^{-n}), \\
\Phi_{ijkl} &= O(r^{-n}), \quad \Phi_{ij}^A = O(r^{-n}), \\
\Psi'_{ijk} &= O(r^{2-n}), \\
\Omega'_{ij} &= O(r^{2-n}).
\end{aligned} \tag{68}$$

Here the leading term is of type III.

In both the above cases we have $(n-3)\Psi'_{ijk} = \tilde{R}\Psi_{[j}^{(n)}\delta_{k]i}$.

3.2.3 Case $n = 4$

In four dimensions, we recover the standard asymptotic behaviour [9, 10], i.e.,

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) \quad (\nu \geq 5), \\
\Psi_{ijk} &= O(r^{-4}), \\
\Phi_{ijkl} &= O(r^{-3}), \quad \Phi_{ij}^A = O(r^{-3}), \\
\Psi'_{ijk} &= O(r^{-2}), \\
\Omega'_{ij} &= O(r^{-1}).
\end{aligned} \tag{69}$$

In our study, the condition $\nu \geq 5$ followed by analyzing the Ricci and Bianchi equations (where we initially only assumed $\nu > 2$), thanks to $\tilde{R} \neq 0$. Additionally, we observe that if $\nu > 5$ then necessarily $\nu \geq 6$. For $\Psi_{ijk}^{(4)} = 0$ this case reduces to (62).

3.3 Case (iii): $\Psi_{ijk} = O(r^{-3})$, $\Omega_{ij} = o(r^{-3})$ ($n > 4$)

Again the behaviour of the Ricci rotation coefficients and derivative operators is the same as in case (i).⁵

3.3.1 Case $\beta_c = -2$

Here in general one has ($n \geq 5$)

$$\begin{aligned}
\Omega_{ij} &= O(r^{-4}), \\
\Psi_{ijk} &= O(r^{-3}), \quad \Psi_i = O(r^{-4}), \\
\Phi_{ijkl} &= O(r^{-2}), \quad \Phi = O(r^{-3}), \quad \Phi_{ij}^A = O(r^{-3}), \\
\Psi'_{ijk} &= O(r^{-1}), \quad \Psi'_i = O(r^{-2}), \\
\Omega'_{ij} &= O(1),
\end{aligned} \tag{70}$$

with $\Phi_{ij}^{S(2)} = \frac{\tilde{R}}{2}\Omega_{ij}^{(4)}$, $\Psi'_{ijk}{}^{(1)} = \frac{\tilde{R}}{2}\Psi_{ijk}^{(3)}$, $\Psi'_i{}^{(2)}$ can be expressed in terms of $\Omega_{ij}^{(4)}$ and $\Psi_{ijk}^{(3)}$ thanks to (B6, [18]), $\Omega_{ij}^{(0)} = \left(\frac{\tilde{R}}{2}\right)^2 \Omega_{ij}^{(4)}$ and

$$(n-4)\Phi_{ki}^{A(3)} = \Phi_{klij}^{(2)}b_{[lj]} + \xi_j^{A0}\Psi_{[ki]j,A}^{(3)} + 2l_{1j}\Psi_{[ki]j}^{(3)} + \Psi_{[ki]l}^{(3)}m_{jj}^l + \Psi_{j[lk]}^{(3)}m_{ij}^l + \tilde{R}\Omega_{j[k}^{(4)}b_{i]j}.$$

The leading term is type N. In the limit $\Psi_{ijk}^{(3)} = 0$ this reduces to case (57).

If Ω_{ij} has a faster fall-off one finds for $n > 5$ (as in section 3.1 the range $4 < \nu < 5$ is forbidden by imposing (B5, [18]) and (B12, [18])); see section 3.3.2 for the case $n = 5$)

⁵In order to arrive at (46) in the present case one needs to use also (10) and (42) and thus to observe that although (B9, [18]) gives $\Psi'_{ijk} = O(r^{-1})$, from its trace one gets $\Psi'_i = O(r^{-2})$ (see also (70)–(72) below).

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) & (\nu \geq 5), \\
\Psi_{ijk} &= O(r^{-3}), & \Psi_i = O(r^{-4}), \\
\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S = O(r^{-3}), & \Phi = O(r^{-4}), & \Phi_{ij}^A = O(r^{-3}) & (n > 5), & (71) \\
\Psi'_{ijk} &= O(r^{-1}), & \Psi'_i = O(r^{-2}), \\
\Omega'_{ij} &= O(r^{-1}),
\end{aligned}$$

where from (B5, [18]) $\Phi_{ij}^{S(3)} = -\Psi_{(ij)l}^{(3)} l_{l1}$, from (B12, [18])

$$\begin{aligned}
(n-4)\Phi_{ki}^{A(3)} &= \Phi_{klij}^{(2)} b_{[lj]} + \xi_j^{A0} \Psi_{[kij]j,A}^{(3)} + 2l_{1j} \Psi_{[kij]j}^{(3)} + \Psi_{[kij]l}^{(3)} m_{jj}^l + \Psi_{j[k]l}^{(3)} m_{ij}^l, \\
-\Phi_{klij}^{(2)} b_{(lj)} &= \xi_j^{A0} \Psi_{(kij)j,A}^{(3)} + [(n-6)l_{j1} + 2l_{1j}] \Psi_{(kij)j}^{(3)} + \Psi_{(kij)l}^{(3)} m_{jj}^l + (2\Psi_{l(k|j}^{(3)} + \Psi_{j[l(k|}^{(3)}) m_{ij}^l + \frac{\tilde{R}}{2}(n-4)\Omega_{ik}^{(5)},
\end{aligned}$$

and $\Omega_{ij}^{\prime(1)}$ can be expressed (using the trace of (B13, [18])) in terms of $\Omega_{ik}^{(5)}$ and $\Psi_{ijk}^{(3)}$.

The leading term is of type III(a) and ℓ can be a single WAND. If $\Psi_{ijk}^{(3)} = 0$ this reduces to (58) for $5 \leq \nu \leq n$ and to (63) for $\nu > n$.

3.3.2 Case $\beta_c < -2$

For $n = 5$, or for $n > 5$ with $\Phi_{ijkl}^{(2)} = 0$, instead of (71) one has

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) & (\nu \geq 5), \\
\Psi_{ijk} &= O(r^{-3}), & \Psi_i = O(r^{-4}), \\
\Phi_{ijkl} &= O(r^{-3}), & \Phi = O(r^{-3}), & \Phi_{ij}^A = O(r^{-3}) & (n \geq 5), & (72) \\
\Psi'_{ijk} &= O(r^{-1}), & \Psi'_i = O(r^{-2}), \\
\Omega'_{ij} &= O(r^{-1}),
\end{aligned}$$

where $\Phi_{ijkl}^{(3)}$ can be expressed in terms of $\Omega_{ij}^{(5)}$ and $\Psi_{ijk}^{(3)}$ using (B12, [18]) (or (B13, [18])). The leading term is of type III(a). Again $\Psi_{ijk}^{\prime(1)} = \frac{\tilde{R}}{2} \Psi_{ijk}^{(3)}$.

4 Case $\tilde{R} = 0$

4.1 Case (i): $\Psi_{ijk} = O(r^{-\nu})$, $\Omega_{ij} = O(r^{-\nu})$ ($\nu > 2$)

4.1.1 Weyl components of b.w. 0

In this case, at the leading order of (B12, [18]) we can have only (some of) the terms $O(r^{\beta_c-1})$, $O(r^{\beta-1})$, $O(r^{-1-\nu})$. The same is true for the antisymmetric part of (B5, [18]), while the leading order terms of the symmetric part of (B5, [18]) can only be $O(r^{\beta_c-1})$, $O(r^{\beta-1})$, $O(r^{-\nu})$. Here, we are mainly interested in studying the case when the leading terms of (B12, [18]) are $O(r^{\beta_c-1})$, $O(r^{\beta-1})$, i.e., $\beta_c > -\nu$ or $\beta > -\nu$. (In all the remaining cases, the asymptotic behaviour of b.w. zero components can be represented by $\Phi_{ijkl} = O(r^{-\nu})$, $\Phi_{ij}^A = O(r^{-\nu})$, $\Omega_{ij} = O(r^{-\nu})$, with $\nu > 2$. The behaviour of higher b.w. components is given in section 4.1.5 below.)

By combining (B12, [18]) and (B5, [18]) we arrive at the following possibilities, also depending on the value of ν and of n :

(A) $\beta_c = -2$: there are several possibilities, i.e.,

A1:

$$\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = o(r^{-2}), \quad \Phi_{ij}^A = o(r^{-2}), \quad \Omega_{ij} = O(r^{-\nu}) \quad (n > 5, 2 < \nu \leq 3). \quad (73)$$

A2:

$$\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = O(r^{-3}), \quad \Phi = O(r^{-\nu}), \quad \Phi_{ij}^A = O(r^{-3}), \quad \Omega_{ij} = O(r^{-\nu}) \quad (n > 5, 3 < \nu < 4). \quad (74)$$

The (anti)symmetric parts of the trace of (B12, [18]) (using (10)) give $(n-4)\Phi_{ij}^{A(3)} = \Phi_{ijkl}^{(2)} b_{[kl]}$ and $(n-6)\Phi_{ki}^{S(3)} = \Phi_{kl ij}^{(2)} b_{(lj)}$. In the special case $\Phi_{ijkl}^{(2)} b_{[kl]} = 0$ thus Φ_{ij}^A goes to zero faster, namely $\Phi_{ij}^A = O(r^{-\nu})$.

A3:

$$\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = O(r^{-3}), \quad \Phi = O(r^{-4}), \quad \Phi_{ij}^A = O(r^{-3}), \quad \Omega_{ij} = O(r^{-4}) \quad (n > 5). \quad (75)$$

As above $(n-4)\Phi_{ij}^{A(3)} = \Phi_{ijkl}^{(2)} b_{[kl]}$ and $(n-6)\Phi_{ki}^{S(3)} = \Phi_{kl ij}^{(2)} b_{(lj)}$ but here with the latter (B5, [18]) further gives $\Phi_{kl ij}^{(2)} b_{(lj)} = -(n-6)(l_{11}\Omega_{ki}^{(4)} + \frac{1}{2}X^{A0}\Omega_{ki,A}^{(4)} + \Omega_{s(k}^{(4)} m_{i)1}^s)$. Here Ω_{ij} can go to zero faster than indicated, i.e., $\Omega_{ij} = O(r^{-\nu})$ with $\nu > 4$, but in that case clearly also Φ_{ij}^S does (namely, $\Phi_{ij}^S = O(r^{1-\nu})$ for $4 < \nu < 5$ and $\Phi_{ij}^S = O(r^{-4})$ for $\nu \geq 5$ – in particular, for $\nu > 5$ the symmetric part of (B5, [18]) gives $\Phi_{ij}^{S(4)}$ in terms of $\Phi_{ij}^{A(3)}$).

If $\Phi_{ijkl}^{(2)} b_{[kl]} = 0$ we obtain the following two subcases, depending on whether $\nu \neq n$ or $\nu = n$.

A4:

$$\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = O(r^{1-\nu}), \quad \Phi = O(r^{-\nu}), \quad \Phi_{ij}^A = O(r^{-\nu}), \quad \Omega_{ij} = O(r^{-\nu}) \quad (n > 5, \nu \geq 4, \nu \neq n), \quad (76)$$

with $\Phi_{ijkl}^{(2)} b_{[kl]} = 0$ and $(n-6)\Phi_{ki}^{S(3)} = \Phi_{kl ij}^{(2)} b_{(lj)}$ (if $\nu = 4$) or $\Phi_{kl ij}^{(2)} b_{(lj)} = 0$ (if $\nu > 4$). For $\nu > n$ this can be seen as a subcase of (77) with $\Phi^{(n-1)} = 0$.

A5:

$$\Phi_{ijkl} = O(r^{-2}), \quad \Phi_{ij}^S = O(r^{1-n}), \quad \Phi_{ij}^A = O(r^{-n}), \quad \Omega_{ij} = O(r^{-n}) \quad (n > 5), \quad (77)$$

with $\Phi_{ijkl}^{(2)} b_{[kl]} = 0$ and $\Phi_{kl ij}^{(2)} b_{(lj)} = 0$. Ω_{ij} can go to zero faster than indicated, with no effect on the fall-off of Φ_{ij}^S . If $\nu > n$ then $(2-n)\Phi_{ij}^{S(n-1)} + \Phi^{(n-1)}\delta_{ij} = 0$.

(B) $\beta_c = -n/2$:

$$\Phi_{ijkl} = O(r^{-n/2}), \quad \Phi = O(r^{-\nu}), \quad \Phi_{ij}^A = O(r^{-\nu}), \quad \Omega_{ij} = O(r^{-\nu}) \quad (n > 4, \frac{n}{2} < \nu \leq 1 + \frac{n}{2}) \quad (78)$$

with $(n-4)\Phi_{ijkl}^{(n/2)} = 4(\Phi_{il}^{S(n/2)}\delta_{kj} - \Phi_{jl}^{S(n/2)}\delta_{ki})$. Note that here Ω_{ij} cannot become $o(r^{-n/2-1})$ as long as $\Phi_{ijkl} = O(r^{-n/2})$. In the special case $\nu = 1 + n/2$, from (B5, [18]) we obtain $(n-2)\Phi_{ij}^{S(n/2)} = -2X^{A0}\Omega_{ij,A}^{(n/2+1)} - (n-2)l_{11}\Omega_{ij}^{(n/2+1)} - 4\Omega_{s(j}^{(n/2+1)} m_{i)1}^s$.

(C) $\beta_c = 1 - n$: similarly as in section 3.1, one has to distinguish between the cases $n > 4$ and $n = 4$, i.e.,

$$\text{if } n > 4: \quad \Phi_{ijkl} = O(r^{1-n}), \quad \Phi_{ij}^A = o(r^{1-n}), \quad \Omega_{ij} = O(r^{-\nu}) \quad (\nu > n-1), \quad (79)$$

$$\text{if } n = 4: \quad \Phi_{ijkl} = O(r^{-3}), \quad \Phi_{ij}^A = O(r^{-3}), \quad \Omega_{ij} = O(r^{-\nu}) \quad (\nu > 3), \quad (80)$$

with (for $n \geq 4$) $(n-2)(n-3)\Phi_{ijkl}^{(n-1)} = 4\Phi^{(n-1)}\delta_{j[k}\delta_{l]i}$ and $(2-n)\Phi_{ij}^{S(n-1)} + \Phi^{(n-1)}\delta_{ij} = 0$. In (79) we have $\Phi_{ij}^A = O(r^{-\nu})$ for $n-1 < \nu < n$ and $\Phi_{ij}^A = O(r^{-n})$ for $\nu \geq n$.

Again, see point (i) of section 2.3 for the behaviour of Ψ_{ijk} in all the above cases. As shown above, in all cases except (73) we have $\nu > 3$, which enables us (thanks to (10)) to specialize (29) to

$$N_{ij} = \frac{n_{ij}}{r} + O(r^{-2}) \quad (\text{except for (73)}). \quad (81)$$

Similarly as for $\tilde{R} \neq 0$ (cf. section 3.1.1), since in all permitted cases one has $\Phi = o(r^{-2})$ and $\Phi_{ij}^A = o(r^{-2})$, for L_{11} , U , M_{j1} one obtains the refined equations that follow by setting $\tilde{R} = 0$ in (43), (44), (45) (in contrast to (81) this applies also when $2 < \nu \leq 3$).

4.1.2 Ricci rotation coefficients of b.w. -2 and Weyl components of b.w. -1

Let us analyse (11f, [20]) and (B6, [18]), (B9, [18]) and (B1, [18]) in all the possible cases listed above. Similarly as in section 3.1.2, it is easy to conclude from (B9, [18]) that we always have $\Psi'_{ijk} = O(r^{-2})$ (or faster, see more details below), which enables one to use (11f, [20]) to obtain

$$N_{i1} = O(1). \quad (82)$$

Using (B9, [18]), (B6, [18]), (B1, [18]) one arrives at the following results (the numbering corresponds to that of section 4.1.1).

(A) For the five subcases we find, respectively,

$$\text{A1: } \Psi'_{ijk} = O(r^{-2}).$$

$$\text{A2: } \Psi'_{ijk} = O(r^{-2}), \Psi'_i = O(r^{-3}).$$

$$\text{A3: } \Psi'_{ijk} = O(r^{-2}), \Psi'_i = O(r^{-3}).$$

$$\text{A4: } \Psi'_{ijk} = O(r^{-2}), \Psi'_i = O(r^{1-\nu}).$$

$$\text{A5: } \Psi'_{ijk} = O(r^{-2}), \Psi'_i = O(r^{1-n}).$$

In all cases except A1 we have $\Psi'_{ijk}{}^{(2)} = -\Phi_{isjk}^{(2)} l_{s1}$ (in case A1, if $\nu = 3$ then (B6, [18]) gives $\Psi'_{ijk}{}^{(2)}$ in terms of $\Omega_{ij}^{(3)}$, $\Psi_{ijk}^{(3)}$, and $\Phi_{isjk}^{(2)}$).

(B) We have $\Psi'_{ijk} = O(r^{-n/2})$ for any $n \geq 6$, and for $n = 5$ provided $3 < \nu \leq \frac{7}{2}$ (in both cases (B9, [18]) enables one to express $\Psi'_{ijk}{}^{(n/2)}$ in terms of $\Phi_{ij}^{S(n/2)}$. If, instead, $n = 5$ and $\frac{5}{2} < \nu \leq 3$ we have $\Psi'_{ijk} = O(r^{-2})$).

(C)

$$\text{if } n > 4 : \quad \Psi'_{ijk} = O(r^{1-n}), \quad (83)$$

$$\text{if } n = 4 : \quad \Psi'_{ijk} = O(r^{-2}). \quad (84)$$

For $n > 4$, (B9, [18]) gives $(n-3)\Psi'_{ijk}{}^{(n-1)} = 2\Psi'_{[j}{}^{(n-1)}\delta_{k]i}$, with $(n-2)\Psi'_{i}{}^{(n-1)} = -(n-1)\Phi^{(n-1)}l_{1i} - \xi_i^{A0}\Phi_{,A}^{(n-1)}$.

4.1.3 Weyl components of b.w. -2

Using (B4, [18]) and (B13, [18]) we arrive at

(A) For the five subcases we find, respectively,

$$\text{A1: } \Omega'_{ij} = O(r^\sigma), \text{ with } -2 < \sigma < \max\{1 - \nu, 1 + \beta\} < -1 \text{ (recall (26)).}$$

$$\text{A2-A5: } \Omega'_{ij} = O(r^{-2}), \text{ with } \Omega'_{ij}{}^{(2)} = -3l_{11}\Phi_{ij}^{S(3)} - X^{A0}\Phi_{ij,A}^{S(3)} - 2\Phi_{s(j}^{S(3)}m_{i)1}^s - \Psi'_{(ij)k}{}^{(2)}l_{k1} \text{ (note that in some of these cases } \Phi_{ij}^{S(3)} = 0).$$

(B) In all cases ($n \geq 5$) we have

$$\Omega'_{ij} = O(r^{1-n/2}), \quad (85)$$

with $(n-4)\Omega'_{ij}{}^{(n/2-1)} = -nl_{11}\Phi_{ij}^{S(n/2)} - 2X^{A0}\Phi_{ij,A}^{S(n/2)} - 4\Phi_{s(j)}^{S(n/2)}m_{i)1}^s$. In the special case $\nu = 1 + n/2$ this can be written in terms of $\Omega_{ij}^{(n/2+1)}$ using the form of $\Phi_{ij}^{S(n/2)}$ given in the above section 4.1.1.

(C)

$$\text{if } n > 4 : \quad \Omega'_{ij} = o(r^{2-n}), \quad (86)$$

$$\text{if } n = 4 : \quad \Omega'_{ij} = O(r^{-1}). \quad (87)$$

In order to obtain the above behaviour, in the $n > 4$ case it is also necessary to recall that at the leading order $\Phi_{ij}^S \propto \delta_{ij}$ (cf. section 4.1.1).

4.1.4 Summary of case (i)

In all cases given here we have

$$\begin{aligned} \Omega_{ij} &= O(r^{-\nu}), & (\nu > 2) \\ \Psi_{ijk} &= O(r^{-\nu}). \end{aligned} \quad (88)$$

This will not be repeated every time below, where we will give only possible further restrictions on ν . See also sections 4.1.1–4.1.3 for relations among the leading order terms of various boost weight.

(A) Here we have $n > 5$ and the following possible behaviours (cf. section 4.1.1 for a few further special subcases).

A1:

$$\begin{aligned} \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= o(r^{-2}), & \Phi_{ij}^A &= o(r^{-2}) & (n > 5, 2 < \nu \leq 3), \\ \Psi'_{ijk} &= O(r^{-2}), \\ \Omega'_{ij} &= O(r^\sigma) & \text{with } -2 < \sigma < \max\{1 - \nu, 1 + \beta\} < -1. \end{aligned} \quad (89)$$

A2:

$$\begin{aligned} \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-3}), & \Phi &= O(r^{-\nu}), & \Phi_{ij}^A &= O(r^{-3}) & (n > 5, 3 < \nu < 4), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{-3}), \\ \Omega'_{ij} &= O(r^{-2}). \end{aligned} \quad (90)$$

A3:

$$\begin{aligned} \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-3}), & \Phi &= O(r^{-4}), & \Phi_{ij}^A &= O(r^{-3}) & (n > 5, \nu \geq 4), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{-3}), \\ \Omega'_{ij} &= O(r^{-2}), \end{aligned} \quad (91)$$

with the further restrictions $\Phi_{ij}^S = O(r^{1-\nu})$ for $4 \leq \nu < 5$ and $\Phi_{ij}^S = O(r^{-4})$ for $\nu \geq 5$.

A4:

$$\begin{aligned} \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{1-\nu}), & \Phi &= O(r^{-\nu}), & \Phi_{ij}^A &= O(r^{-\nu}) & (n > 5, \nu \geq 4, \nu \neq n), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{1-\nu}), \\ \Omega'_{ij} &= O(r^{-2}). \end{aligned} \quad (92)$$

A5:

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}) & (n > 5, \nu \geq n), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{1-n}), \\ \Omega'_{ij} &= O(r^{-2}).\end{aligned}\tag{93}$$

None of the above five cases can describe asymptotically flat spacetimes, cf. [15]. In cases A2–A5, the leading term at infinity falls off as $1/r^2$ and it is of type II(abd). In cases A3–A5, ℓ can be a multiple WAND, cf. also the results of [30]. Examples in case A5 are Robinson-Trautman Ricci-flat spacetime [24].

(B) For any $n > 5$, we have

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-n/2}), & \Phi &= O(r^{-\nu}), & \Phi_{ij}^A &= O(r^{-\nu}) & \left(n > 5, \frac{n}{2} < \nu \leq 1 + \frac{n}{2}\right), \\ \Psi'_{ijk} &= O(r^{-n/2}), \\ \Omega'_{ij} &= O(r^{1-n/2}).\end{aligned}\tag{94}$$

Note that here ℓ cannot be a WAND. The leading term at infinity falls off as $1/r^{n/2-1}$ and it is of type N. At order $1/r^{n/2}$ the type becomes II(acd) (as follows from section 4.1.1).

For $n = 5$ the same behaviour applies if $3 < \nu \leq \frac{7}{2}$, while $\Psi'_{ijk} = O(r^{-2})$ if $\frac{5}{2} < \nu \leq 3$ (the other terms being unchanged).

If we take for b.w. +2 components $\nu = 1 + \frac{n}{2}$ and additionally *assume* that

$$\Omega_{ij} = \frac{\Omega_{ij}^{(n/2+1)}}{r^{n/2+1}} + \frac{\Omega_{ij}^{(n/2+2)}}{r^{n/2+2}} + o(r^{-n/2-2}),\tag{95}$$

then (B4, [18]) with (B5, [18]) show that the subleading term of Ω'_{ij} is of order $O(r^{-n/2})$, which with (94) implies the following peeling-off behaviour

$$C_{abcd} = \frac{N_{abcd}}{r^{n/2-1}} + \frac{II_{abcd}}{r^{n/2}} + o(r^{-n/2}) \quad (n \geq 5).\tag{96}$$

This result is in agreement with the conclusions of [15] for asymptotically flat spacetimes (and extends it to asymptotics along twisting null geodesics). However, in order to obtain higher order terms one would need to make further assumptions on how Ω_{ij} can be expanded, which goes beyond the analysis of the present paper (however, recall that it is precisely at a higher order in (96) that [15] found a qualitative difference between five and higher dimensions). In five dimensions, a permitted behaviour more general than (96) is described in section 4.3.2 below (it does not appear here because it belongs to case (iii)).

In view of [15], we conclude that the above behaviour (94) includes *radiative spacetimes* that are asymptotically flat in the Bondi definition [16,17] (which is equivalent [15] to the conformal definition [11,12] in even dimensions).

If one takes $\nu > 1 + \frac{n}{2}$ in (94), this reduces to (99) if $1 + \frac{n}{2} < \nu \leq n - 1$, to (97) if $n - 1 < \nu \leq n$, and to (105) if $\nu > n$.

(C) For $n > 4$ the fall-off is

$$\begin{aligned}\Phi_{ijkl} &= O(r^{1-n}), & \Phi_{ij}^A &= o(r^{1-n}) & (n > 4, \nu > n - 1), \\ \Psi'_{ijk} &= O(r^{1-n}), \\ \Omega'_{ij} &= o(r^{2-n}),\end{aligned}\tag{97}$$

with $\Phi_{ij}^A = O(r^{-\nu})$ for $n - 1 < \nu < n$ and $\Phi_{ij}^A = O(r^{-n})$ for $\nu \geq n$. Here ℓ can become a multiple WAND, cf. [27, 30]. This behaviour is compatible with the results of [15] for asymptotically flat spacetimes, in the case of *vanishing radiation*. In particular, it includes asymptotically flat spacetimes for which ℓ is a multiple WAND [27, 30], such as Ricci flat Robinson-Trautman spacetimes [24] (e.g., Schwarzschild-Tangherlini black holes), and Kerr-Schild spacetimes [26] with a non-degenerate Kerr-Schild vector⁶ (e.g., Myers-Perry black holes).

For $n = 4$ we have instead

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-3}), & \Phi_{ij}^A &= O(r^{-3}) & (n = 4, \nu > 3), \\ \Psi'_{ijk} &= O(r^{-2}), \\ \Omega'_{ij} &= O(r^{-1}),\end{aligned}\tag{98}$$

where the leading $1/r$ term is of type N.

4.1.5 Special subcase $\beta_c = \beta = -\nu$

In addition, there is the case $\beta_c = \beta = -\nu$ (briefly mentioned in section 4.1.1 above but not explicitly studied in sections 4.1.2 and 4.1.3), for which one easily arrives for $n > 4$ at (note that (82) still applies here)

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-\nu}), & \Phi_{ij}^A &= O(r^{-\nu}), & (n > 4) \\ \Psi'_{ijk} &= O(r^{-2}) \quad \text{if } 2 < \nu \leq 3, & \Psi'_{ijk} &= O(r^{-\nu}) \quad \text{if } \nu > 3, \\ \Omega'_{ij} &= o(r^{1-\nu}) \quad \text{if } \nu \neq \frac{n}{2}, & \Omega'_{ij} &= O(r^{1-n/2}) \quad \text{if } \nu = \frac{n}{2},\end{aligned}\tag{99}$$

with $X^{A0}\Omega_{ij,A}^{(\nu)} + (\nu - 2)l_{11}\Omega_{ij}^{(\nu)} + 2\Omega_{s(j}^{(\nu)}m_{i)1}^s = 0$. ℓ cannot be a WAND. The above conditions on Ω'_{ij} have been obtained by using (B4, [18]) and the trace of (B13, [18]).

For $n = 4$ one finds instead

$$\begin{aligned}\Phi_{ijkl} &= O(r^{-\nu}), & \Phi_{ij}^A &= O(r^{-\nu}), & (n = 4, \nu > 2) \\ \Psi'_{ijk} &= O(r^{-2}), \\ \Omega'_{ij} &= O(r^{-1}),\end{aligned}\tag{100}$$

which is asymptotically of type N. For $\nu > 4$, this is a subcase of (107) having $\Phi_{ijkl}^{(3)} = 0$, $\Phi_{ij}^{A(3)} = 0$ and $\Psi_{ijk}^{(4)} = 0$.

4.2 Case (ii): $\Psi_{ijk} = O(r^{-n})$, $\Omega_{ij} = o(r^{-n})$

The behaviour of the Ricci rotation coefficients and derivative operators is the same as in case (i) (in particular, (29), (31), (32), (34) and (82) still apply).

4.2.1 Case $\beta_c = -2$, $n > 5$

All the following cases can occur only for $n > 5$.

$$\begin{aligned}\Omega_{ij} &= o(r^{-n}), \\ \Psi_{ijk} &= O(r^{-n}), \\ \Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-4}), & \Phi_{ij}^A &= O(r^{-3}), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{-3}), \\ \Omega'_{ij} &= O(r^{-2}),\end{aligned}\tag{101}$$

⁶For these one finds $\Omega'_{ij} = O(r^{1-n})$. Note that in order to explicitly verify this using the general expressions given in [26] one should recall to enforce the vacuum equation $R_{11} = 0$, cf. [32]. The same comment applies to the (A)dS Kerr-Schild spacetimes [29] mentioned in section 3.1.4.

with $(n-4)\Phi_{ij}^{A(3)} = \Phi_{ikjl}^{(2)}b_{[kl]}$ and $\Phi_{ikjl}^{(2)}b_{(kl)} = 0$. Here ℓ can be a single WAND. For $\Psi_{ijk}^{(n)} = 0$ this case reduces to (91) (with $\nu > n$).

If $\Phi_{ikjl}^{(2)}b_{[kl]} = 0$ (in particular, if ℓ is twistfree) the following subcase arises:

$$\begin{aligned}
\Omega_{ij} &= o(r^{-n}), \\
\Psi_{ijk} &= O(r^{-n}), \\
\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}), \\
\Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{1-n}), \\
\Omega'_{ij} &= O(r^{-2}),
\end{aligned} \tag{102}$$

with $(2-n)\Phi_{ij}^{S(n-1)} + \Phi^{(n-1)}\delta_{ij} = 0$.

As a further ‘‘subcase’’, if $\Phi_{ij}^{S(n-1)} = 0$ we obtain, depending on the value of ν ,

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) & (n < \nu \leq n+1), \\
\Psi_{ijk} &= O(r^{-n}), \\
\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{1-\nu}), & \Phi &= O(r^{-n}), & \Phi_{ij}^A &= O(r^{-n}), \\
\Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{1-\nu}), \\
\Omega'_{ij} &= O(r^{-2}),
\end{aligned} \tag{103}$$

or

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) & (\nu > n+1), \\
\Psi_{ijk} &= O(r^{-n}), \\
\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-n}), & \Phi_{ij}^A &= O(r^{-n}), \\
\Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{1-n}), \\
\Omega'_{ij} &= O(r^{-2}),
\end{aligned} \tag{104}$$

In all the above cases $\Psi'_{ijk}^{(2)} = -\Phi_{isjk}^{(2)}l_{s1}$ and $\Omega'_{ij}^{(2)} = -\Psi'_{(ij)k}^{(2)}l_{k1} = \Phi_{isjk}^{(2)}l_{s1}l_{k1}$. The asymptotically leading term is of type II(abd) but it reduces to type D(abd) if a particular frame with $l_{i1} = 0$ is employed cf. the comments at the end of section 2.3. The terms $\Phi_{ijkl} = O(r^{-2})$ violate the asymptotically flat conditions [15].

4.2.2 Case $\beta_c < -2$, $n > 4$

If $\Phi_{ijkl}^{(2)} = 0$ one is left with

$$\begin{aligned}
\Omega_{ij} &= o(r^{-n}), \\
\Psi_{ijk} &= O(r^{-n}), \\
\Phi_{ijkl} &= O(r^{1-n}), & \Phi_{ij}^A &= O(r^{-n}), \\
\Psi'_{ijk} &= O(r^{1-n}), & \Psi'_i &= O(r^{1-n}), \\
\Omega'_{ij} &= o(r^{2-n}),
\end{aligned} \tag{105}$$

with $(n-2)(n-3)\Phi_{ijkl}^{(n-1)} = 4\Phi^{(n-1)}\delta_{j[k}\delta_{m]i}$, $(n-3)\Psi_{ijk}^{(n-1)} = 2\Psi_{[j}^{(n-1)}\delta_{k]i}$, $(n-2)\Psi_i^{(n-1)} = -(n-1)\Phi^{(n-1)}l_{i1}$, and where ℓ can be a single WAND. This behaviour is compatible with the results of [15] for asymptotically flat spacetimes, in the case of vanishing radiation. For $\Psi_{ijk}^{(n)} = 0$, this case reduces to (97) (with $\nu > n$).

If $\Phi^{(n-1)} = 0$ this reduces to

$$\begin{aligned}
\Omega_{ij} &= o(r^{-n}), \\
\Psi_{ijk} &= O(r^{-n}), \\
\Phi_{ijkl} &= O(r^{-n}), & \Phi_{ij}^A &= O(r^{-1-n}), \\
\Psi'_{ijk} &= O(r^{-n}), \\
\Omega'_{ij} &= O(r^{1-n}).
\end{aligned} \tag{106}$$

The asymptotically leading term is of type N.

4.2.3 Case $n = 4$

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) & (\nu > 4), \\
\Psi_{ijk} &= O(r^{-4}), \\
\Phi_{ijkl} &= O(r^{-3}), & \Phi_{ij}^A &= O(r^{-3}), \\
\Psi'_{ijk} &= O(r^{-2}), \\
\Omega'_{ij} &= O(r^{-1}).
\end{aligned} \tag{107}$$

The above behaviour agrees with the well-known results of [3] (where it was assumed $\nu = 5$). For $\Psi_{ijk}^{(4)} = 0$ this case reduces to (98) (with $\nu > 4$). See [4] for results also at the subleading order.

4.3 Case (iii): $\Psi_{ijk} = O(r^{-3})$, $\Omega_{ij} = o(r^{-3})$ ($n > 4$)

Again the behaviour of the Ricci rotation coefficients and derivative operators is the same as in case (i).

4.3.1 Case $n > 5$

In more than five dimensions we generically have $\beta_c = -2$, giving rise to

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) & (\nu > 3), \\
\Psi_{ijk} &= O(r^{-3}), & \Psi_i &= o(r^{-3}), \\
\Phi_{ijkl} &= O(r^{-2}), & \Phi_{ij}^S &= O(r^{-3}), & \Phi &= o(r^{-3}), & \Phi_{ij}^A &= O(r^{-3}), \\
\Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{-3}), \\
\Omega'_{ij} &= O(r^{-2}),
\end{aligned} \tag{108}$$

where $\Psi_i = O(r^{-\nu})$, $\Phi = O(r^{-\nu})$ for $3 < \nu \leq 4$ while $\Psi_i = O(r^{-4})$, $\Phi = O(r^{-4})$ for $\nu > 4$ and

$$\begin{aligned}
(n-4)\Phi_{ki}^{A(3)} &= \Phi_{klij}^{(2)} b_{[lj]} + \xi_j^{A0} \Psi_{[ki]j,A}^{(3)} + 2l_{1j} \Psi_{[ki]j}^{(3)} + \Psi_{[ki]l}^{(3)} m_{jj}^l + \Psi_{j[lk]}^{(3)} m_{ij}^l, \\
(n-6)\Phi_{ki}^{S(3)} &= \Phi_{klij}^{(2)} b_{(lj)} + \xi_j^{A0} \Psi_{(ki)j,A}^{(3)} + 2l_{1j} \Psi_{(ki)j}^{(3)} + \Psi_{(ki)l}^{(3)} m_{jj}^l + (2\Psi_{l(k|j]}^{(3)} + \Psi_{j l(k|}^{(3)}) m_{ij}^l.
\end{aligned}$$

Here ℓ can be a single WAND and the asymptotically leading term is of type II(abd). For $\Psi_{ijk}^{(3)} = 0$, this case reduces for $3 < \nu < 4$ to (90) (with $\nu > n$), for $4 \leq \nu \leq n$ to (91) and for $\nu > n$ to (101).

A subcase with $\Phi_{ijkl}^{(2)} = 0$ is also possible, giving

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) & (\nu > 3), \\
\Psi_{ijk} &= O(r^{-3}), & \Psi_i &= o(r^{-3}), \\
\Phi_{ijkl} &= O(r^{-3}), & \Phi &= o(r^{-3}), & \Phi_{ij}^A &= O(r^{-3}), \\
\Psi'_{ijk} &= O(r^{-2}), & \Psi'_i &= O(r^{-3}), \\
\Omega'_{ij} &= O(r^{-2}),
\end{aligned} \tag{109}$$

with the same behaviour as above for Ψ_i and Φ . In this case the leading term at infinity is of type III(a).

Neither of the above behaviours can represent asymptotically flat spacetimes since the fall-off of the Weyl tensor is too slow [15].

4.3.2 Case $n = 5$

In five dimensions, we generically have

$$\begin{aligned}\Omega_{ij} &= O(r^{-\nu}) & (3 < \nu \leq \frac{7}{2}), \\ \Psi_{ijk} &= O(r^{-3}), & \Psi_i = O(r^{-\nu}), \\ \Phi_{ijkl} &= O(r^{-5/2}), & \Phi = O(r^{-\nu}), & \Phi_{ij}^A = O(r^{-3}), \\ \Psi'_{ijk} &= O(r^{-2}), & \Psi'_i = O(r^{-3}), \\ \Omega'_{ij} &= O(r^{-3/2}),\end{aligned}\tag{110}$$

with $\Phi_{ki}^{A(3)} = \xi_j^{A0} \Psi_{[ki]j,A}^{(3)} + 2l_{1j} \Psi_{[ki]j}^{(3)} + \Psi_{[ki]l}^{(3)} m_{jj}^l + \Psi_{jl[k}^{(3)} m_{i]j}^l$, $\Psi'_{ijk}^{(2)}$ can be expressed in terms of $\Psi_{ijk}^{(3)}$ using (B6, [18]) and $\Omega'_{ij}^{(3/2)} = -5l_{11} \Phi_{ij}^{S(5/2)} - 2X^{A0} \Phi_{ij,A}^{S(5/2)} - 4\Phi_{s(j}^{S(5/2)} m_{i)1}^s$. If $\nu = 7/2$ this can be rewritten using $3\Phi_{ij}^{S(5/2)} = -2X^{A0} \Omega_{ij,A}^{(7/2)} - 3l_{11} \Omega_{ij}^{(7/2)} - 4\Omega_{s(j}^{(7/2)} m_{i)1}^s$. Recalling the comments following (94), one finds that the same behaviour (110) holds in fact for the full range $\frac{5}{2} < \nu \leq \frac{7}{2}$ (unless $\Psi_{ijk}^{(3)} = 0$). In all cases here ℓ cannot be a WAND, and the asymptotically leading term is of type N.

Note an important difference with the behaviour (94) with $n = 5$: after the leading type N term, the subleading term in (110) is of type III(a) (it was of type II(acd) in (94)). If we assume for Ω_{ij} a fall-off as in (95), this shows that the subleading term of Ω'_{ij} is of order $O(r^{-2})$, thus leading to the qualitatively different peeling-off behaviour

$$C_{abcd} = \frac{N_{abcd}}{r^{3/2}} + \frac{III_{abcd}}{r^2} + o(r^{-2}) \quad (n = 5).\tag{111}$$

However, according to [15] this behaviour is not permitted in asymptotically flat spacetimes. For the latter one thus concludes that $\Psi_{ijk}^{(3)} = 0$ (in which case (110) reduces to (94) with $n = 5$) is a necessary boundary condition in five dimensions. This is perhaps not surprising since $\Psi_{ijk}^{(3)} = 0$ already in four dimensions (where $\Psi_{ijk} = O(r^{-4})$ [3], cf. also (107) above).

If $\nu > 7/2$ the asymptotic behaviour is described by (109) (in which cases ℓ can be a single WAND).

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