Compressible fluid flows driven by stochastic forcing

Eduard Feireisl
based on joint work with D.Breit (Edinburgh), M.Hofmanová (Bielefeld)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

PDE workshop, Stralsund, April 8–11, 2018

The research leading to these results has received funding from the European Research Council
under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

Driven Navier-Stokes/Euler system

Field equations

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho)\mathrm{d}t = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\mathrm{d}t + \boxed{\varrho \mathbf{G}(x, \varrho, \mathbf{u})\mathrm{d}W}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Stochastic forcing

$$\varrho \mathbf{G}(x, \varrho, \mathbf{u})\mathrm{d}W = \sum_{k=1}^{\infty} \varrho \mathbf{G}_k(x, \varrho, \mathbf{u})\mathrm{d}\beta_k$$

Iconic examples

$$\mathbf{G}_k = \mathbf{f}_k(x), \ \mathbf{G}_k = \mathbf{u} d_k(x) - \text{“stochastic damping”}$$

Initial and boundary conditions

(Random) initial data

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0$$

Spatial domain

$$Q \subset R^N,\ \text{or ``flat'' torus}\ Q = \mathcal{T}^N = \left([0,1]|_{\{0,1\}}\right)^N,\ N = (1), 2, 3$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial Q} = 0 \text{ impermeability}$$

$$\mathbf{u} \times \mathbf{n}|_{\partial Q} = 0 \text{ no-slip}$$

$$[\mathbb{S} \cdot \mathbf{n}] \times \mathbf{n}|_{\partial Q} = 0 \text{ complete slip}$$

Weak (PDE) formulation

Field equations

$$\left[\int_Q \varrho\phi \, \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_Q \varrho\mathbf{u} \cdot \nabla_x \phi \, \mathrm{d}x\mathrm{d}t,$$

$$\left[\int_Q \varrho\mathbf{u} \cdot \phi \, \mathrm{d}x\right]_{t=0}^{t=\tau} - \int_0^\tau \int_Q \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \phi + p(\varrho)\mathrm{div}_x \phi \, \mathrm{d}x\mathrm{d}t$$

$$= - \int_0^\tau \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \phi \, \mathrm{d}x\mathrm{d}t + \boxed{\int_0^\tau \left(\int_Q \varrho\mathbf{G} \cdot \phi \, \mathrm{d}x\right) \mathrm{d}W}$$

$\phi = \phi(x) -$ a smooth test function

Stochastic integral (Itô's formulation)

$$\int_0^\tau \left(\int_Q \varrho\mathbf{G} \cdot \phi \, \mathrm{d}x\right) \mathrm{d}W = \sum_{k=1}^\infty \int_0^\tau \left(\int_Q \varrho\mathbf{G}_k \cdot \phi \, \mathrm{d}x\right) \mathrm{d}\beta_k$$

Admissibility

Energy inequality

$$-\int_0^T \partial_t\psi \int_Q \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right] \mathrm{d}x\mathrm{d}t + \int_0^T \psi \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t$$

$$\leq \psi(0) \int_Q \left[\frac{|(\varrho\mathbf{u})_0|^2}{2\varrho_0} + P(\varrho_0)\right] \mathrm{d}x$$

$$+\frac{1}{2}\int_0^T \psi \left(\int_Q \sum_{k\geq 1} \varrho |\mathbf{G}_k(x,\varrho,\mathbf{u})|^2 \ \mathrm{d}x \right) \mathrm{d}t + \int_0^T \psi \mathrm{d}M_E$$

$$\psi \geq 0, \ \psi(T) = 0, \ P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z$$

Strong vs. martingale solutions

Strong solutions

- the functions ϱ, $\mathbf{u}$ are differentiable a.s., the equations are satisfied in the classical sense
- the probability space uniquely determined

Martingale solutions

- solutions defined on a different, typically, the standard probability space
- the white noise as well as the initial data coincide with the originals in law

Existence theory

Local existence of strong solutions [Kim [2011]], [Breit, EF, Hofmanová [2017]]

If the initial data are smooth, then the problem admits local-in-time smooth solutions. Solutions exist up to a (maximal) positive *stopping time*. The life-span is a random variable.

Global existence for the Navier–Stokes system [Breit, Hofmanová [2015]

The Navier–Stokes system admits global–in–time martingale solutions for

$$p(\varrho) \approx \varrho^{\gamma}, \ \gamma > \frac{N}{2}$$

Relative energy inequality

Relative energy

$$\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_Q \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)\right] \,\mathrm{d}x$$

Relative energy inequality

$$-\int_0^T \partial_t\psi \; \mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \,\mathrm{d}t$$

$$+\int_0^T \psi \int_Q \mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \,\mathrm{d}x\mathrm{d}t$$

$$\leq \psi(0)\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(0) + \int_0^T \psi \mathrm{d}M_{RE} + \int_0^T \psi \mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \mathrm{d}t$$

Test functions

$$\mathrm{d}r = D_t^d r \,\mathrm{d}t + \mathbb{D}_t^s r \,\mathrm{d}W, \;\; \mathrm{d}\mathbf{U} = D_t^d \mathbf{U} \,\mathrm{d}t + \mathbb{D}_t^s \mathbf{U} \,\mathrm{d}W$$

Remainder

Remainder term

$$\mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_Q \varrho \Big(D_t^d \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x$$

$$+ \int_Q \big((r - \varrho) P''(r) D_t^d r + \nabla_x P'(r) (r \mathbf{U} - \varrho \mathbf{u}) \big) \, \mathrm{d}x$$

$$- \int_Q \mathrm{div}_x \mathbf{U} (p(\varrho) - p(r)) \, \mathrm{d}x$$

$$+ \frac{1}{2} \sum_{k \geq 1} \int_Q \varrho \Big| \mathbf{G}_k (\varrho, \varrho \mathbf{u}) - [\mathbb{D}_t^s \mathbf{U}]_k \Big|^2 \, \mathrm{d}x$$

$$+ \frac{1}{2} \sum_{k \geq 1} \int_Q \varrho P'''(r) |[\mathbb{D}_t^s r]_k|^2 \, \mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_Q p''(r) |[\mathbb{D}_t^s r]_k|^2 \, \mathrm{d}x$$

$$+ \int_Q \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, \mathrm{d}x$$

Weak–strong uniqueness

Weak–strong uniqueness [Breit, EF, Hofmanová [2016]]

Pathwise uniqueness.
A weak and strong solutions defined on the same probability space and emanating from the same initial data coincide as long as the latter exists

Uniqueness in law.
If a weak and strong solution are defined on a different probability space, then their *laws* are the same provided the laws of the initial data are the same

Stationary solutions to the Navier–Stokes system

Basic hypotheses

- $$|\mathbf{G}_k| + |\nabla \mathbf{G}_k| \approx \alpha_k, \ \sum_{k>0} \alpha_k^2 < \infty$$
- $$p(\varrho) \approx \varrho^\gamma, \ \gamma > \frac{N}{2}$$
- complete slip/no slip boundary conditions

Stationary solutions [Breit, EF, Hofmanová, Maslowski] [2017]

For a given (deterministic) mass

$$M = \int_Q \varrho \ \mathrm{d}x > 0$$

the Navier–Stokes system admits a stationary martingale solution.

Weak (PDE) solutions to the Euler system

Infinitely many weak (PDE) solutions, Breit, EF, Hofmanová [2017]

Let $T > 0$ and the initial data

$$\varrho_0 \in C^3(Q),\ \varrho_0 > 0,\ \mathbf{u}_0 \in C^3(Q)$$

be given.
There exists a sequence of *strictly positive* stopping times

$$\tau_M > 0,\ \tau_M \to \infty$$

a.s. such that the initial–value problem for the *compressible Euler system* possesses infinitely many solutions defined in $(0, T \wedge \tau_M)$. Solutions are adapted to the filtration associated to the Wiener process W.