On singular limits for inviscid fluid flows

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Singular limits

Primitive system

$$\partial_t \mathbf{U}_\varepsilon + \mathcal{B}_\varepsilon \mathbf{U}_\varepsilon + \mathcal{A}(\mathbf{U}_\varepsilon) = 0, \ \mathbf{U}_\varepsilon(0) = \mathbf{U}_{0,\varepsilon}$$

Decomposition

$$\mathbf{U}_\varepsilon = \mathbf{V}_\varepsilon + \underbrace{\mathbf{W}_\varepsilon}, \ \mathcal{B}_\varepsilon \mathbf{V}_\varepsilon = 0,$$

Oscillatory part

$$\partial_t \mathbf{W}_\varepsilon + \mathcal{B}_\varepsilon \mathbf{W}_\varepsilon = 0$$

$$\partial_t \mathbf{V}_\varepsilon + \mathcal{A}(\mathbf{V}_\varepsilon + \mathbf{W}_\varepsilon) = 0$$

Target system

$$\partial_t \mathbf{U} + \mathcal{A}(\mathbf{U}) = 0, \ \mathbf{U}(0,\cdot) = \mathbf{U}_0$$

Ill vs. well prepared initial data

$$\mathbf{U}_{0,\varepsilon} = \mathbf{V}_{0,\varepsilon} + \mathbf{W}_{0,\varepsilon}, \ \mathbf{V}_{0,\varepsilon} \to \mathbf{U}, \ \mathbf{W}_0 \begin{cases} \to 0 \\ \to \mathbf{W}_{0,\varepsilon} \end{cases}$$

Primitive system

Euler system - standard variables

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x \left(\varrho \mathbf{u} \otimes \mathbf{u}\right) + \nabla_x p(\varrho, \vartheta) = \varrho \nabla_x F$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \mathbf{u} \right] + \mathrm{div}_x \left(p(\varrho, \vartheta) \mathbf{u} \right)$$
$$= \varrho \nabla_x F \cdot \mathbf{u}$$

Conservative variables - scaling

Polytropic EOS

$$\mathbf{m} = \varrho \mathbf{u}, \ \ E_\varepsilon = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \frac{1}{\varepsilon^2}\varrho e, \ \ p = (\gamma - 1)\varrho e = (\gamma - 1)\left(E_\varepsilon - \frac{|\mathbf{m}|^2}{\varrho}\right)$$

Euler system - conservative variables

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0,$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}\right) + \boxed{\frac{1}{\varepsilon^2}} \nabla_x p = \boxed{\frac{1}{\varepsilon^2}} \varrho \nabla_x F$$

$$\partial_t E_\varepsilon + \mathrm{div}_x \left[\left(E_\varepsilon + \frac{1}{\varepsilon^2} p\right)\frac{\mathbf{m}}{\varrho}\right] = \frac{1}{\varepsilon^2}\nabla_x F \cdot \mathbf{m}$$

Solution class for the primitive system

Classical solutions

Existence on a short time interval the length of which may depend on ε. Results of this type by Klainerman, Majda, Schochet, Alazard and many others

Weak solutions

Global existence not known. Problem is ill posed e.g. in L^∞ for some initial data

Why to go measure–valued?

Motto: The larger (class) the better

- Universal limits of *numerical* schemes
- Limits of more complex physical systems - vanishing viscosity/heat conductivity limit
- Global existence for “any” data

Weak–strong uniqueness

A (DMV) solution coincides with a smooth solution with the same initial data as long as the latter solution exists

Entropy

Gibbs' relation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy equation (inequality)

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) \boxed{\geq} 0$$

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) \geq 0.$$

Thermodynamic stability - standard variables

$$\frac{\partial p}{\partial \varrho} > 0, \ \ \frac{\partial e}{\partial \vartheta} > 0$$

Thermodynamic stability - conservative variables

$$\mathcal{S}: \ (\varrho, \mathbf{m}, E) \mapsto \varrho s(\varrho, \mathbf{m}, E) \text{ concave function}$$

Entropy renormalization

Entropy in the polytropic case

$$s = S\left(\frac{p}{\varrho^\gamma}\right) = S\left((\gamma - 1)\frac{E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}}{\varrho^\gamma}\right)$$

Renormalization

$$S_\chi = \chi \circ s,\ \partial_t(\varrho S_\chi) + \mathrm{div}_x(\varrho S_\chi \mathbf{u}) \geq 0$$

Levels of renormalization

- **“Isentropic”**

$$\chi''(s) \leq 0$$

- **Standard dissipative**

$$\chi'(s) \geq 0,\ \chi''(s) \leq 0$$

- **Vanishing viscosity**

$$\chi(s) = s$$

Relative energy

Ballistic free energy

$$H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta \varrho s(\varrho,\vartheta)$$

Relative energy in standard variables

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle|\tilde\varrho,\tilde\vartheta,\tilde{\mathbf{u}}\right)$$

$$= \frac{1}{2}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2 + H_{\tilde\vartheta}(\varrho,\vartheta) - \frac{\partial H(\tilde\varrho,\tilde\vartheta)}{\partial\varrho}(\varrho-\tilde\varrho) - H_{\tilde\vartheta}(\tilde\varrho,\tilde\vartheta)$$

Relative energy in the conservative variables

$$\mathcal{E}\left(\varrho,E,\mathbf{m}\middle|\tilde\varrho,\tilde E,\tilde{\mathbf{m}}\right)$$

$$= -\tilde\vartheta\Big[\mathcal{S}(\varrho,E,\mathbf{m}) - \nabla_{\varrho,E,\mathbf{m}}\mathcal{S}(\tilde\varrho,\tilde E,\tilde{\mathbf{m}})\cdot\left(\varrho-\tilde\varrho, E-\tilde E, \mathbf{m}-\tilde{\mathbf{m}}\right)$$

$$-\mathcal{S}(\tilde\varrho,\tilde E,\tilde{\mathbf{m}})\Big], \qquad e(\tilde\varrho,\tilde\vartheta) = \tilde E - \frac{1}{2}\frac{|\tilde{\mathbf{m}}|^2}{\tilde\varrho}$$

Relative energy inequality

$$
\begin{aligned}
&\left[\int_\Omega \mathcal{E} \left(\varrho, \mathbf{m}, E \middle| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\
&\quad \leq - \int_0^\tau \int_\Omega \left[\varrho s(\varrho, \mathbf{m}, E) \partial_t \tilde{\vartheta} + s(\varrho, \mathbf{m}, E) \mathbf{m} \cdot \nabla_x \tilde{\vartheta} \right] \ \mathrm{d}x\mathrm{d}t \\
&\qquad + \int_0^\tau \int_\Omega \left[(\varrho \tilde{\mathbf{u}} - \mathbf{m}) \cdot \partial_t \tilde{\mathbf{u}} + \frac{(\varrho \tilde{\mathbf{u}} - \mathbf{m}) \otimes \mathbf{m}}{\varrho} : \nabla_x \tilde{\mathbf{u}} \right] \ \mathrm{d}x\mathrm{d}t \\
&\qquad - (\gamma - 1) \int_0^\tau \int_\Omega \left[\left(E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} \right) \mathrm{div}_x \tilde{\mathbf{u}} \right] \ \mathrm{d}x\mathrm{d}t \\
&\qquad + \int_0^\tau \int_\Omega \left[\varrho \partial_t \tilde{\vartheta} s(\tilde{\varrho}, \tilde{\vartheta}) + \mathbf{m} \cdot \nabla_x \tilde{\vartheta} s(\tilde{\varrho}, \tilde{\vartheta}) \right] \ \mathrm{d}x\mathrm{d}t \\
&\qquad + \int_0^\tau \int_\Omega \left[(\tilde{\varrho} - \varrho) \frac{1}{\tilde{\varrho}} \partial_t p(\tilde{\varrho}, \tilde{\vartheta}) - \mathbf{m} \cdot \frac{1}{\tilde{\varrho}} \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \right] \ \mathrm{d}x\mathrm{d}t
\end{aligned}
$$

Dissipative measure–valued (DMV) solutions

Parameterized measure

$$\underbrace{\mathcal{F}}_{\text{phase space}} = \left\{ \varrho \geq 0,\ \mathbf{m} \in R^3,\ E \in [0, \infty) \right\}, \qquad \underbrace{Q_T}_{\text{physical space}} = (0, T) \times \Omega$$

$$\{\mathcal{V}_{t,x}\}_{(t,x) \in Q_T},\ \ Y_{t,x} \in \mathcal{P}(\mathcal{F})$$

Field equations

$$\partial_t \left\langle \mathcal{V}_{t,x}; \varrho \right\rangle + \mathrm{div}_x \left\langle \mathcal{V}_{t,x}; \mathbf{m} \right\rangle = 0$$

$$\partial_t \left\langle \mathcal{V}_{t,x}; \mathbf{m} \right\rangle + \mathrm{div}_x \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle + \nabla_x \left\langle \mathcal{V}_{t,x}; p \right\rangle = D_x \mu_C$$

$$\partial_t \int_\Omega \left\langle \mathcal{V}_{t,x}; E \right\rangle \ \mathrm{d}x + \mathcal{D} = 0,\ \ \partial_t \left\langle \mathcal{V}_{t,x}; \varrho s \right\rangle + \mathrm{div}_x \left\langle \mathcal{V}_{t,x}; s\mathbf{m} \right\rangle \geq 0$$

Compatibility

$$\int_0^\tau \int_\Omega |\mu_C| \ \mathrm{d}x\mathrm{d}t \leq C \int_0^\tau \mathcal{D}\mathrm{d}t$$

Relative energy inequality for (DMV) solutions

$$
\begin{aligned}
&\Big[\int_\Omega \left\langle \mathcal{V}_{t,x}; \mathcal{E}\left(\varrho, \mathbf{m}, E \Big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \right\rangle \ \mathrm{d}x \Big]_{t=0}^{t=\tau} + \mathcal{D}(\tau) \\
&\leq - \int_0^\tau \int_\Omega \Big[\langle \mathcal{V}_{t,x}; \varrho s(\varrho, \mathbf{m}, E) \rangle \, \partial_t \tilde{\vartheta} + \langle \mathcal{V}_{t,x}; s(\varrho, \mathbf{m}, E)\mathbf{m} \rangle \cdot \nabla_x \tilde{\vartheta} \Big] \ \mathrm{d}x \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \tilde{\mathbf{u}} - \mathbf{m} \rangle \cdot \partial_t \tilde{\mathbf{u}} + \left\langle \mathcal{V}_{t,x}; \frac{(\varrho \tilde{\mathbf{u}} - \mathbf{m}) \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \tilde{\mathbf{u}} \right] \ \mathrm{d}x \mathrm{d}t \\
&\quad - (\gamma - 1) \int_0^\tau \int_\Omega \left[\left\langle \mathcal{V}_{t,x}; E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} \right\rangle \mathrm{div}_x \tilde{\mathbf{u}} \right] \ \mathrm{d}x \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \Big[\langle \mathcal{V}_{t,x}; \varrho \rangle \, \partial_t \tilde{\vartheta} s(\tilde{\varrho}, \tilde{\vartheta}) + \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \tilde{\vartheta} s(\tilde{\varrho}, \tilde{\vartheta}) \Big] \ \mathrm{d}x \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \tilde{\varrho} - \varrho \rangle \, \frac{1}{\tilde{\varrho}} \partial_t p(\tilde{\varrho}, \tilde{\vartheta}) - \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \frac{1}{\tilde{\varrho}} \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \right] \ \mathrm{d}x \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega \nabla_x \tilde{\mathbf{u}} : \mathrm{d}\mu_C,
\end{aligned}
$$

Limits of Euler flows with strong stratification

Scaled Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \frac{1}{\varepsilon^2} \varrho \nabla_x F,$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \varrho e(\varrho, \vartheta) \right) \mathbf{u} \right]$$

$$+ \mathrm{div}_x \left(\frac{1}{\varepsilon^2} p(\varrho, \vartheta) \mathbf{u} \right) = \frac{1}{\varepsilon^2} \varrho \nabla_x F \cdot \mathbf{u}.$$

Geometry

$$\Omega = \mathcal{T}^2 \times (0,1), \ \mathcal{T}^2 = [0,1]|_{\{0,1\}} - \text{the two dimensional torus}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Isothermal case

Boyle-Marriot EoS - stationary problem

$$p = \varrho\vartheta,\ F = F(z) = -z,\ \nabla_x p = \varrho \nabla_x F$$

Isothermal case

$$\nabla_x(\varrho_s \overline{\Theta}) = -\varrho_s \nabla_x F,\ \varrho_s = \exp\left(-\frac{z}{\overline{\Theta}}\right),\ \overline{\Theta} > 0$$

Well–prepared initial data

$$\varrho_{0,\varepsilon} = \varrho_s + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vartheta_{0,\varepsilon} = \overline{\Theta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \mathbf{u}_{0,\varepsilon}$$

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)} + \|\vartheta^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^\infty(\Omega;R^N)} \leq c,$$

$$\varrho^{(1)}_{\varepsilon} \to 0,\ \vartheta^{(1)}_{0,\varepsilon} \to 0,\ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0,$$

$$\mathbf{U}_0 \in W^{k,2}(\Omega; R^3),\ k > 3,\ \mathbf{U}_0 = [U^1_0, U^2_0, 0],\ \mathrm{div}_h \mathbf{U}_0 = 0.$$

Target problem - isothermal case

Limit velocity

$$\mathbf{U} = \mathbf{U}(t, x_h, z),\ x_h \in \mathcal{T}^2,\ z \in (0,1)$$

Incompressible Euler system in $2D$

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_h \mathbf{U} + \nabla_x \Pi = 0,\ \mathrm{div}_h \mathbf{U} = 0 \text{ in } \mathcal{T}^2,\ z \text{ fixed}$$

Stratified initial data

$$\mathbf{U}(0,x) = \mathbf{U}_0(x_h, z) = [U_0^1(x_h, z), U_0^2(x_h, z), 0]$$

Singular limit (MV) $\to$ strong

Convergence to the target system

Let $\{\mathcal{V}^{\varepsilon}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$, $\mathcal{D}^{\varepsilon}$ be a family of dissipative measure–valued solutions to the scaled system scaled Euler system, with the well prepared initial data

$$\mathcal{V}^{\varepsilon}_{0,x} = \delta_{\varrho_{0,\varepsilon},\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon},c_v\varrho_{0,\varepsilon}\vartheta_{0,\varepsilon}}.$$

Then

$$\mathcal{D}^{\varepsilon} \to 0 \text{ in } L^{\infty}(0,T),$$

and

$$\mathcal{V}^{\varepsilon} \to \delta_{\varrho_s,\varrho_s\mathbf{U},c_v\varrho_s\overline{\Theta}} \text{ in } L^{\infty}(0,T;L^1(\Omega;\mathcal{M}^+(\mathcal{F})_{\mathrm{weak}-(*)})),$$

where $[\varrho_s,\overline{\Theta}]$ is the static state and $\mathbf{U}$ is the unique solution to the incompressible 2D Euler system

Isentropic case

Stationary problem

$$p = \varrho\vartheta, \ \ F = F(z) = -z$$

$$s_s(\varrho_s, \vartheta_s) = \overline{s}, \ \ \exp\left((\gamma - 1)\overline{s}\right) \nabla_x(\varrho_s^\gamma) = -\varrho_s \nabla_x F$$

$$p_s = \exp\left((\gamma - 1)\overline{s}\right) \varrho_s^\gamma$$

Target problem - isentropic case

Anelastic Euler system

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla_x \Pi = 0, \ \mathrm{div}(\varrho_s \mathbf{U}) = 0, \ x \in \Omega$$

Singular limit (MV) $\to$ strong

Let

$$\mathbf{U}_0 \in W^{k,2}(\Omega; R^N), k \geq N, \ \mathrm{div}_x(\varrho_s \mathbf{U}_0) = 0, \ \mathbf{U}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Suppose that the anelastic Euler system with initial datum $\mathbf{U}_0$ admits a unique strong solution $\mathbf{U}$ defined on a maximal time interval $[0, T_{\max})$.
Let $\{\mathcal{V}^{\varepsilon}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$, $0 < T < T_{\max}$, be a family of (DMV) solutions such that

$$\mathcal{V}^{\varepsilon}_{0,x} \left\{ [\varrho, \mathbf{m}, p] \ \Big| \left| \frac{\varrho - \varrho_s}{\varepsilon} \right| + \left| \frac{\mathbf{m}}{\varrho} - \mathbf{U}_0 \right| + \left| \frac{p - p_s}{\varepsilon} \right| \leq M_\varepsilon(x) \right\} = 1 \ \text{ for a.a. } \ x \in \Omega,$$

where

$$\|M_\varepsilon\|_{L^\infty(\Omega)} \leq c \qquad \text{and} \qquad M_\varepsilon \to 0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0.$$

Suppose that the initial entropy satisfies

$$\mathcal{V}^{\varepsilon}_{0,x} \left\{ [\varrho, \mathbf{m}, p] \ \Big| \bar{s} - \varepsilon^{2+\alpha} \leq s(\varrho, p) \leq \bar{s} + \varepsilon^{2+\alpha} \right\} = 1 \ \text{ for a.a. } \ x \in \Omega, \ \alpha > 0$$

Then

$$\mathcal{D}^\varepsilon \to 0 \quad \text{in } L^\infty(0, T) \text{ as } \varepsilon \to 0,$$

and

$$\mathcal{V}^\varepsilon \to \delta_{\tilde{\varrho}, \tilde{\varrho}\mathbf{U}, \tilde{p}} \quad \text{in } L^\infty(0, T, L^q(\Omega; \mathcal{M}^+(\mathcal{F}))) \text{ as } \varepsilon \to 0 \text{ for any } 1 \leq q < \infty$$

Ill prepared initial data

Isentropic Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = 0$$

$$\Omega = R^N, \ N = 2, 3, \ \varrho \to \overline{\varrho} > 0, \ \mathbf{u} \to 0 \text{ as } |x| \to \infty$$

Ill prepared initial data

$$\varrho(0, \cdot) = \varrho_{0,\varepsilon}, \ \varrho_{0,\varepsilon} \approx \overline{\varrho} + \varepsilon \boxed{s_\varepsilon} \to s \text{ in } L^1 \cap L^\infty(R^N)$$

$$\mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 + \boxed{\nabla_x \Phi} \text{ in } L^2 \cap L^\infty(R^N, R^N)$$

Target system, acoustic waves

Incompressible Euler system

$$\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U}) + \nabla_x \Pi = 0, \ \mathrm{div}_x \mathbf{U} = 0 \text{ in } R^N$$

Acoustic waves

$$\varepsilon \partial_t s_\varepsilon + \mathrm{div}_x(\overline{\varrho} \nabla_x \Phi) = 0$$

$$\varepsilon \partial_t \nabla_x \Phi + \frac{p'(\overline{\varrho})}{\overline{\varrho}} \nabla_x s = 0$$

Dispersive estimates (Strichartz's estimates)

Strichartz estimates

$$\|s(\tau,\cdot)\|^2_{L^p(R^3)} + \|\nabla_x \Phi(\tau,\cdot)\|^2_{L^p(R^3;R^3)}$$

$$\leq c\left(1+\frac{\tau}{\varepsilon}\right)^{(N-1)\left(\frac{1}{p}-\frac{1}{q}\right)} \left[\|s_0\|^2_{W^{k,q}(R^3)} + \|\nabla_x \Phi_0\|^2_{W^{k,q}(R^3;R^3)}\right],$$

$$k \geq N\left(\frac{1}{q}-\frac{1}{p}\right),\ 2 \leq p \leq \infty,\ \frac{1}{p}+\frac{1}{q}=1$$