Reliable numerical methods for elliptic partial differential eigenvalue problems

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Reliable numerical methods

To compute (approximate) solution is not sufficient.
We should provide an information about the error.

Can we provide
a guaranteed upper bound?

$$\|u - u_h\| \leq \eta$$

Sinking of the Sleipner A offshore platform in 1991, Norway. The failure resulted from inaccurate NASTRAN calculations.

Babuška, Verfürth, Ainsworth, Rannacher, Repin, . . .

Eigenvalue problems

Laplace eigenvalue problem

$$\begin{aligned} -\Delta u_n &= \lambda_n u_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega \end{aligned}$$

Finite element method

- Very flexible (various domains, high order, various problems, ...)
- Converges with optimal speed
- Adaptive mesh refinement
- Nice theory

Guaranteed upper bound

$$\lambda_n \leq \lambda_{h,n}$$

Can we dream about anything else?

Eigenvalue problems

Laplace eigenvalue problem

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Finite element method

- Very flexible (various domains, high order, various problems, ...)
- Converges with optimal speed
- Adaptive mesh refinement
- Nice theory

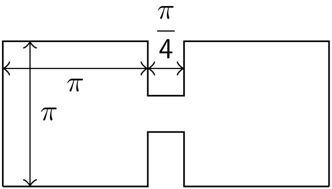
Guaranteed upper bound

$$? \le \lambda_n \le \lambda_{h,n}$$

Can we dream about anything else? Lower bounds!

Example – dumbbell

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega$$
$$u_n = 0 \quad \text{on } \partial\Omega$$



[Trefethen, Betcke 2006]

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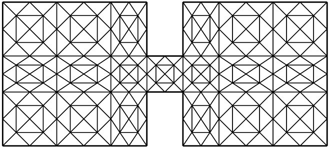
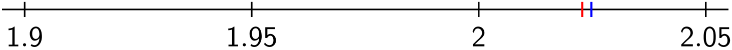
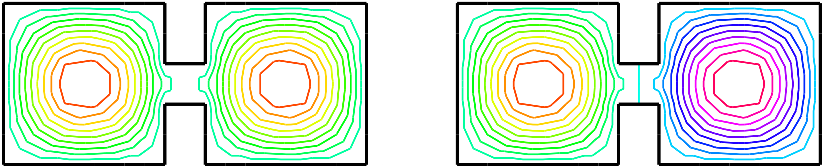




$\lambda_1 \approx 2.02280$ $\lambda_2 \approx 2.02481$



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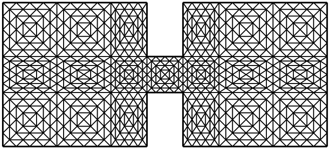

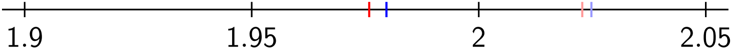

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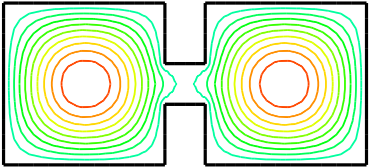

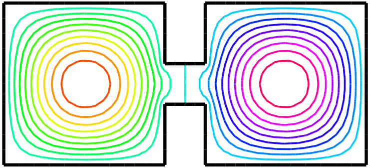

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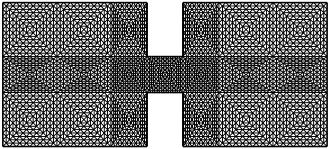

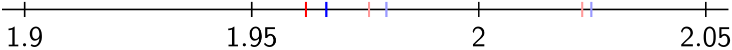

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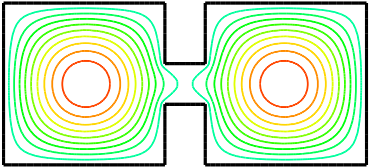

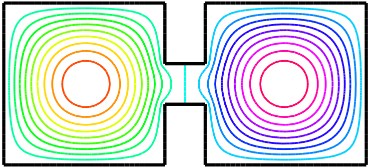

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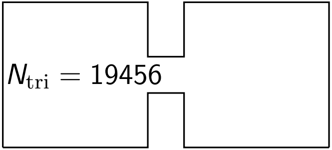

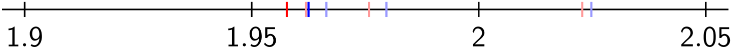

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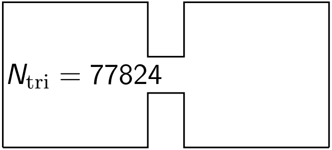

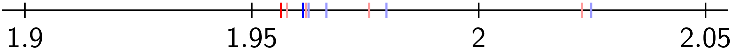

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Example – dumbbell

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
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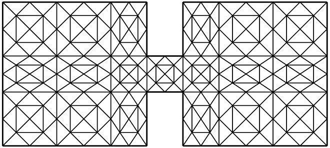
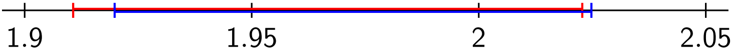
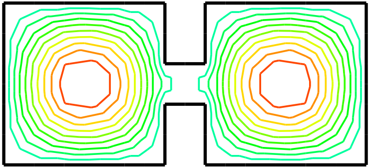
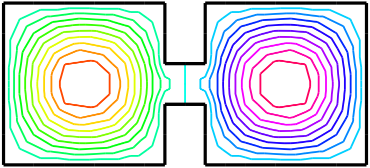




$1.91067 \le \lambda_1 \le 2.02280$ $\quad$ $1.91981 \le \lambda_2 \le 2.02481$





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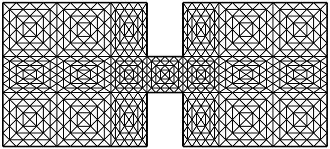

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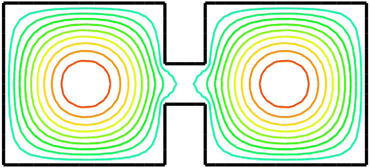

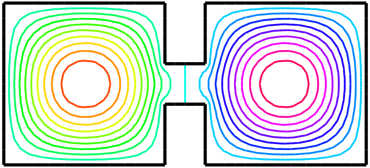

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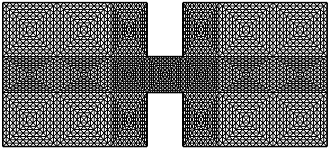

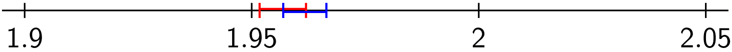

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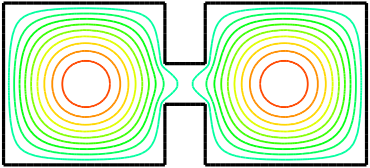

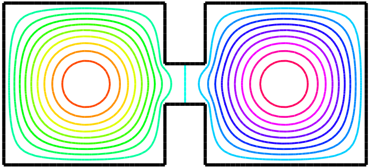

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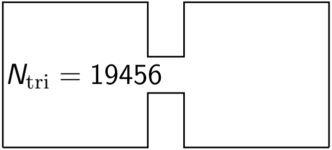

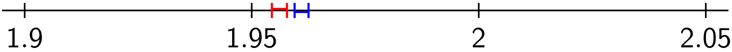

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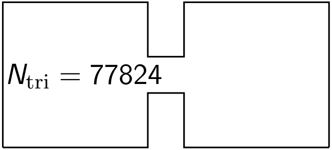

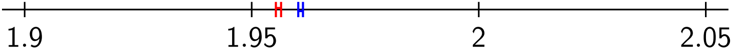

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Outline

1. Motivation
2. Theory
   2.1 Existence
   2.2 Min-max principle
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2. Theory

2.1 Existence

Abstract formulation

Eigenvalue problem Find eigenvalue λ_n and eigenfunction $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V.$$

- V is a Hilbert space.
- $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are two bilinear forms on V.

Example

$$\begin{aligned} -\Delta u_n &= \lambda_n u_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak formulation

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in V$$

- $V = H_0^1(\Omega)$
- $a(u, v) = (\nabla u, \nabla v)$
- $b(u, v) = (u, v)$

Hilbert-Schmidt theorem

$$Su_n = \mu_n u_n$$

Let

- V be a Hilbert space
- $S : V \to V$ be linear, bounded, compact, self-adjoint operator

Then

- there is a countable sequence of nonzero real eigenvalues of S (repeated according to their multiplicity): $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots > 0$, $\lim_{n\to\infty} \mu_n = 0$
- eigenfunctions u_n form an orthonormal basis of range S (range S is closed)
- $V = (\ker S) \oplus (\text{range}\, S)$

Assumptions

Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$: $\quad a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$

- V is a real Hilbert space
- $a(\cdot,\cdot)$ is continuous, bilinear, symmetric, V-elliptic
- $b(\cdot,\cdot)$ is continuous, bilinear, symmetric, positive semidefinite
- $\|v\|_a = a(v,v)^{1/2}$ is the norm induced by $a(\cdot,\cdot)$
- $|v|_b = b(v,v)^{1/2}$ is the seminorm induced by $b(\cdot,\cdot)$
- $|\cdot|_b$ is compact with respect to $\|\cdot\|_a$,
  i.e. from any sequence bounded in $\|\cdot\|_a$, we can extract a subsequence which is Cauchy in $|\cdot|_b$

Existence

Theorem. There exists (at most) countable sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty$$

and the corresponding eigenfunctions can be normalized to satisfy

$$b(u_i, u_j) = \delta_{ij} \quad \forall i, j = 1, 2, \dots .$$

Proof

- Solution operator $S : V \to V$: $a(Su, v) = b(u, v) \quad \forall v \in V$
- $a(u_n, v) = \lambda_n \underbrace{b(u_n, v)}_{a(Su_n, v)} \quad \forall v \in V \quad \Leftrightarrow \quad Su_n = \dfrac{1}{\lambda_n} u_n$
- Exercise: compactness of $|\cdot|_b$ with respect to $\|\cdot\|_a$ is equivalent to compactness of S
- Hilbert-Schmidt theorem: $\mu_1 \ge \mu_2 \ge \mu_3 \ge \cdots > 0$, $\mu_n = 1/\lambda_n$

□

Existence

Theorem. There exists (at most) countable sequence of eigenvalues

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and the corresponding eigenfunctions can be normalized to satisfy

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Note

$$\frac{1}{\lambda_i} a(u_i, u_j) = \delta_{ij} \quad \forall i, j = 1, 2, \ldots$$

Orthonormal basis of eigenfunctions

Theorem. The space V can be decomposed as

$$V = \mathcal{K} \oplus \mathcal{M},$$

where $\mathcal{K} = \{v \in V : |v|_b = 0\}$ and $\mathcal{M} = \operatorname{span}\{u_1, u_2, \dots\}$.
Moreover,

$$\begin{aligned} a(u,v) &= 0 \quad \forall u \in \mathcal{K},\ \forall v \in \mathcal{M}, \\ b(u,v) &= 0 \quad \forall u \in \mathcal{K},\ \forall v \in V. \qquad (*) \end{aligned}$$

Proof

- $(*)$ follows from $b(u,v) \le |u|_b |v|_b = 0$
- Hilbert-Schmidt theorem: $V = (\ker S) \oplus \mathcal{M}$
  Now, $\ker S = \mathcal{K}$, because
  (a) $u \in \mathcal{K} \Rightarrow 0 = b(u,v) = a(Su,v)\ \forall v \in V$
  $\Rightarrow Su = 0 \Rightarrow u \in \ker S$
  (b) $u \in \ker S \Rightarrow 0 = a(Su,u) = b(u,u) = |u|_b^2 \Rightarrow u \in \mathcal{K}$

Orthonormal basis of eigenfunctions

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Proof

- Express $v \in \mathcal{M}$ as $v = \sum_{n=1}^{\infty} c_n u_n$ and

$$a(u, v) = \sum_{n=1}^{\infty} c_n a(u, u_n) = \sum_{n=1}^{\infty} c_n \lambda_n b(u, u_n) \overset{(*)}{=} 0.$$

□

Parseval's identities

Theorem. For all $v \in V$, there are unique $v^{\mathcal{K}} \in \mathcal{K}$ and $v^{\mathcal{M}} \in \mathcal{M}$ such that

$$v = v^{\mathcal{K}} + v^{\mathcal{M}}, \quad v^{\mathcal{M}} = \sum_{n=1}^{\infty} c_n u_n, \quad c_n = b(v^{\mathcal{M}}, u_n) = b(v, u_n)$$

$$|v|_b^2 = \sum_{n=1}^{\infty} |b(v, u_n)|^2,$$

$$\|v\|_a^2 = \|v^{\mathcal{K}}\|_a^2 + \|v^{\mathcal{M}}\|_a^2 \quad \text{with } \|v^{\mathcal{M}}\|_a^2 = \sum_{n=1}^{\infty} \lambda_n |b(v, u_n)|^2.$$

Proof

- $v = v^{\mathcal{K}} + v^{\mathcal{M}} = v^{\mathcal{K}} + \sum_{n=1}^{\infty} c_n u_n$
- $|v|_b^2 = b(v, v^{\mathcal{K}} + \sum_{n=1}^{\infty} c_n u_n) = \sum_{n=1}^{\infty} c_n b(v, u_n)$
- $\|v\|_a^2 = \|v^{\mathcal{M}}\|_a^2 + \|v^{\mathcal{K}}\|_a^2$ and $\|v^{\mathcal{M}}\|_a^2 = \sum_{n=1}^{\infty} \lambda_n c_n^2$

$\square$

Example 1: Dirichlet Laplacian

$$
\begin{aligned}
-\Delta u_n &= \lambda_n u_n \quad \text{in } \Omega \\
u_n &= 0 \qquad\quad \text{on } \partial\Omega
\end{aligned}
$$

Weak formulation: Find $\lambda_n \in \mathbb{R}$, $u_n \in H^1_0(\Omega) \setminus \{0\}$:

$$
(\nabla u_n, \nabla v) = \lambda_n (I u_n, I v) \quad \forall v \in H^1_0(\Omega),
$$

where $I : H^1_0(\Omega) \to L^2(\Omega)$ is the identity operator.

- $V = H^1_0(\Omega)$
- $a(u, v) = (\nabla u, \nabla v)$ ... cont., bilin., sym., V-elliptic
- $b(u, v) = (u, v)$ ... cont., bilin., sym., pos. def.
- Compactness: I is a compact operator by Rellich theorem.
  Definition: I is compact if from a sequence $\{v_i\} \subset H^1_0(\Omega)$ bounded in $\|\nabla v\|_{L^2(\Omega)} \leq C$ we can extract a subsequence such that $\{I v_{i_j}\}$ is Cauchy in $L^2(\Omega)$.

Example 1: Dirichlet Laplacian

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Exact solution for an interval $\Omega = (0, L)$

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad u_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Easy to verify

$$u_n'(x) = \frac{n\pi}{L}\cos\frac{n\pi x}{L}$$

$$u_n''(x) = -\frac{n^2\pi^2}{L^2}\sin\frac{n\pi x}{L} = -\frac{n^2\pi^2}{L^2}u_n(x)$$

Is it complete?

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$$

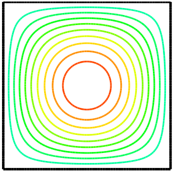
Exact solution for a square $\Omega = (0, \pi)^2$

$$\lambda_{k,\ell} = k^2 + \ell^2, \quad u_{k,\ell}(x, y) = \sin(kx)\sin(\ell y), \quad k, \ell = 1, 2, \ldots$$

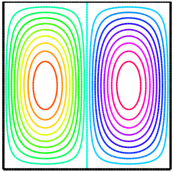
$$
\begin{aligned}
\lambda_1 &= 2\ (k = 1,\ \ell = 1) & \lambda_6 &= 10\ (k = 1,\ \ell = 3) \\
\lambda_2 &= 5\ (k = 2,\ \ell = 1) & \lambda_7 &= 13\ (k = 3,\ \ell = 2) \\
\lambda_3 &= 5\ (k = 1,\ \ell = 2) & \lambda_8 &= 13\ (k = 2,\ \ell = 3) \\
\lambda_4 &= 8\ (k = 2,\ \ell = 2) & \lambda_9 &= 17\ (k = 4,\ \ell = 1) \\
\lambda_5 &= 10\ (k = 3,\ \ell = 1) & \lambda_{10} &= 17\ (k = 1,\ \ell = 4)
\end{aligned}
$$

Example: Square

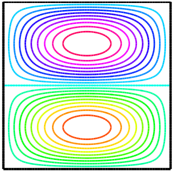
$\lambda_1 = 2,\ u_1(x, y) = \sin(x)\sin(y)$

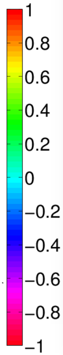


$\lambda_2 = 5,\ u_2(x, y) = \sin(2x)\sin(y)$



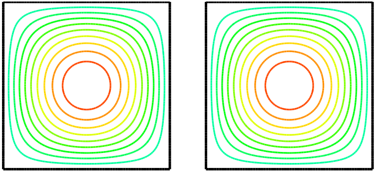
$\lambda_3 = 5,\ u_3(x, y) = \sin(x)\sin(2y)$



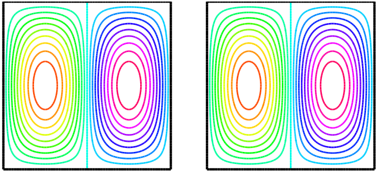


Example: Two squares

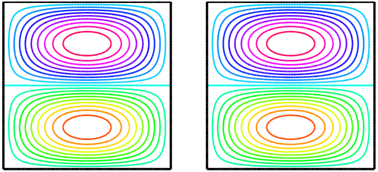
$\lambda_1 = 2$

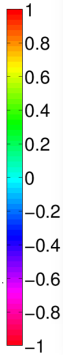


$\lambda_2 = 5$

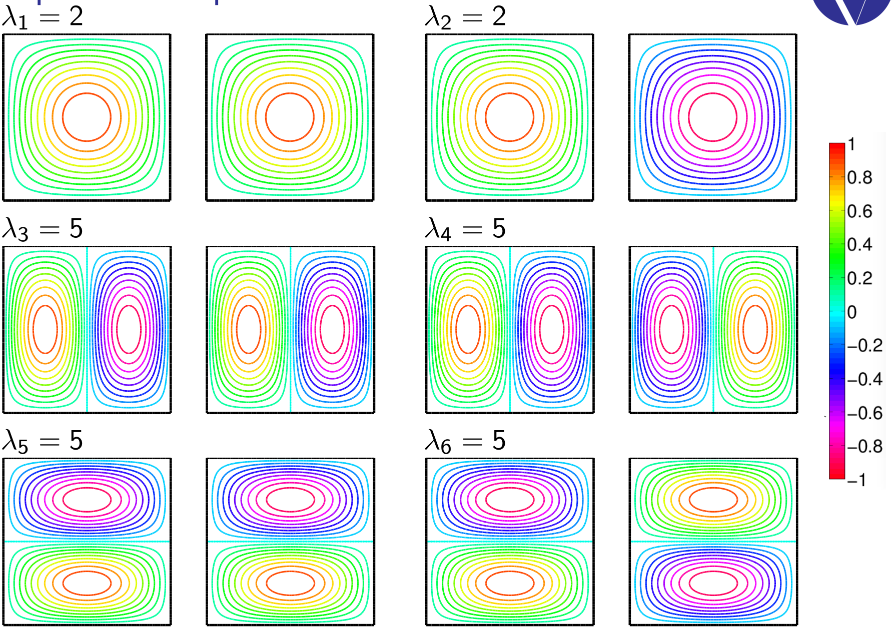


$\lambda_3 = 5$

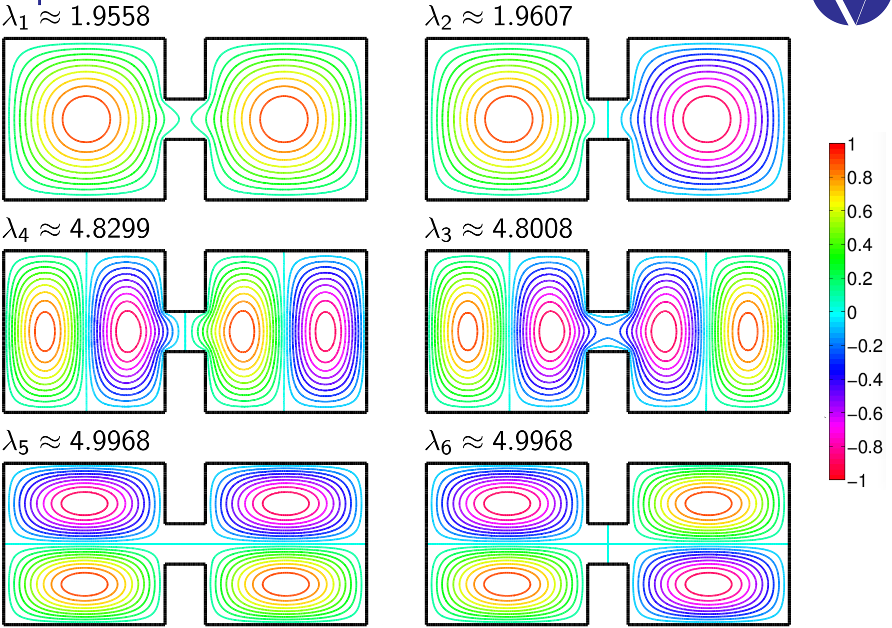




Example: Two squares



Example: Dumbbell



2. Theory

2.2 Min-max principle

Minimum principle

Rayleigh quotien: $R(v) = \dfrac{a(v,v)}{b(v,v)} = \dfrac{\|v\|_a^2}{|v|_b^2}$

Theorem. Numbers $0 < \lambda_1 \le \lambda_2 \le \cdots$ and functions $u_1, u_2, \cdots \in V \setminus \{0\}$ are eigenpairs of

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

if and only if

$$\lambda_1 = \min_{v\in V,\ |v|_b\neq 0} R(v) \qquad u_1 = \operatorname*{arg\,min}_{v\in V,\ |v|_b\neq 0} R(v),$$

$$\lambda_n = \min_{v\in \mathcal{M}_{n-1}^{\perp}} R(v) \qquad u_n = \operatorname*{arg\,min}_{v\in \mathcal{M}_{n-1}^{\perp}} R(v),$$

where $\mathcal{M}_{n-1} = \operatorname{span}\{u_1, u_2, \ldots, u_{n-1}\}$,

$$\begin{aligned}\mathcal{M}_{n-1}^{\perp} &= \{v \in \mathcal{M} : b(v, u_i) = 0,\ \forall i = 1, 2, \ldots, n-1\}\\ &= \{v \in V : b(v, u_i) = 0,\ \forall i = 1, 2, \ldots, n-1\\ &\qquad\qquad \text{and } |v|_b \neq 0\}.\end{aligned}$$

Minimum principle

Proof. (Including $n = 1$).

$\Rightarrow$ Let $a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$.

Then $u_n \in \mathcal{M}_{n-1}^{\perp}$, $\lambda_n = R(u_n)$, and thus $\min_{\mathcal{M}_{n-1}^{\perp}} R(v) \leq \lambda_n$.

If $v \in \mathcal{M}_{n-1}^{\perp}$ then $v^{\mathcal{K}} = 0$, $c_i = b(v, u_i) = 0$ for $i = 1, \dots, n-1$, and

$$R(v) = \frac{\|v\|_a^2}{|v|_b^2} = \frac{\sum_{i=n}^{\infty} \lambda_i c_i^2}{\sum_{i=n}^{\infty} c_i^2} \geq \lambda_n \frac{\sum_{i=n}^{\infty} c_i^2}{\sum_{i=n}^{\infty} c_i^2} = \lambda_n$$

$\Leftarrow$ The minimum is attained: $\exists u_n \in \mathcal{M}_{n-1}^{\perp} : \quad \lambda_n = R(u_n)$.

Let $t \in \mathbb{R}$, $v \in \mathcal{M}_{n-1}^{\perp}$ and $\varphi(t) = R(u_n + tv)$.

Derivative $\varphi'(0)$ exists and

$$\varphi'(0) = \frac{2}{|u_n|_b} \left(a(u_n, v) - \frac{\|u_n\|_a^2}{|u_n|_b^2} b(u_n, v) \right)$$

Since $\varphi(t)$ has a minimum at $t = 0$, we have $\varphi'(0) = 0$.

If $v = u_i$, $i = 1, 2, \dots, n-1$, then

$$b(u_n, u_i) = 0 \text{ and } a(u_n, u_i) = \lambda_i b(u_n, u_i) = 0.$$

(Courant–Fischer–Weyl) Min-max principle

Theorem.

$$\lambda_n = \min_{v \in \mathcal{M}_{n-1}^{\perp}} R(v) = \min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v)$$

where $\mathcal{V}^{(n)}$ is the set of all n-dimensional subspaces of $\mathcal{M}$.
Moreover, the mininum is attaind for $E = \mathrm{span}\{u_1, \ldots, u_n\}$.
Proof. (Induction over n.)
$n = 1$: Since $R(\alpha v) = R(v)$ for all $\alpha \neq 0$, we have

$$\min_{E \in \mathcal{V}^{(1)}} \max_{v \in E} R(v) = \min_{v \in \mathcal{M}} R(v) = \min_{v \in V,\ |v|_b \neq 0} R(v)$$

(Courant–Fischer–Weyl) Min-max principle

Theorem.

$$\lambda_n = \min_{v \in \mathcal{M}_{n-1}^{\perp}} R(v) = \min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v)$$

where $\mathcal{V}^{(n)}$ is the set of all n-dimensional subspaces of $\mathcal{M}$.
Moreover, the mininum is attaind for $E = \operatorname{span}\{u_1, \dots, u_n\}$.

Proof. (Induction over n.)
$n > 1$: Let $\widetilde{\mathcal{V}}^{(n)} \subset \mathcal{V}^{(n)}$ be a set of all spaces $\widetilde{E}^z = \operatorname{span}\{u_1, \dots, u_{n-1}, z\}$, where $b(z, u_i) = 0$ for $i = 1, \dots, n-1$.

$$\min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v) \le \min_{\widetilde{E}^z \in \widetilde{\mathcal{V}}^{(n)}} \max_{v \in \widetilde{E}^z} R(v) = \min_{z \in \mathcal{M}_{n-1}^{\perp}} \max_{v \in \widetilde{E}^z} R(v) \overset{(!)}{=} \min_{z \in \mathcal{M}_{n-1}^{\perp}} R(z)$$

To prove (!), let $v \in \widetilde{E}^z$, $|v|_b = |z|_b = 1$. Thus,
$v = \alpha z + \sum_{i=1}^{n-1} c_i u_i$, $|v|_b^2 = \alpha^2 + \sum_{i=1}^{n-1} c_i^2 = 1$, and

$$R(v) = \|v\|_a^2 = \alpha^2 \|z\|_a^2 + \sum_{i=1}^{n-1} c_i^2 \|u_i\|_a^2 \le \left(\alpha^2 + \sum_{i=1}^{n-1} c_i^2\right) \|z\|_a^2 = R(z),$$

because $z \in \mathcal{M}_{i-1}^{\perp}$ for all $i = 1, 2, \dots, n-1$ and $R(u_i) \le R(z)$.

(Courant–Fischer–Weyl) Min-max principle

Theorem.

$$\lambda_n = \min_{v \in \mathcal{M}_{n-1}^{\perp}} R(v) = \min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v)$$

where $\mathcal{V}^{(n)}$ is the set of all n-dimensional subspaces of $\mathcal{M}$.
Moreover, the mininum is attaind for $E = \text{span}\{u_1, \ldots, u_n\}$.

Proof. (Induction over n.)

$n > 1$: (cont'd)

Let $E \in \mathcal{V}^{(n)}$.

There exists $z \in E : |z|_b \neq 0$ and $b(z, u_i) = 0$ for $i = 1, 2, \ldots, n-1$.

$$\max_{v \in E} R(v) \geq R(z) \geq \min_{z \in \mathcal{M}_{n-1}^{\perp}} R(z)$$

□

Example 2: Neumann Laplacian

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega$$
$$\frac{\partial u_n}{\partial \nu} = 0 \qquad \text{on } \partial\Omega$$

Weak formulation: Find $\lambda_n \in \mathbb{R}$, $u_n \in H^1(\Omega) \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in H^1(\Omega)$$

Problem: $u_0 \equiv 1$, $\lambda_0 = 0$
$\Rightarrow$ bilinear form $a(u, v) = (\nabla u, \nabla v)$ is not $H^1(\Omega)$-elliptic.

- $V = \{v \in H^1(\Omega) : \int_\Omega v = 0\}$
- $a(u, v) = (\nabla u, \nabla v)$ ... cont., bilin., sym., V-elliptic
- $b(u, v) = (u, v)$ ... cont., bilin., sym., pos. def.
- Compactness: by Rellich theorem.

Example 2: Neumann Laplacian

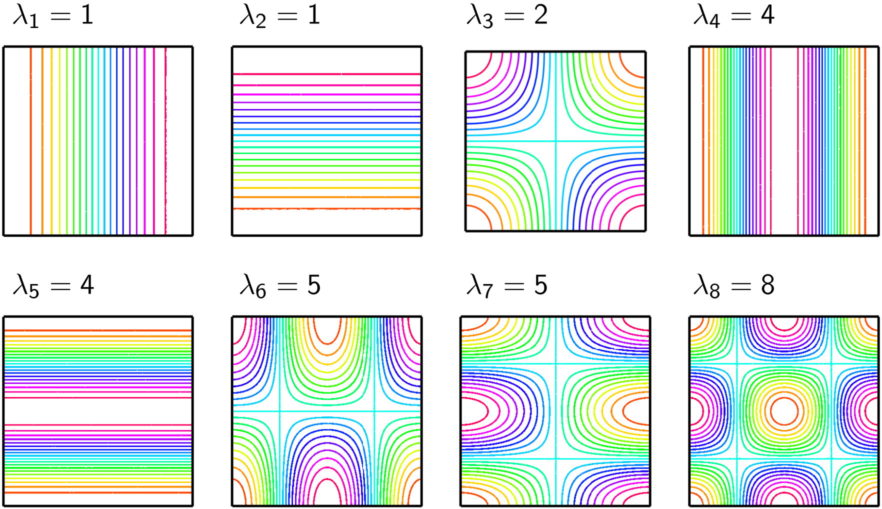
$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega$$
$$\frac{\partial u_n}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

Exact solution for a square $\Omega = (0, \pi)^2$

$$\lambda_{k,\ell} = k^2 + \ell^2, \quad u_{k,\ell}(x, y) = \cos(kx)\cos(\ell y), \quad k, \ell = 0, 1, 2, \ldots$$

$$\lambda_0 = 0 \ (k = 0, \ \ell = 0) \qquad \lambda_5 = 4 \ (k = 0, \ \ell = 2)$$
$$\lambda_1 = 1 \ (k = 1, \ \ell = 0) \qquad \lambda_6 = 5 \ (k = 2, \ \ell = 1)$$
$$\lambda_2 = 1 \ (k = 0, \ \ell = 1) \qquad \lambda_7 = 5 \ (k = 1, \ \ell = 2)$$
$$\lambda_3 = 2 \ (k = 1, \ \ell = 1) \qquad \lambda_8 = 8 \ (k = 2, \ \ell = 2)$$
$$\lambda_4 = 4 \ (k = 2, \ \ell = 0) \qquad \lambda_9 = 9 \ (k = 3, \ \ell = 0)$$

Example 2: Neumann Laplacian



Example 3: Steklov eigenvalue problem

$$-\Delta u_n + u_n = 0 \qquad \text{in } \Omega$$
$$\frac{\partial u_n}{\partial \nu} = \lambda_n u_n \quad \text{on } \partial\Omega$$

Weak formulation: Find $u_n \in H^1(\Omega)$, $\|u_n\|_{L^2(\partial\Omega)} \neq 0$, and $\lambda_n \in \mathbb{R}$:

$$(\nabla u_n, \nabla v) + (u_n, v) = \lambda_n(\gamma u_n, \gamma v)_{\partial\Omega} \quad \forall v \in H^1(\Omega)$$

- $V = H^1(\Omega)$, $V = \mathcal{K} \oplus \mathcal{M}$, $\mathcal{K} = \{v \in H^1(\Omega) : \gamma v = 0 \text{ on } \partial\Omega\}$
  $\mathcal{M} = \{v \in H^1(\Omega) : \gamma v \neq 0 \text{ on } \partial\Omega\}$
- $a(u,v) = (\nabla u, \nabla v) + (u,v)$ ... cont., bilin., sym., V-elliptic
- $b(u,v) = (u,v)_{\partial\Omega}$ ... cont., bilin., sym., pos. semidefinite
- Compactness:
  Trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ is compact
  [Kufner, John, Fučík 1997], [Biegert 2009]

Example 3: Steklov eigenvalue problem

$$-\Delta u_n + u_n = 0 \qquad \text{in } \Omega$$

$$\frac{\partial u_n}{\partial \nu} = \lambda_n u_n \quad \text{on } \partial\Omega$$

Exact solution for a square $\Omega = (-L, L)^2$

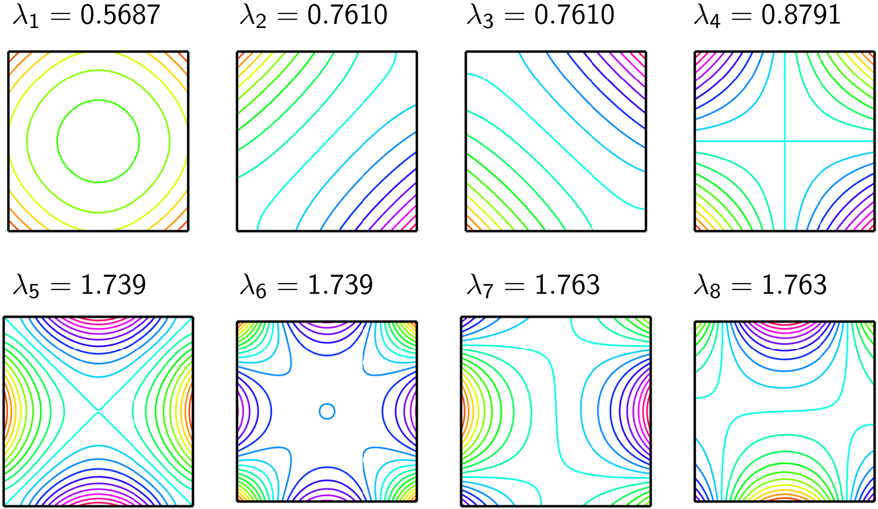
$$\lambda_1 = \frac{\sqrt{2}}{2} \tanh\left(\frac{\sqrt{2}}{2} L\right), \quad u_1(x, y) = \cosh\left(\frac{\sqrt{2}}{2} x\right) \cosh\left(\frac{\sqrt{2}}{2} y\right)$$

$$\lambda_2 = ?$$

$$\lambda_3 = ?$$

$$\lambda_4 = \frac{\sqrt{2}}{2} \coth\left(\frac{\sqrt{2}}{2} L\right), \quad u_4(x, y) = \sinh\left(\frac{\sqrt{2}}{2} x\right) \sinh\left(\frac{\sqrt{2}}{2} y\right)$$

Example 3: Steklov eigenvalue problem



Optimal constants

Abstract eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$:

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Theorem

$$|v|_b \leq \lambda_1^{-1/2} \|v\|_a \quad \forall v \in V, \quad \text{with equality for } v = u_1.$$

Proof

Let $v \in V$.

$$\lambda_1 = \min_{w \in V} \frac{\|w\|_a^2}{|w|_b^2} \leq \frac{\|v\|_a^2}{|v|_b^2} \quad \Leftrightarrow \quad |v|_b^2 \leq \lambda_1^{-1} \|v\|_a^2$$

□

Optimal constants

Abstract eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$:

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Theorem

$$|v|_b \le \lambda_1^{-1/2} \|v\|_a \quad \forall v \in V, \quad \text{with equality for } v = u_1.$$

Example 1: Dirichlet Laplacian.

$V = H_0^1(\Omega)$, $\quad \|v\|_a = \|\nabla v\|_{L^2(\Omega)} \quad |v|_b = \|v\|_{L^2(\Omega)}$

Corollary 1. The optimal constant in Friedrichs inequality

$$\|v\|_{L^2(\Omega)} \le C_{\mathrm{F}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega) \quad \text{is} \quad C_{\mathrm{F}} = \lambda_1^{-1/2},$$

where λ_1 is the principal eigenvalue of the Dirichlet Laplacian.

- $\Omega = (0, L) \quad \Rightarrow \quad C_{\mathrm{F}} = \frac{L}{\pi}$
- $\Omega = (0, L_1) \times (0, L_2) \quad \Rightarrow \quad C_F = \frac{1}{\pi} \left(\frac{1}{L_1^2} + \frac{1}{L_2^2} \right)^{-1}$

Optimal constants

Abstract eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$:

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Theorem

$$|v|_b \leq \lambda_1^{-1/2} \|v\|_a \quad \forall v \in V, \quad \text{with equality for } v = u_1.$$

Example 2: Neumann Laplacian.

$V = \{v \in H^1(\Omega) : \int_\Omega v \,\mathrm{d}x = 0\}, \quad \|v\|_a = \|\nabla v\|_{L^2(\Omega)}, \quad |v|_b = \|v\|_{L^2(\Omega)}$

Corollary 2. The optimal constant in Poincaré inequality

$$\|v\|_{L^2(\Omega)} \leq C_{\mathrm{P}} \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1(\Omega), \int_\Omega v \,\mathrm{d}x = 0, \quad \text{is} \quad C_{\mathrm{P}} = \lambda_1^{-1/2},$$

where λ_1 is the principal eigenvalue of the Neumann Laplacian.

- $\Omega = (0, L_1) \times (0, L_2) \quad \Rightarrow \quad C_{\mathrm{P}} = \dfrac{\max\{L_1, L_2\}}{\pi}$

Optimal constants

Abstract eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$:

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Theorem

$$|v|_b \leq \lambda_1^{-1/2} \|v\|_a \quad \forall v \in V, \quad \text{with equality for } v = u_1.$$

Example 3: Steklov eigenvalue problem.

$$V = H^1(\Omega), \quad \|v\|_a^2 = \|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2, \quad |v|_b = \|v\|_{L^2(\partial\Omega)}$$

Corollary 3. The optimal constant in trace inequality

$$\|v\|_{L^2(\partial\Omega)} \leq C_{\mathrm{T}} \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega) \quad \text{is} \quad C_{\mathrm{T}} = \lambda_1^{-1/2},$$

where λ_1 is the principal eigenvalue of the Steklov problem.

- $\Omega = (-L, L)^2 \quad \Rightarrow \quad C_{\mathrm{T}} = \left(\sqrt{2} \coth(\sqrt{2} L/2)\right)^{1/2}$

3. Numerical methods

3.1 Discretization

Rayleigh-Ritz (Galerkin) method

Eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in V \setminus \{0\}$:

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Finite dimensional subspace: $V_h \subset V$, $\dim V_h = N < \infty$.

Discrete eigenvalue problem: Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

Properties

Discrete eigenvalue problem: Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

- $0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \cdots \leq \lambda_{h,N}$
- $\frac{1}{\lambda_{h,i}} a(u_{h,i}, u_{h,j}) = b(u_{h,i}, u_{h,j}) = \delta_{ij} \quad \forall i, j = 1, 2, \ldots, N.$
- Minimum principle:

$$\lambda_{h,1} = \min_{v_h \in V_h,\ |v_h|_b \neq 0} R(v_h) \qquad u_{h,1} = \arg\min_{v_h \in V_h,\ |v_h|_b \neq 0} R(v_h),$$

$$\lambda_{h,n} = \min_{v_h \in S_h^{n-1}} R(v_h) \qquad u_{h,n} = \arg\min_{v_h \in S_h^{n-1}} R(v_h),$$

where $S_h^{n-1} = \{v_h \in V_h : |v_h|_b \neq 0 \text{ and } b(v_h, u_{h,i}) = 0 \quad \forall i = 1, 2, \ldots, n-1\}.$

Properties

Discrete eigenvalue problem: Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

- $0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \cdots \leq \lambda_{h,N}$
- $\dfrac{1}{\lambda_{h,i}} a(u_{h,i}, u_{h,j}) = b(u_{h,i}, u_{h,j}) = \delta_{ij} \quad \forall i,j = 1,2,\ldots,N.$
- Min-max principle:

$$\lambda_{h,n} = \min_{E_h \in \mathcal{V}_h^{(n)}} \max_{v_h \in E_h} R(v_h)$$

  where $\mathcal{V}_h^{(n)}$ is the set of all n-dimensional subspaces of V_h.

- Theorem.

$$\lambda_n \leq \lambda_{h,n}, \quad n = 1,2,\ldots,N$$

  Proof.

$$\mathcal{V}_h^{(n)} \subset \mathcal{V}^{(n)} \quad \Rightarrow \quad \lambda_n = \min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v) \leq \lambda_{h,n} \qquad \square$$

How to compute

Discrete eigenvalue problem: Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h \quad (*)$$

Theorem. Let $\varphi_1, \ldots, \varphi_N$ be a basis of V_h.

$$(*) \quad \Leftrightarrow \quad A\mathbf{x}_n = \lambda_{h,n} B\mathbf{x}_n,$$

where $A_{ij} = a(\varphi_j, \varphi_i)$ and $B_{ij} = b(\varphi_j, \varphi_i)$.

Proof. Use $u_{h,n} = \sum_{j=1}^N x_{n,j}\varphi_j$ and $v_h = \varphi_i$ and get

$$\sum_{j=1}^N a(\varphi_j, \varphi_i) x_{n,j} = \lambda_{h,n} \sum_{j=1}^N b(\varphi_j, \varphi_i) x_{n,j}$$

□

Finite element method

Triangulation:

- $\mathcal{T}_h$ is a set of closed and disjoint simplices (elements)
- $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$
- face-to-face
- discretization parameter:
  $h = \max_{K \in \mathcal{T}_h} h_K$, $h_K = \operatorname{diam} K$

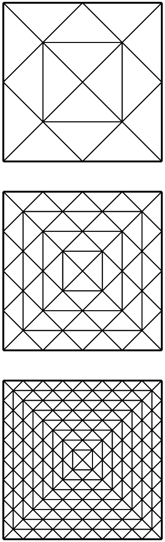
Family of triangulations:

$\mathcal{F} = \{\mathcal{T}_h\}$ such that $\forall h_0 > 0 \; \exists \mathcal{T}_h \in \mathcal{F} : h < h_0$.

Regular family:

$$\exists C > 0 \; \forall \mathcal{T}_h \in \mathcal{F} \; \forall K \in \mathcal{T}_h : \frac{h_K}{\varrho_K} \leq C,$$

where ϱ_K is the in-radius of K

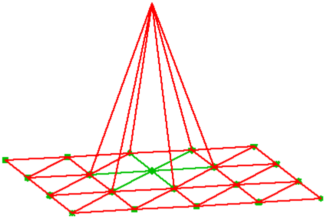


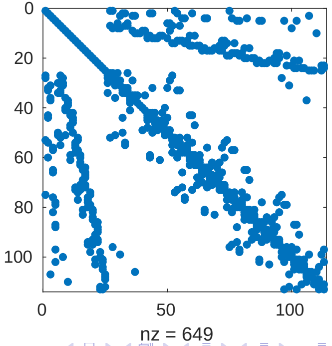
Finite element basis functions

Finite element space: $V_h = \{v_h \in V : v_h|_K \in \mathbb{P}^1(K)\ \forall K \in \mathcal{T}_h\}$

Basis functions: $\varphi_i(\mathbf{z}_j) = \delta_{ij}$, where $\mathbf{z}_j$ is a node (vertex) of $\mathcal{T}_h$

- $\operatorname{supp}\varphi_i$ is small
- If $\mathbf{z}_i$ and $\mathbf{z}_j$ are not neighbours then
  $A_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \,\mathrm{d}x = \int_{\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_i} \nabla\varphi_j \cdot \nabla\varphi_i \,\mathrm{d}x = 0$
- A is sparse





3. Numerical methods

3.2 Convergence of the FEM (for Laplacian)

[Boffi 2010]

Convergence for Laplacian

Strong formulation:

$$\begin{aligned} -\Delta u_n &= \lambda_n u_n && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak formulation: Find $\lambda_n \in \mathbb{R}$, $u_n \in H_0^1(\Omega) \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in H_0^1(\Omega)$$

Finite element method:

$V_h = \{v_h \in H_0^1(\Omega) : v_h|_K \in \mathbb{P}^1(K)\ \forall K \in \mathcal{T}_h\}$

Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

$$(\nabla u_{h,n}, \nabla v_h) = \lambda_{h,n}(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

Convergence:

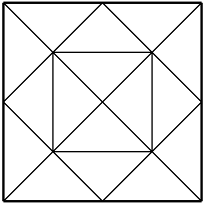
$$|\lambda_n - \lambda_{h,n}| \leq Ch^2$$

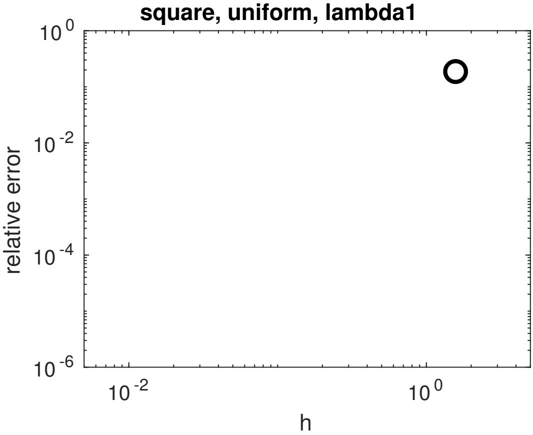
$$\|\nabla u_n - \nabla u_{h,n}\|_0 \leq Ch$$

Example

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = (0,\pi)^2$$
$$u_n = 0 \qquad \text{on } \partial\Omega$$

$$\text{rel\_err} = \frac{|\lambda_n - \lambda_{h,n}|}{\lambda_n}$$





Example

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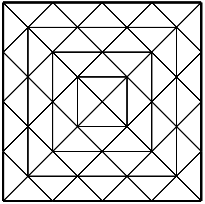

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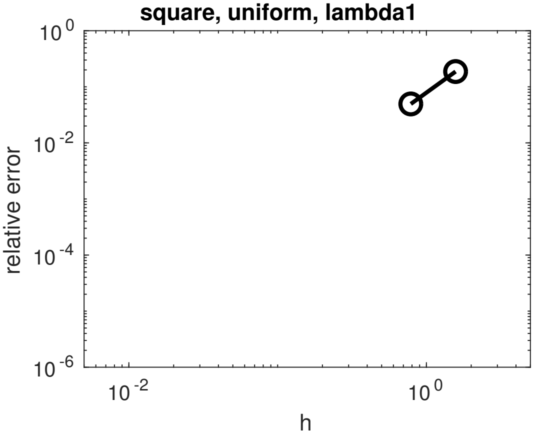

Example

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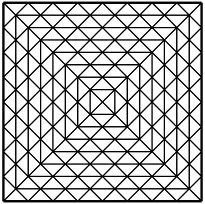

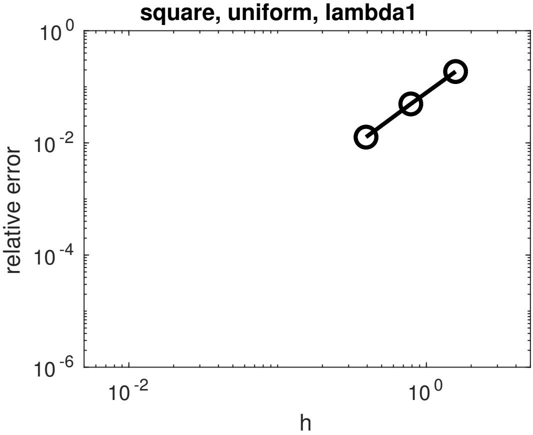

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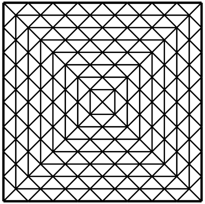

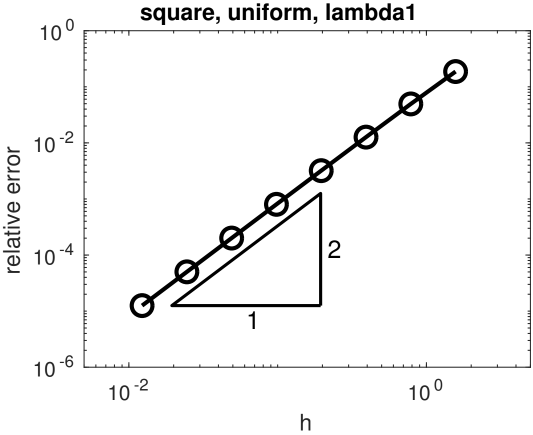

Example

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
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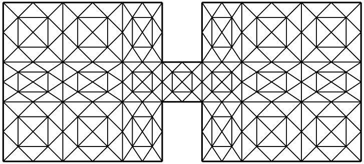
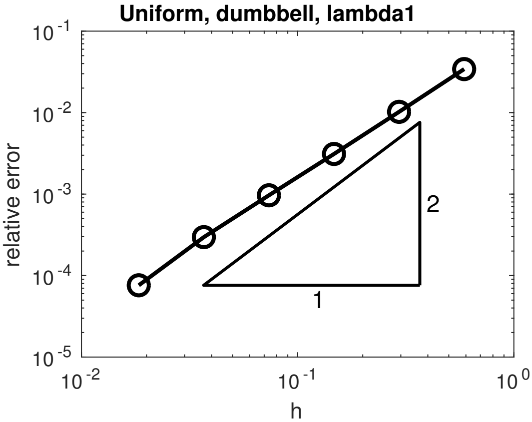
$$\text{rel\_err} \approx \frac{|\lambda_n^{\mathrm{ref}} - \lambda_{h,n}|}{\lambda_n^{\mathrm{ref}}}$$





Interpolation theorem

Interpolation: $\pi_h : C(\overline{\Omega}) \to V_h$

$$\pi_h v(\mathbf{z}_i) = v(\mathbf{z}_i) \quad \text{for all nodes } \mathbf{z}_i \text{ of the mesh } \mathcal{T}_h.$$

Theorem. Let $\Omega \subset \mathbb{R}^2$ (or $\mathbb{R}^3$). Let $\mathcal{F}$ be a regular family of triangulations of Ω. Then there exists $C > 0$ and $h_0 > 0$ such that for all $\mathcal{T}_h \in \mathcal{F}$ with $h \le h_0$ we have

$$\|v - \pi_h v\|_1 \le Ch|v|_{H^2(\Omega)} \quad \forall v \in H^2(\Omega).$$

[Ciarlet 1978]

Regularity: If Ω is convex and $\Omega \subset \mathbb{R}^2$ then

$$u_n \in H^2(\Omega)$$

and

$$|v|_{H^2(\Omega)} \le C\|\Delta v\|_0 \quad \forall v \in H^2(\Omega)$$

[Brenner, Scott 1994]

Error of elliptic projection

Elliptic projection: $P_h : H_0^1(\Omega) \to V_h$

$$P_h v \in V_h : \quad (\nabla v - \nabla P_h v, \nabla v_h) = 0 \quad \forall v_h \in V_h$$

Theorem. Let $\Omega \subset \mathbb{R}^2$ be convex. Then

$$\|\nabla v - \nabla P_h v\|_0 \leq Ch\|\Delta v\|_0 \quad \forall v \in H^2(\Omega),$$

$$\|v - P_h v\|_0 \leq Ch^2\|\Delta v\|_0 \quad \forall v \in H^2(\Omega).$$

Proof

$$\|\nabla v - \nabla P_h v\|_0 = \inf_{v_h \in V_h} \|\nabla v - \nabla v_h\|_0 \leq \|\nabla v - \nabla \pi_h v\|_0$$

$$\leq Ch|v|_{H^2(\Omega)} \leq Ch\|\Delta v\|_0$$

Aubin-Nitsche duality technique □

Convergence of eigenvalues

Theorem. Let Ω be a convex polygon. Let $\mathcal{F}$ be a regular family of triangulations of Ω. Then for all n there exists $C(n) > 0$ and $h_0 > 0$ such that for all meshes $\mathcal{T}_h \in \mathcal{F}$ with $h < h_0$ we have

$$|\lambda_n - \lambda_{h,n}| \leq C(n)h^2$$

Proof

- $E = \text{span}\{u_1, \ldots, u_n\}, \quad E_h = P_h E \quad (\dim E_h = n \text{ for } h \leq h_0)$
- Discrete min-max principle with E_h:

$$\lambda_{h,n} \leq \max_{v \in E_h} \frac{\|\nabla v\|_0^2}{\|v\|_0^2} = \max_{v \in E} \frac{\|\nabla P_h v\|_0^2}{\|P_h v\|_0^2} \leq \max_{v \in E} \frac{\|\nabla v\|_0^2}{\|P_h v\|_0^2}$$
$$= \max_{v \in E} \frac{\|\nabla v\|_0^2}{\|v\|_0^2} \frac{\|v\|_0^2}{\|P_h v\|_0^2} \leq \lambda_n \max_{v \in E} \frac{\|v\|_0^2}{\|P_h v\|_0^2}$$

- It remains to bound $\frac{\|v\|_0^2}{\|P_h v\|_0^2}$ for $v \in E$.

Convergence of eigenvalues

Theorem. Let Ω be a convex polygon. Let $\mathcal{F}$ be a regular family of triangulations of Ω. Then for all n there exists $C(n) > 0$ and $h_0 > 0$ such that for all meshes $\mathcal{T}_h \in \mathcal{F}$ with $h < h_0$ we have

$$|\lambda_n - \lambda_{h,n}| \leq C(n)h^2$$

Proof

- To bound $\frac{\|v\|_0^2}{\|P_h v\|_0^2}$, consider $v \in E$.
- Regularity result $\Rightarrow$ $v \in H^2(\Omega)$:
$$\|v - P_h v\|_0 \leq Ch^2 \|\Delta v\|_0 \leq C\lambda_n h^2 \|v\|_0$$
- $\Rightarrow \quad \|P_h v\|_0 \geq \|v\|_0 - \|v - P_h v\|_0 \geq \|v\|_0 (1 - C\lambda_n h^2)$
- Hence,
$$\lambda_{h,n} \leq \lambda_n \max_{v \in E} \frac{\|v\|_0^2}{\|P_h v\|_0^2} \leq \lambda_n \left(\frac{1}{1 - C\lambda_n h^2} \right)^2$$
$$\leq \lambda_n (1 + 2C\lambda_n h^2)^2 \leq \lambda_n (1 + 6C\lambda_n h^2)$$

Convergence of simple eigenfunctions

Definition: Let λ_n be simple (i.e. $\lambda_n \neq \lambda_i \ \forall i \neq n$). Define

$$\varrho_{h,n} = \max_{i \neq n} \frac{\lambda_n}{|\lambda_n - \lambda_{h,i}|}$$

Theorem. Let λ_n be simple. Let $n \leq \dim V_h$. Let $\|u_n\|_0 = \|u_{h,n}\|_0 = 1$ and let $u_{h,n}$ has a correct sign. Then

$$\|u_n - u_{h,n}\|_0 \leq 2(1 + \varrho_{h,n})\|u_n - P_h u_n\|_0 \quad (\leq Ch^2)$$

$$\|\nabla u_n - \nabla u_{h,n}\|_0^2 = \lambda_n \|u_n - u_{h,n}\|_0^2 + \lambda_{h,n} - \lambda_n \quad (\leq Ch^2)$$

Proof of the last equality:

$$\begin{aligned}
\|\nabla u_n - \nabla u_{h,n}\|_0^2 &= \|\nabla u_n\|_0^2 - 2(\nabla u_n, \nabla u_{h,n}) + \|\nabla u_{h,n}\|_0^2 \\
&= \lambda_n - 2\lambda_n(u_n, u_{h,n}) + \lambda_n - \lambda_n + \lambda_{h,n} \\
&= \lambda_n \|u_n - u_{h,n}\|_0^2 - \lambda_n + \lambda_{h,n}
\end{aligned}$$

General convergence theorem

Theorem [Boffi 2010]. Let $n \le \dim V_h$. Then

$$\lambda_{h,n} - \lambda_n \le C(n) \sup_{\substack{v \in \mathrm{span}\{u_1,\dots,u_n\} \\ \|v\|_0 = 1}} \|v - P_h v\|_{H^1(\Omega)}.$$

Moreover, if the multiplicity of λ_n is m, so that

$$\lambda_n = \cdots = \lambda_{n+m-1} \quad \text{and} \quad \lambda_n \ne \lambda_i \text{ for } i \ne n, \dots, n+m-1,$$

then there exists $\tilde{u}_{h,n} \in \mathrm{span}\{u_{h,n}, \dots, u_{h,n+m-1}\}$ such that

$$\|u_n - \tilde{u}_{h,n}\|_0 \le C(n) \|u_n - P_h u_n\|_0$$

$$\|u_n - \tilde{u}_{h,n}\|_{H^1(\Omega)} \le C(n) \sup_{\substack{v \in \mathrm{span}\{u_1,\dots,u_{n+m-1}\} \\ \|v\|_0 = 1}} \|v - P_h v\|_{H^1(\Omega)}$$

3. Numerical methods

3.3 Advanced approaches

Higher-order finite elements

Lapace eigenvalue problem:

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega$$
$$u_n = 0 \quad \text{on } \partial\Omega$$

Weak formulation: Find $\lambda_n \in \mathbb{R}$, $u_n \in H_0^1(\Omega) \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in H_0^1(\Omega),$$

Higher-order finite element method:

$V_h = \{v_h \in H_0^1(\Omega) : v_h|_K \in \mathbb{P}^p(K)\ \forall K \in \mathcal{T}_h\}$

Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

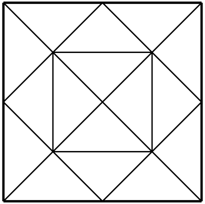
$$(\nabla u_{h,n}, \nabla v_h) = \lambda_{h,n} (u_{h,n}, v_h) \quad \forall v_h \in V_h,$$

Convergence: If $u_n \in H^{p+1}(\Omega)$ then

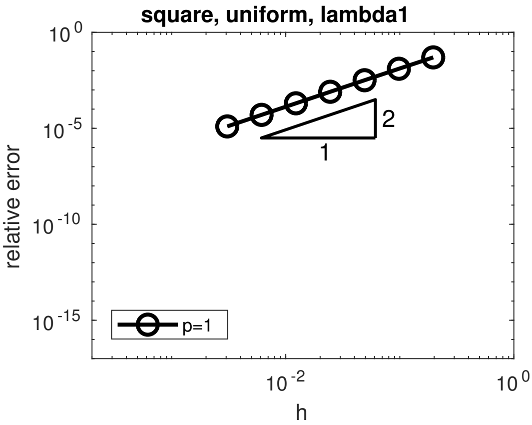
$$|\lambda_n - \lambda_{h,n}| \leq Ch^{2p}$$
$$\|\nabla u_n - \nabla u_{h,n}\|_0 \leq Ch^p$$

Example – square

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = (0,\pi)^2$$
$$u_n = 0 \quad \text{on } \partial\Omega$$



$$\text{rel\_err} = \frac{|\lambda_n - \lambda_{h,n}|}{\lambda_n}$$



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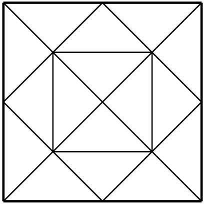

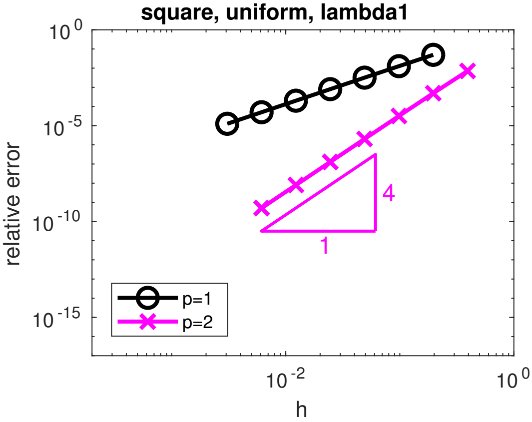

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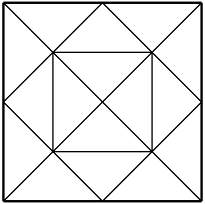

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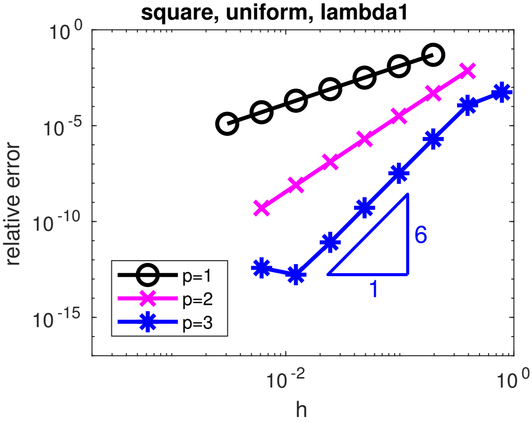

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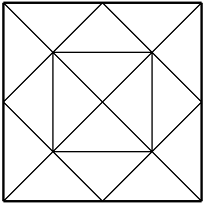

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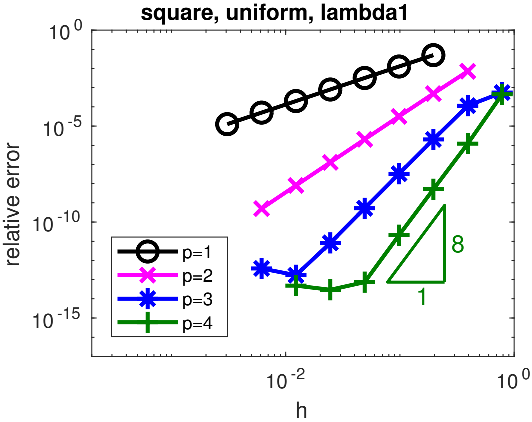

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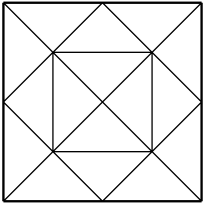

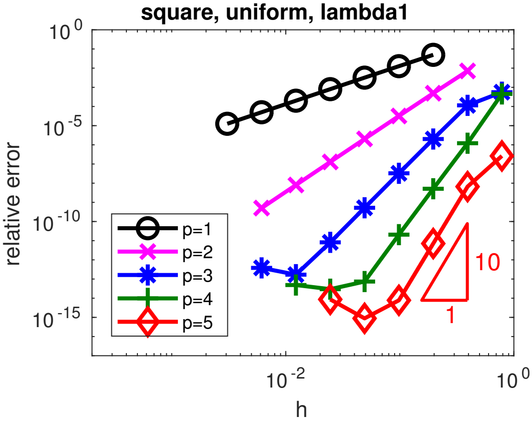

Example – dumbbell

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
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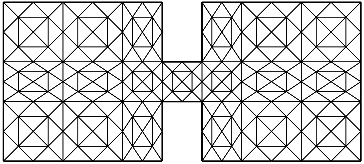
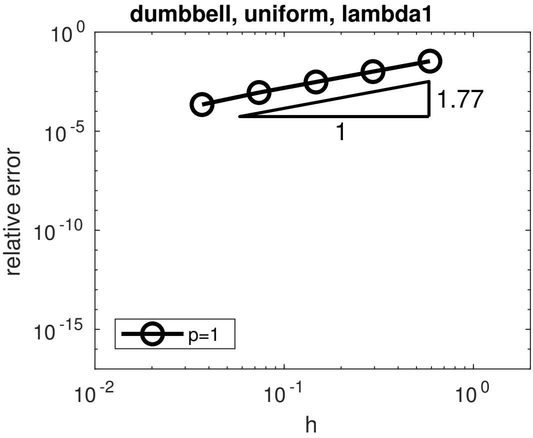


$$\text{rel\_err} \approx \frac{|\lambda_n^{\text{ref}} - \lambda_{h,n}|}{\lambda_n^{\text{ref}}}$$



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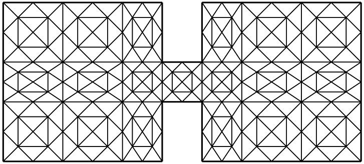

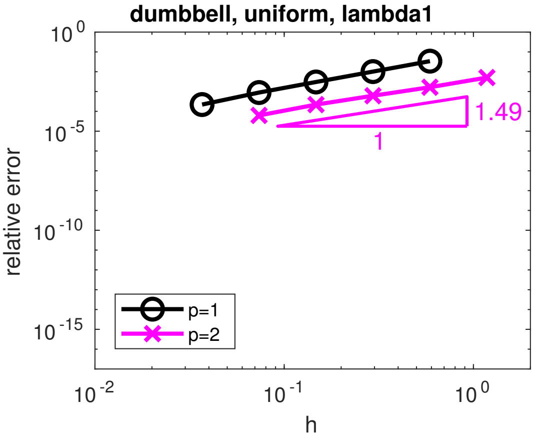

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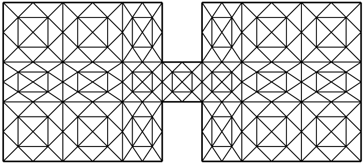

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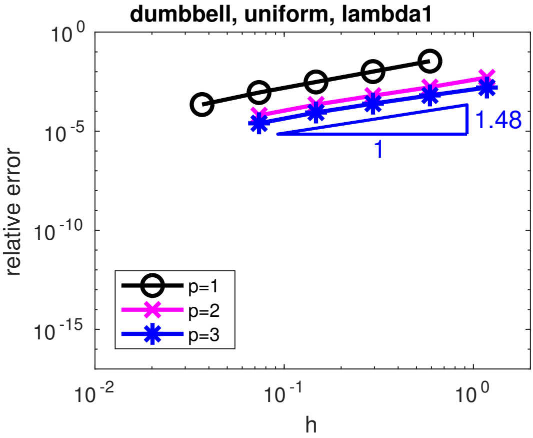

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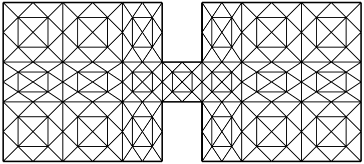

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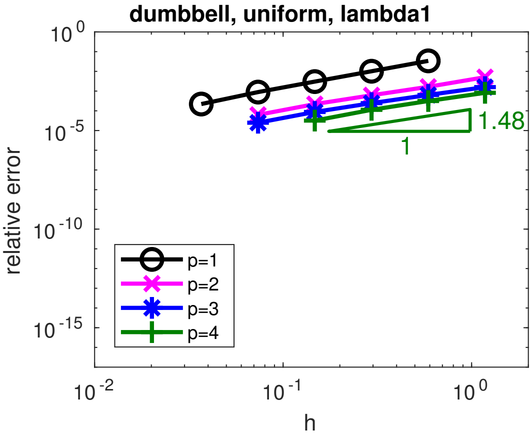

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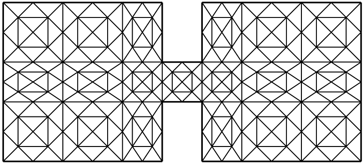

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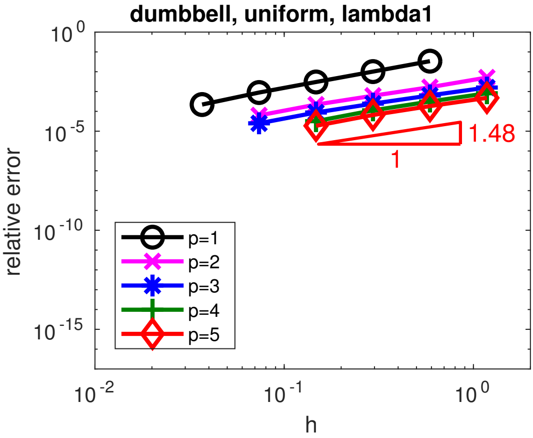
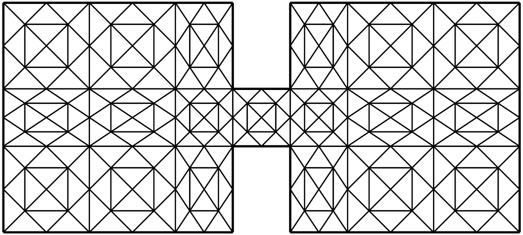


Adaptive finite element method

Uniform refinement



Adaptive finite element method

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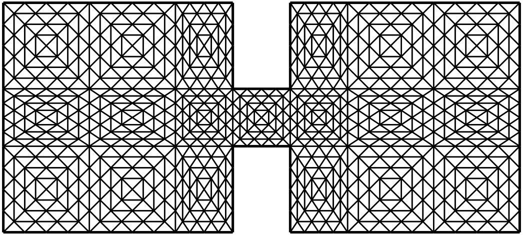

Adaptive finite element method

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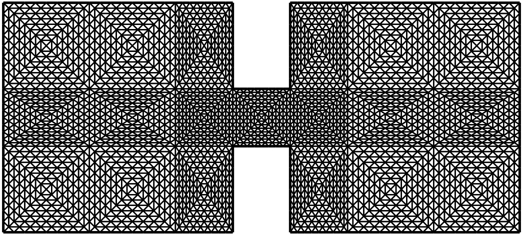
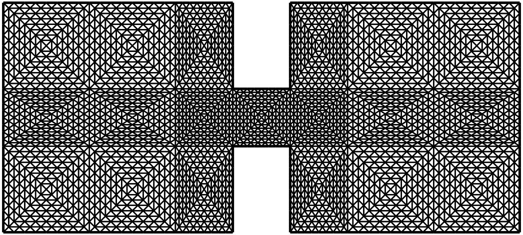
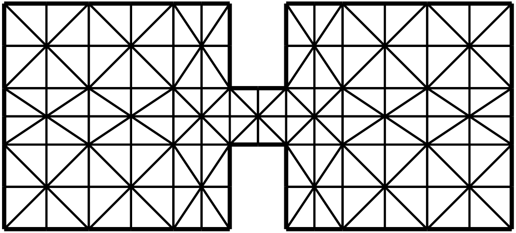


Adaptive finite element method

Uniform refinement

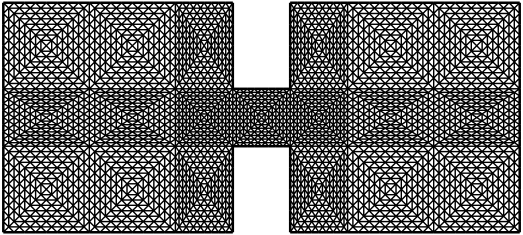


Adaptive refinement



Adaptive finite element method

Uniform refinement



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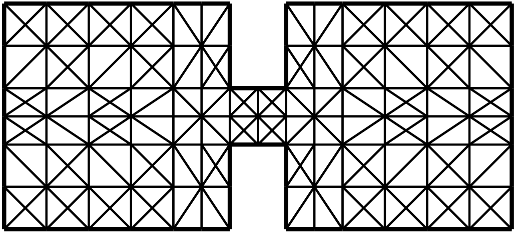

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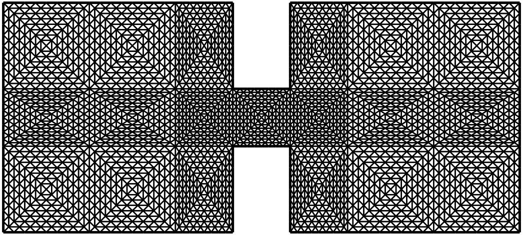

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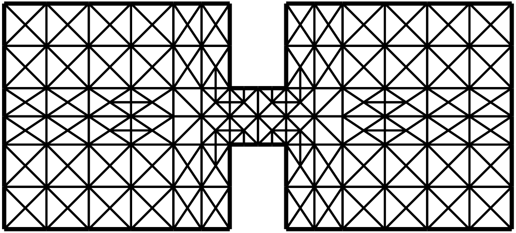

Adaptive finite element method

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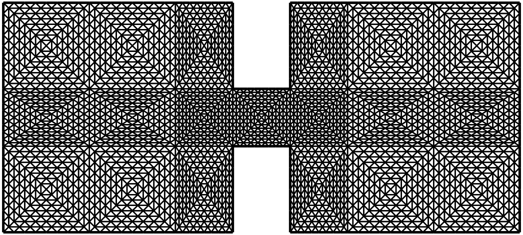

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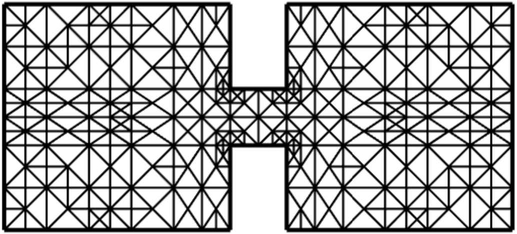
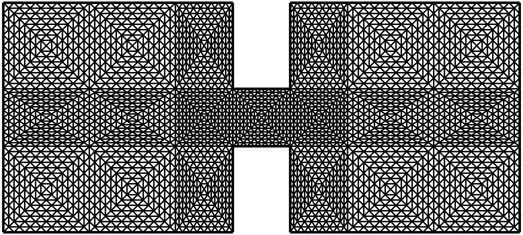


Adaptive finite element method

Uniform refinement



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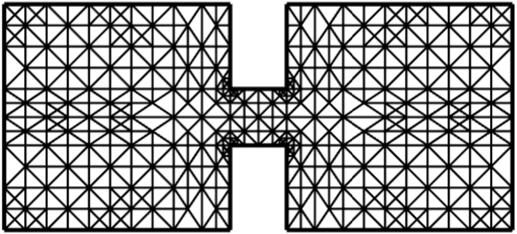

Adaptive algorithm

1. Construct initial mesh $\mathcal{T}_h$.
2. Solve. Compute $\lambda_{h,i}$, $u_{h,i}$.
3. Estimate.
   - Compute error indicators η_K for all $K \in \mathcal{T}_h$.

5. Mark. Mark elements with large η_K. [Dörfler 1996]
   Sort: $\eta_{K_1} \geq \eta_{K_2} \geq \cdots \geq \eta_{K_N}$ and find the smallest N^*:

$$\sum_{i=1}^{N^*} \eta_{K_i}^2 \geq \Theta \sum_{i=1}^{N} \eta_{K_i}^2, \quad 0 < \Theta < 1 \quad \Rightarrow \quad \text{mark } \eta_{K_1}, \dots, \eta_{K_N^*}$$

6. Refine. Refine marked elements and construct new mesh $\mathcal{T}_h$.
7. Go to 2.

Adaptive algorithm

1. Construct initial mesh $\mathcal{T}_h$.
2. Solve. Compute $\lambda_{h,i}$, $u_{h,i}$.
3. Estimate.
   - Compute error indicators η_K for all $K \in \mathcal{T}_h$.
   - Compute error estimator $\eta = \lambda_{h,i} - \ell_i$.
4. Stopping criterion. If $\eta \leq \mathrm{TOL} \quad \Rightarrow \quad \mathrm{STOP}$
5. Mark. Mark elements with large η_K. [Dörfler 1996]
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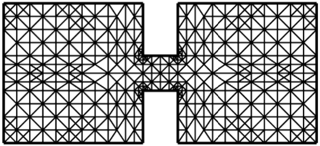
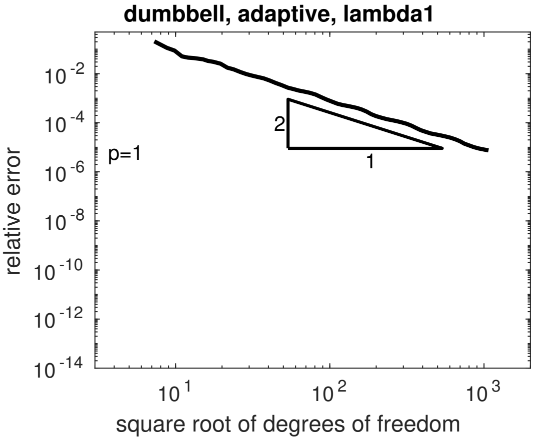
Example – dumbbell

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
$$u_n = 0 \quad \text{on } \partial\Omega$$



$$\text{rel\_err} \approx \frac{|\lambda_n^{\text{ref}} - \lambda_{h,n}|}{\lambda_n^{\text{ref}}}$$



$$h \approx N_{\text{dof}}^{-1/d}$$

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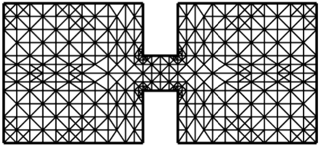

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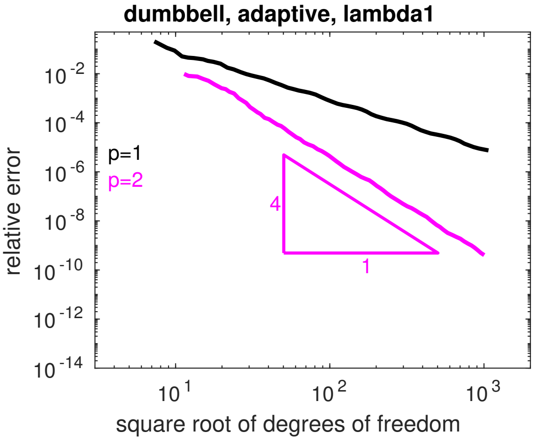

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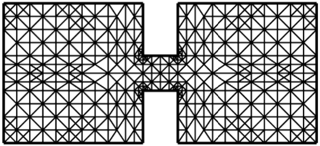

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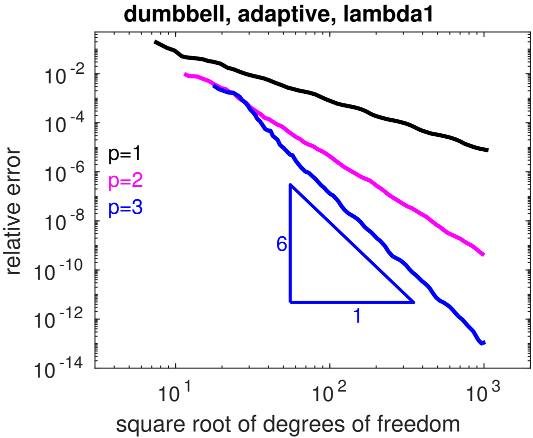

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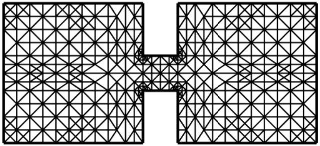

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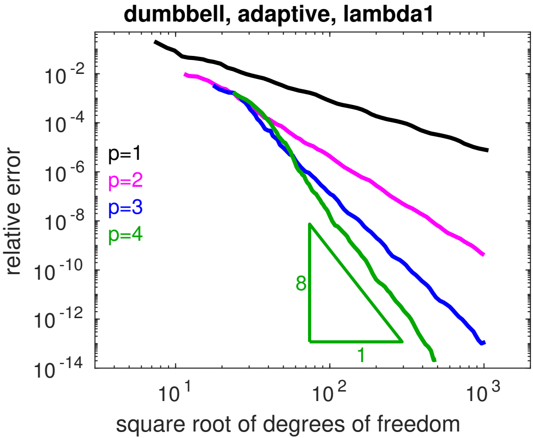

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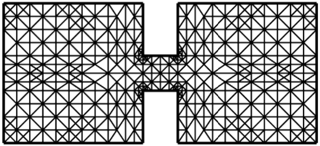
$$



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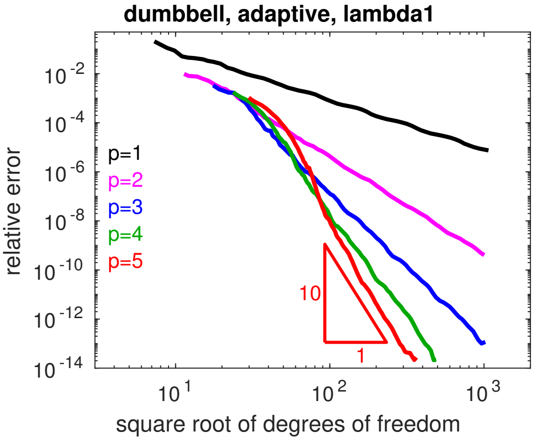
$$



$$
h \approx N_{\mathrm{dof}}^{-1/d}
$$

4. Lower bounds on eigenvalues

4.1 Weinstein's bound

Introduction

Eigenvalue problem:
Find λ_n and $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Rayleigh-Ritz (Galerkin) method: Let $V_h \subset V$, $\dim V_h = N < \infty$.
Find $\lambda_{h,n} \in \mathbb{R}$ and $u_{h,n} \in V_h \setminus \{0\}$:

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

Min-Max principle:

$$\lambda_n \leq \lambda_{h,n}$$

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Min-Max principle:

$$? \leq \lambda_n \leq \lambda_{h,n}$$

Lower bounds – history

Standard (conforming) approach:
Temple (1928), Weinstein (1937), Kato (1949),
Lehmann (1949), Goerisch (1985), . . .

Nonconforming FEM:
Carstensen, Gedicke, Gallistl (2014), Xuefeng LIU (2015), . . .

Many results: M.G. Armentano, G. Barrenechea, H. Behnke,
R.G. Duran, L. Grubišić, Jun Hu, J.R. Kuttler, Y.A. Kuznetsov,
Fubiao Lin, Qun Lin, M. Plum, S.I. Repin, V.G. Sigillito,
M. Vohralík, Hehu Xie, Yidu Yang, Zhimin Zhang, . . . *many others*

Recall

Find $\lambda_n \in \mathbb{R}$ and $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

- V is a Hilbert space.
- $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are two bilinear forms on V.
- $V = \mathcal{K} \oplus \mathcal{M}$
- $\mathcal{K} = \{v \in V : |v|_b = 0\}$
- $\mathcal{M} = \text{span}\{u_1, u_2, \dots\}$
- $v = v^{\mathcal{K}} + v^{\mathcal{M}}$
- $v^{\mathcal{M}} = \sum_{n=1}^{\infty} c_n u_n, \quad c_n = b(v^{\mathcal{M}}, u_n) = b(v, u_n)$
- $|v|_b^2 = \sum_{n=1}^{\infty} |b(v, u_n)|^2$
- $\|v\|_a^2 = \|v^{\mathcal{K}}\|_a^2 + \|v^{\mathcal{M}}\|_a^2 \quad \text{with } \|v^{\mathcal{M}}\|_a^2 = \sum_{n=1}^{\infty} \lambda_n |b(v, u_n)|^2$

Weinstein's bound

Theorem

Let $\lambda_* \in \mathbb{R}$ and $u_* \in V \setminus \{0\}$ be arbitrary and $w \in V$ be given by

$$a(w, v) = a(u_*, v) - \lambda_* b(u_*, v) \quad \forall v \in V.$$

Then

$$\min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \le \frac{\|w\|_a^2}{|u_*|_b^2}.$$

Proof: $w = w^{\mathcal{K}} + w^{\mathcal{M}}$

$$\|w^{\mathcal{M}}\|_a^2 = \sum_{j=1}^{\infty} \lambda_j |b(w, u_j)|^2 = \sum_{j=1}^{\infty} \frac{|a(w, u_j)|^2}{\lambda_j}$$

$$= \sum_{j=1}^{\infty} \frac{|a(u_*, u_j) - \lambda_* b(u_*, u_j)|^2}{\lambda_j} = \sum_{j=1}^{\infty} \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} |b(u_*, u_j)|^2$$

Thus,

$$\|w\|_a^2 \ge \|w^{\mathcal{M}}\|_a^2 \ge \min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \sum_{j=1}^{\infty} |b(u_*, u_j)|^2 \qquad \square$$

Weinstein's bound

Corollary: Let λ_n has multiplicity m, i.e.,
$\lambda_{n-1} \neq \lambda_n = \cdots = \lambda_{n+m-1} \neq \lambda_{n+m}$. If

$$\sqrt{\lambda_{n-1}\lambda_n} \leq \lambda_* \leq \sqrt{\lambda_n \lambda_{n+m}} \qquad \text{(closeness)}$$

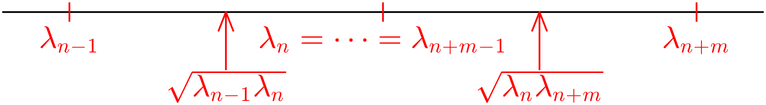
and

$$\|w\|_a \leq \eta$$

then

$$\ell_n \leq \lambda_n,$$

where $\ell_n = \dfrac{1}{4|u_*|_b^2}\left(-\eta + \sqrt{\eta^2 + 4\lambda_*|u_*|_b^2}\right)^2.$



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Proof: Clearly,

$$\frac{(\lambda_n - \lambda_*)^2}{\lambda_n} = \min_j \frac{|\lambda_j - \lambda_*|^2}{\lambda_j} \leq \frac{\|w\|_a^2}{|u_*|_b^2} \leq \frac{\eta^2}{|u_*|_b^2}$$

and solve for λ_n.

Complementary upper bound on the residual

Laplace eigenvalue problem: Find λ_n and $u_n \in H_0^1(\Omega) \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in H_0^1(\Omega)$$

Definition. Flux $\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$ is equilibrated if $-\operatorname{div} \mathbf{q} = \lambda_* u_*$.

Theorem. If $\mathbf{q}$ is an equilibrated flux then

$$\|\nabla w\|_0 \leq \eta = \|\nabla u_* - \mathbf{q}\|_0.$$

Proof: Let $v \in H_0^1(\Omega)$, then

$$\begin{aligned}(\nabla w, \nabla v) &= (\nabla u_*, \nabla v) - \lambda_*(u_*, v) - (\operatorname{div} \mathbf{q}, v) - (\mathbf{q}, \nabla v) \\ &= (\nabla u_* - \mathbf{q}, \nabla v) - (\lambda_* u_* + \operatorname{div} \mathbf{q}, v) \\ &\leq \|\nabla u_* - \mathbf{q}\|_0 \|\nabla v\|_0 \qquad \square\end{aligned}$$

[Neittaanmäki, Repin 2004], [Repin 2008], [Braess, Schöberl, 2008], [Ainsworth, Vejchodský 2011,2014], [Vohralík at al.]

Avoiding equilibration

Shifted eigenvalue problem:

$$\underbrace{(\nabla u_n, \nabla v) + \gamma(u_n, v)}_{a_\gamma(u_n,v)} = (\lambda_n + \gamma)(u_n, v) \quad \forall v \in H_0^1(\Omega)$$

Theorem. Let $\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$ and $\gamma > 0$. Then

$$\|\nabla w\|_0 \le \|w\|_{a_\gamma} \le \eta, \quad \eta^2 = \|\nabla u_* - \mathbf{q}\|_0^2 + \frac{1}{\gamma}\|\lambda_* u_* + \mathrm{div}\,\mathbf{q}\|_0^2$$

Proof:

$$\begin{aligned} a_\gamma(w, v) &= (\nabla u_*, \nabla v) - \lambda_*(u_*, v) - (\mathrm{div}\,\mathbf{q}, v) - (\mathbf{q}, \nabla v) \\ &= (\nabla u_* - \mathbf{q}, \nabla v) - (\lambda_* u_* + \mathrm{div}\,\mathbf{q}, v) \\ &\le \|\nabla u_* - \mathbf{q}\|_0 \|\nabla v\|_0 + \gamma^{-1/2}\|\lambda_* u_* + \mathrm{div}\,\mathbf{q}\|_0 \; \gamma^{1/2}\|v\|_0 \\ &\le \left(\|\nabla u_* - \mathbf{q}\|_0^2 + \gamma^{-1}\|\lambda_* u_* + \mathrm{div}\,\mathbf{q}\|_0^2\right)^{1/2} \left(\|\nabla v\|_0^2 + \gamma\|v\|_0^2\right)^{1/2} \end{aligned}$$

Thus, $\|w\|_{a_\gamma}^2 \le \|\nabla u_* - \mathbf{q}\|_0^2 + \gamma^{-1}\|\lambda_* u_* + \mathrm{div}\,\mathbf{q}\|_0^2$ $\square$

How to compute $\mathbf{q}$?

Global flux reconstruction: Find $\mathbf{q}_h \in \mathbf{W}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ minimizing

$$\eta^2 = \|\nabla u_* - \mathbf{q}_h\|_0^2 + \frac{1}{\gamma}\|\lambda_* u_* + \mathrm{div}\,\mathbf{q}_h\|_0^2$$

FEM space:

$V_h = \{v_h \in V : v_h|_K \in \mathbb{P}^1(K)\ \forall K \in \mathcal{T}_h\}$

FEM approximation:

$u_* = u_{h,n} \in V_h,\ \lambda_* = \lambda_{h,n}$

Raviart-Thomas space:

$\mathbf{RT}_1(K) = [\mathbb{P}^1(K)]^2 \oplus \mathbf{x}\mathbb{P}^1(K)$ (local)

$\mathbf{W}_h = \{\mathbf{q}_h \in \mathbf{H}(\mathrm{div}, \Omega) : \mathbf{q}_h|_K \in \mathbf{RT}_1(K) \quad \forall K \in \mathcal{T}_h\}$ (global)

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Global flux reconstruction: Find $\mathbf{q}_h \in \mathbf{W}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ minimizing

$$\eta^2 = \|\nabla u_* - \mathbf{q}_h\|_0^2 + \frac{1}{\gamma}\|\lambda_* u_* + \mathrm{div}\,\mathbf{q}_h\|_0^2$$

Euler-Lagrange equations:

$$(\mathbf{q}_h, \mathbf{w}_h) + \frac{1}{\gamma}(\mathrm{div}\,\mathbf{q}_h, \mathrm{div}\,\mathbf{w}_h) = (\nabla u_*, \mathbf{w}_h) - \frac{1}{\gamma}(\lambda_* u_*, \mathrm{div}\,\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{W}_h$$

Equivalent to linear system:

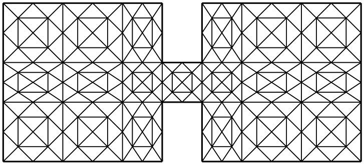
$$M\mathbf{y} = F,$$

where $\mathbf{q}_h = \sum_j y_j \boldsymbol{\psi}_j$, $\quad M_{ij} = (\boldsymbol{\psi}_j, \boldsymbol{\psi}_i) + \frac{1}{\gamma}(\mathrm{div}\,\boldsymbol{\psi}_j, \mathrm{div}\,\boldsymbol{\psi}_i)$,

$$F_i = (\nabla u_*, \boldsymbol{\psi}_i) - \frac{1}{\gamma}(\lambda_* u_*, \mathrm{div}\,\boldsymbol{\psi}_i)$$

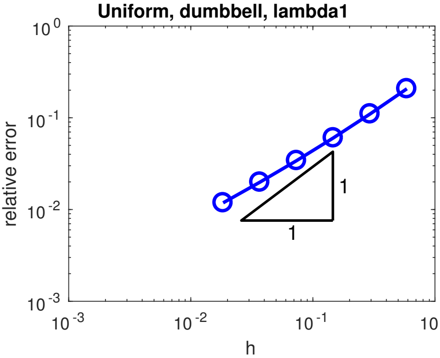
Example: dumbbell

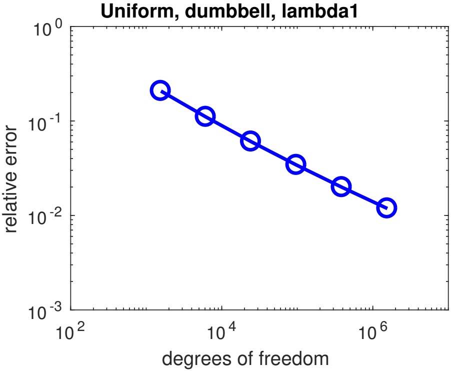
$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
$$u_n = 0 \quad \text{on } \partial\Omega$$



$$\text{rel\_err} = \frac{|\lambda_n - \lambda_{h,n}|}{\lambda_n} \le \frac{\lambda_{h,n} - \ell_n}{\ell_n}$$

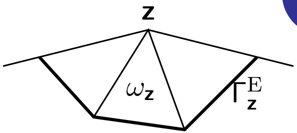
$$\gamma = 10^{-6}$$





Local flux reconstruction

Flux reconstruction:



$$\mathbf{q}_h = \sum_{\mathbf{z}\in\mathcal{N}_h} \mathbf{q}_z$$

Local problems: Find $\mathbf{q}_z \in \mathbf{W}_z$ minimizing

$$\|\varphi_{\mathbf{z}}\nabla u_* - \mathbf{q}_{\mathbf{z}}\|^2_{L^2(\omega_{\mathbf{z}})} + \frac{1}{\gamma}\|\lambda_*\varphi_{\mathbf{z}}u_* + \operatorname{div}\mathbf{q}_{\mathbf{z}}\|^2_{L^2(\omega_{\mathbf{z}})}$$

Euler-Lagrange equations:

$$(\mathbf{q}_{\mathbf{z}},\mathbf{w}_h)_{\omega_{\mathbf{z}}} + \frac{1}{\gamma}(\operatorname{div}\mathbf{q}_{\mathbf{z}},\operatorname{div}\mathbf{w}_h)_{\omega_{\mathbf{z}}} = (\varphi_{\mathbf{z}}\nabla u_*,\mathbf{w}_h)_{\omega_{\mathbf{z}}} - \frac{1}{\gamma}(\lambda_*\varphi_{\mathbf{z}}u_*,\operatorname{div}\mathbf{w}_h)_{\omega_{\mathbf{z}}} \quad \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}}$$

Patch of elements: $\omega_{\mathbf{z}} = \bigcup\{K \in \mathcal{T}_h : \mathbf{z}\in K\}$

Partition of unity: $\sum_{\mathbf{z}\in\mathcal{N}_h}\varphi_{\mathbf{z}} = 1$

$\mathbf{W}_{\mathbf{z}} = \{\mathbf{q}\in\mathbf{H}(\operatorname{div},\omega_{\mathbf{z}}) : \mathbf{q}|_K \in \mathbf{RT}_1(K)\ \forall K\subset\omega_{\mathbf{z}},\ \mathbf{q}\cdot\mathbf{n}_{\mathbf{z}} = 0 \text{ on } \Gamma^{\mathrm{E}}_{\mathbf{z}}\}$

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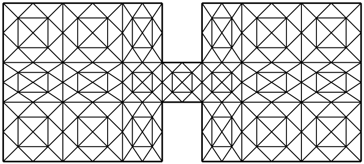

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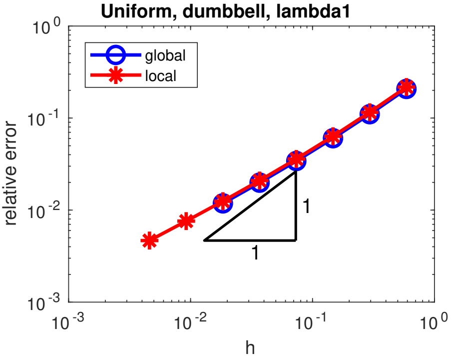
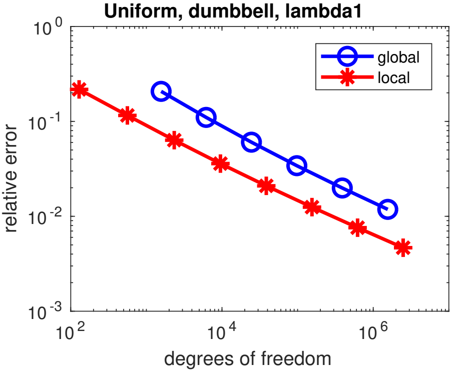
$\gamma = 10^{-6}$

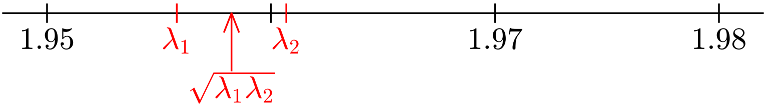




Closeness assumption for dumbbell

$$\sqrt{\lambda_{n-1}\lambda_n} \le \lambda_* \le \sqrt{\lambda_n \lambda_{n+m}} \quad \Rightarrow \quad \ell_n \le \lambda_n$$

Exact eigenvalues: $\lambda_1 = 1.955793794588$, $\lambda_2 = 1.960683031595$

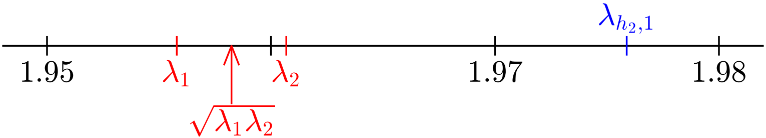


h	ℓ_1	$\lambda_{h,1}$	closeness
$h_1 = 1.1781$	1.6618	2.0228	no

Closeness assumption for dumbbell

$$\sqrt{\lambda_{n-1}\lambda_n} \le \lambda_* \le \sqrt{\lambda_n \lambda_{n+m}} \quad \Rightarrow \quad \ell_n \le \lambda_n$$

Exact eigenvalues: $\lambda_1 = 1.955793794588$, $\lambda_2 = 1.960683031595$



h	ℓ_1	$\lambda_{h,1}$	closeness
$h_1 = 1.1781$	1.6618	2.0228	no
$h_2 = 0.5890$	1.7711	1.9759	no

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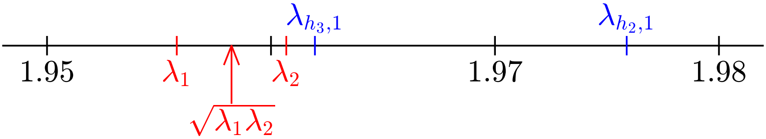

h	ℓ_1	$\lambda_{h,1}$	closeness
$h_1 = 1.1781$	1.6618	2.0228	no
$h_2 = 0.5890$	1.7711	1.9759	no
$h_3 = 0.2945$	1.8449	1.9620	no

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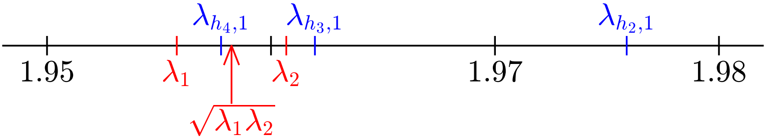

h	ℓ_1	$\lambda_{h,1}$	closeness
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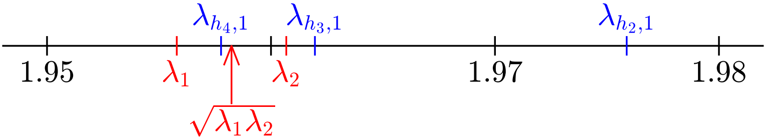
$h_2 = 0.5890$	1.7711	1.9759	no
$h_3 = 0.2945$	1.8449	1.9620	no
$h_4 = 0.1473$	1.8899	1.9578	yes

Closeness assumption for dumbbell

$$\sqrt{\lambda_{n-1}\lambda_n} \le \lambda_* \le \sqrt{\lambda_n \lambda_{n+m}} \quad \Rightarrow \quad \ell_n \le \lambda_n$$

Exact eigenvalues: $\lambda_1 = 1.955793794588$, $\lambda_2 = 1.960683031595$



h	ℓ_1	$\lambda_{h,1}$	closeness
$h_1 = 1.1781$	1.6618	2.0228	no
$h_2 = 0.5890$	1.7711	1.9759	no
$h_3 = 0.2945$	1.8449	1.9620	no
$h_4 = 0.1473$	1.8899	1.9578	yes
$h_5 = 0.0736$	1.9163	1.9565	yes
$h_6 = 0.0368$	1.9319	1.9560	yes
$h_7 = 0.0184$	1.9411	1.9559	yes

Weinstein's bound – summary

- easy to use
- it is a generalization of Bauer–Fike estimates for matrices
- good for general symmetric elliptic problems
- sub-optimal speed of convergence
- a priori information on spectrum needed for guaranteed lower bounds

4. Lower bounds on eigenvalues

4.2 Lehmann–Goerisch method

Lehmann–Goerisch method

General setting:

Find $\lambda_n \in \mathbb{R}$ and $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Lehmann method

Theorem

Let $\lambda_{h,N} < \rho \leq \lambda_{N+1}$

- $\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N \in V$ be linearly independent
- $A_{0,ij} = a(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = b(\tilde{u}_i, \tilde{u}_j)$
- $w_i \in V: \quad a(w_i, v) = b(\tilde{u}_i, v) \quad \forall v \in V$
  $A_{2,ij} = a(w_i, w_j)$
- $\mu_1 \leq \mu_2 \leq \dots \leq \mu_N: \quad (A_0 - \rho A_1)\mathbf{x} = \mu(A_0 - 2\rho A_1 + \rho^2 A_2)\mathbf{x}$

Then $\mu_N < 0$ and

$$\rho - \frac{\rho}{1 - \mu_{N+1-n}} \leq \lambda_n, \quad n = 1, 2, \dots, N.$$

Lehmann–Goerisch method

Theorem

Let $\lambda_{h,N} < \rho \leq \lambda_{N+1}$

- $\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N \in V$ be linearly independent
- $A_{0,ij} = a(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = b(\tilde{u}_i, \tilde{u}_j)$
- X ... vector space
  $\mathcal{B}$ ... positive semidefinite symmetric bilinear form on X
  $T : V \to X$ ... linear operator:
  (a) $\mathcal{B}(Tu, Tv) = a(u, v) \quad \forall u, v \in V$
  (b) $\hat{\mathbf{w}}_i \in X : \quad \mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \quad \forall v \in V$
  (c) $\hat{A}_{2,ij} = \mathcal{B}(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j)$
- $\hat{\mu}_1 \leq \hat{\mu}_2 \leq \dots \leq \hat{\mu}_N : \quad (A_0 - \rho A_1)\hat{\mathbf{x}} = \hat{\mu}(A_0 - 2\rho A_1 + \rho^2 \hat{A}_2)\hat{\mathbf{x}}$

Then $\hat{\mu}_N < 0$ and

$$\rho - \frac{\rho}{1 - \hat{\mu}_{N+1-n}} \leq \lambda_n, \quad n = 1, 2, \dots, N.$$

Proof: Lehmann $\Rightarrow$ Goerisch

It suffices to show that $\hat{A}_2 - A_2$ is positive semidefinite, because

$\Rightarrow$ $\mu_i \leq \hat{\mu}_i < 0$ for all $i = 1, 2, \ldots, N$,

$\Rightarrow$ $\rho - \dfrac{\rho}{1 - \hat{\mu}_{N+1-n}} \leq \rho - \dfrac{\rho}{1 - \mu_{N+1-n}} \leq \lambda_n.$

To show that $\hat{A}_2 - A_2$ is positive semidefinite:
Let $\mathbf{x} \in \mathbb{R}^N$, $\tilde{u} = \sum_{i=1}^N x_i \tilde{u}_i$, $w = \sum_{i=1}^N x_i w_i$, $\hat{\mathbf{w}} = \sum_{i=1}^N x_i \hat{\mathbf{w}}_i$, and

$$0 \leq \mathcal{B}(\hat{\mathbf{w}} - Tw, \hat{\mathbf{w}} - Tw) = \mathcal{B}(\hat{\mathbf{w}}, \hat{\mathbf{w}}) - 2\underbrace{\mathcal{B}(\hat{\mathbf{w}}, Tw)}_{\substack{\overset{(b)}{=} b(\tilde{u}, w) \\ = a(w,w)}} + \underbrace{\mathcal{B}(Tw, Tw)}_{\overset{(a)}{=} a(w,w)}.$$

Thus,

$$0 \leq \mathcal{B}(\hat{\mathbf{w}}, \hat{\mathbf{w}}) - a(w, w) \overset{(c)}{=} \mathbf{x}^T (\hat{A}_2 - A_2)\mathbf{x}.$$

□

Application to Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Setting

- $V = H_0^1(\Omega),\ a(u,v) = (\nabla u, \nabla v) + \gamma(u,v),\ b(u,v) = (u,v)$
- $X = \left[L^2(\Omega)\right]^3$
- $\mathcal{B}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = (\hat{u}_1, \hat{v}_1) + (\hat{u}_2, \hat{v}_2) + \gamma(\hat{u}_3, \hat{v}_3)$
- $Tu = \begin{pmatrix} \nabla u \\ u \end{pmatrix}$

Facts

(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

Application to Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega)$$

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- $Tu = \begin{pmatrix} \nabla u \\ u \end{pmatrix}$

Facts

(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

(b) $\mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \Leftarrow \hat{\mathbf{w}}_i = \begin{pmatrix} \boldsymbol{\sigma}_i \\ \hat{w}_{i,3} \end{pmatrix} \quad \boldsymbol{\sigma}_i \in \mathbf{H}(\mathrm{div}, \Omega)$

$$(\boldsymbol{\sigma}_i, \nabla v) + \gamma(\hat{w}_{i,3}, v) = (\tilde{u}_i, v) \quad \forall v \in V$$

$$-(\mathrm{div}\, \boldsymbol{\sigma}_i, v) + \gamma(\hat{w}_{i,3}, v) = (\tilde{u}_i, v) \quad \forall v \in V$$

$$\hat{w}_{i,3} = \frac{1}{\gamma}(\tilde{u}_i + \mathrm{div}\, \boldsymbol{\sigma}_i)$$

Application to Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Setting

- $V = H_0^1(\Omega),\ a(u,v) = (\nabla u, \nabla v) + \gamma(u,v),\ b(u,v) = (u,v)$
- $X = \left[L^2(\Omega)\right]^3$
- $\mathcal{B}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = (\hat{u}_1, \hat{v}_1) + (\hat{u}_2, \hat{v}_2) + \gamma(\hat{u}_3, \hat{v}_3)$
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(a) $\mathcal{B}(Tu, Tv) = a(u, v)$

(b) $\mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \Leftarrow \hat{\mathbf{w}}_i = \begin{pmatrix} \boldsymbol{\sigma}_i \\ \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i) \end{pmatrix} \quad \boldsymbol{\sigma}_i \in \mathbf{H}(\operatorname{div}, \Omega)$

Application to Laplace eigenvalue problem

$$(\nabla u_i, \nabla v) + \gamma(u_i, v) = (\lambda_i + \gamma)(u_i, v) \quad \forall v \in H_0^1(\Omega)$$

Setting

- $V = H_0^1(\Omega),\ a(u,v) = (\nabla u, \nabla v) + \gamma(u,v),\ b(u,v) = (u,v)$
- $X = \left[L^2(\Omega)\right]^3$
- $\mathcal{B}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = (\hat{u}_1, \hat{v}_1) + (\hat{u}_2, \hat{v}_2) + \gamma(\hat{u}_3, \hat{v}_3)$
- $Tu = \begin{pmatrix} \nabla u \\ u \end{pmatrix}$

Facts

(a) $\mathcal{B}(Tu, Tv) = a(u,v)$

(b) $\mathcal{B}(\hat{\mathbf{w}}_i, Tv) = b(\tilde{u}_i, v) \Leftarrow \hat{\mathbf{w}}_i = \begin{pmatrix} \boldsymbol{\sigma}_i \\ \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i) \end{pmatrix} \quad \boldsymbol{\sigma}_i \in \mathbf{H}(\operatorname{div}, \Omega)$

(c) $\hat{A}_{2,ij} = \mathcal{B}(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j) \ \Leftrightarrow\ \hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(\tilde{u}_i + \operatorname{div} \boldsymbol{\sigma}_i, \tilde{u}_j + \operatorname{div} \boldsymbol{\sigma}_j)$

Application to Laplace eigenvalue problem

Theorem (Lehmann–Goerisch)

Let $\lambda_{h,N} + \gamma < \rho \leq \lambda_{N+1} + \gamma$, $\gamma > 0$

- $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N \in V$ be linearly independent
- $A_{0,ij} = (\nabla \tilde{u}_i, \nabla \tilde{u}_j) + \gamma(\tilde{u}_i, \tilde{u}_j)$
- $A_{1,ij} = (\tilde{u}_i, \tilde{u}_j)$
- $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_N \in \mathbf{H}(\mathrm{div}, \Omega)$ be arbitrary
  $\hat{A}_{2,ij} = (\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j) + \frac{1}{\gamma}(\tilde{u}_i + \mathrm{div}\, \boldsymbol{\sigma}_i, \tilde{u}_j + \mathrm{div}\, \boldsymbol{\sigma}_j)$
- $\hat{\mu}_1 \leq \hat{\mu}_2 \leq \cdots \leq \hat{\mu}_N : \quad (A_0 - \rho A_1)\hat{\mathbf{x}} = \hat{\mu}(A_0 - 2\rho A_1 + \rho^2 \hat{A}_2)\hat{\mathbf{x}}$

Then

$$\ell_n = \rho - \gamma - \frac{\rho}{1 - \hat{\mu}_{N+1-n}} \leq \lambda_n, \quad n = 1, 2, \ldots, N$$

[Behnke, Mertins, Plum, Wieners 2000]

How to find good $\hat{\mathbf{w}}_i$?

Observation: Let $\tilde{u}_i \approx u_i$ and $\tilde{\lambda}_i \approx \lambda_i$.

$\Rightarrow \; a(w_i, v) = b(\tilde{u}_i, v) \approx \frac{1}{\tilde{\lambda}_i} a(\tilde{u}_i, v) \quad \forall v \in V$

$\Rightarrow \; w_i \approx \frac{1}{\tilde{\lambda}_i} \tilde{u}_i$

Need

$\Rightarrow \; \hat{A}_2 \approx A_2$

$\Rightarrow \; \mathcal{B}(\hat{\mathbf{w}}_i, \hat{\mathbf{w}}_j) \approx a(w_i, w_j) = \mathcal{B}(Tw_i, Tw_j)$

$\Rightarrow \; \hat{\mathbf{w}}_i \approx Tw_i \approx \frac{1}{\tilde{\lambda}_i} T\tilde{u}_i$

Natural idea

make $|\frac{1}{\tilde{\lambda}_i} T\tilde{u}_i - \hat{\mathbf{w}}_i|^2_{\mathcal{B}}$ small

For Laplacian: Find $\boldsymbol{\sigma}_{h,i} \in \mathbf{H}(\mathrm{div}, \Omega)$ that

makes $\left\| \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma} - \boldsymbol{\sigma}_{h,i} \right\|_0^2 + \frac{1}{\gamma} \left\| \frac{\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma} + \mathrm{div}\, \boldsymbol{\sigma}_{h,i} \right\|_0^2$ small

Choice of $\boldsymbol{\sigma}_i$ – global

Global minimization:

Find $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h$, $i = 1, 2, \ldots, N$, that minimizes

$$\left\| \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma} - \boldsymbol{\sigma}_{h,i} \right\|_0^2 + \frac{1}{\gamma} \left\| \frac{\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma} + \operatorname{div} \boldsymbol{\sigma}_{h,i} \right\|_0^2$$

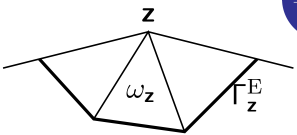
Euler-Lagrange equations:

$$(\boldsymbol{\sigma}_{h,i}, \mathbf{w}_h) + \frac{1}{\gamma}(\operatorname{div} \boldsymbol{\sigma}_{h,i}, \operatorname{div} \mathbf{w}_h) = \left(\frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma}, \mathbf{w}_h \right) - \frac{1}{\gamma} \left(\frac{\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma}, \operatorname{div} \mathbf{w}_h \right) \quad \forall \mathbf{w}_h \in \mathbf{W}_h$$

$$\mathbf{W}_h = \{ \boldsymbol{\sigma}_h \in \mathbf{H}(\operatorname{div}, \Omega) : \boldsymbol{\sigma}_h|_K \in \mathbf{RT}_1(K) \quad \forall K \in \mathcal{T}_h \}$$

Choice of $\boldsymbol{\sigma}_i$ – local

Flux reconstruction:



$$\boldsymbol{\sigma}_{h,i} = \sum_{\mathbf{z}\in\mathcal{N}_h} \boldsymbol{\sigma}_{\mathbf{z},i}$$

Local problems: Find $\boldsymbol{\sigma}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$, $i = 1, 2, \dots, N$ minimizing

$$\left\| \varphi_{\mathbf{z}} \frac{\nabla u_{h,i}}{\lambda_{h,i}+\gamma} - \boldsymbol{\sigma}_{\mathbf{z},i} \right\|_{0,\omega_{\mathbf{z}}}^2 + \frac{1}{\gamma} \left\| \frac{\lambda_{h,i}\varphi_{\mathbf{z}} u_{h,i}}{\lambda_{h,i}+\gamma} + \operatorname{div} \boldsymbol{\sigma}_{\mathbf{z},i} \right\|_{0,\omega_{\mathbf{z}}}^2$$

Euler-Lagrange equations:

$$(\boldsymbol{\sigma}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} + \frac{1}{\gamma} (\operatorname{div} \boldsymbol{\sigma}_{\mathbf{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}}$$

$$= \left(\varphi_{\mathbf{z}} \frac{\nabla u_{h,i}}{\lambda_{h,i}+\gamma}, \mathbf{w}_h \right)_{\omega_{\mathbf{z}}} - \frac{1}{\gamma} \left(\frac{\varphi_{\mathbf{z}} \lambda_{h,i} u_{h,i}}{\lambda_{h,i}+\gamma}, \operatorname{div} \mathbf{w}_h \right)_{\omega_{\mathbf{z}}} \quad \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}}$$

Patch of elements: $\omega_{\mathbf{z}} = \bigcup \{K \in \mathcal{T}_h : \mathbf{z} \in K\}$

Partition of unity: $\sum_{\mathbf{z}\in\mathcal{N}_h} \varphi_{\mathbf{z}} = 1$

$\mathbf{W}_{\mathbf{z}} = \{\boldsymbol{\sigma} \in \mathbf{H}(\operatorname{div}, \omega_{\mathbf{z}}) : \boldsymbol{\sigma}|_K \in \mathbf{RT}_1(K)\ \forall K \subset \omega_{\mathbf{z}},\ \boldsymbol{\sigma}\cdot\mathbf{n}_{\mathbf{z}} = 0 \text{ on } \Gamma_{\mathbf{z}}^{\mathrm{E}}\}$

Comparison of flux reconstructions

Weinstein: Find $\mathbf{q}_{h,i} \in \mathbf{W}_h$ minimizing

$$\|\nabla u_{h,i} - \mathbf{q}_{h,i}\|_0^2 + \frac{1}{\gamma}\|\lambda_{h,i} u_{h,i} + \operatorname{div} \mathbf{q}_{h,i}\|_0^2$$

Lehmann–Goerisch: Find $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h$ minimizing

$$\left\| \frac{\nabla u_{h,i}}{\lambda_{h,i} + \gamma} - \boldsymbol{\sigma}_{h,i} \right\|_0^2 + \frac{1}{\gamma} \left\| \frac{\lambda_{h,i} u_{h,i}}{\lambda_{h,i} + \gamma} + \operatorname{div} \boldsymbol{\sigma}_{h,i} \right\|_0^2$$

Thus,

$$\mathbf{q}_{h,i} = (\lambda_{h,i} + \gamma)\boldsymbol{\sigma}_{h,i}$$

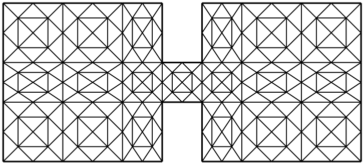
[Vejchodský 2018]

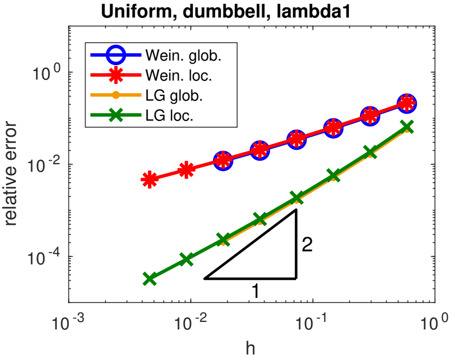
Example: dumbbell

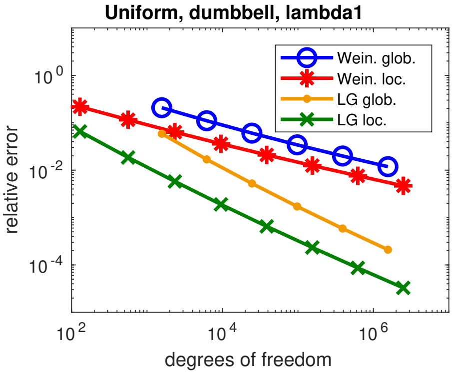
$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
$$u_n = 0 \quad \text{on } \partial\Omega$$

$$\text{rel\_err} = \frac{|\lambda_n - \lambda_{h,n}|}{\lambda_n} \leq \frac{\lambda_{h,n} - \ell_n}{\ell_n}$$

$\gamma = 10^{-6}$







How to get the a priori lower bound ρ?

Monotonicity principle: If $V \subset \widetilde{V}$ then $\mathcal{V}^{(n)} \subset \widetilde{\mathcal{V}}^{(n)}$ and

$$\tilde{\lambda}_n = \min_{E \in \widetilde{\mathcal{V}}^{(n)}} \max_{v \in E} R(v) \leq \min_{E \in \mathcal{V}^{(n)}} \max_{v \in E} R(v) = \lambda_n$$

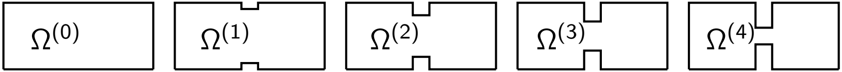
Example 1.

$$\Omega \subset \widetilde{\Omega} \quad \Rightarrow \quad H_0^1(\Omega) \subset H_0^1(\widetilde{\Omega}) \quad \Rightarrow \quad \tilde{\lambda}_n \leq \lambda_n$$

Example 2.

$$H_0^1(\Omega) \subset H^1(\Omega) \quad \Rightarrow \quad \lambda_n^{\text{Neumann}} \leq \lambda_n^{\text{Dirichlet}}$$

Homotopy



Analytically:	$\rho = 12.16$	$\rho = 11.39$	$\rho = 10.77$	$\rho = 9.988$
$12.16 \leq \lambda_{17}^{(0)}$	$\ell_{15} \doteq 11.39$	$\ell_{13} \doteq 10.77$	$\ell_{11} \doteq 9.988$	

[Plum 1990, 1991]

Adaptive mesh refinement

Recall the residual

$$w \in V : \quad (\nabla w, \nabla v) = (\nabla u_{h,i}, \nabla v) - \lambda_{h,i}(u_{h,i}, v) \quad \forall v \in V$$

Recall theorem:

$$\|\nabla w\|_0 \le \eta, \quad \text{where } \eta^2 = \|\nabla u_{h,i} - \mathbf{q}_{h,i}\|^2_{L^2(\Omega)} + \frac{1}{\gamma}\|\lambda_{h,i} u_{h,i} + \operatorname{div} \mathbf{q}_{h,i}\|^2_{L^2(\Omega)}$$

Local error indicators for mesh refinement:

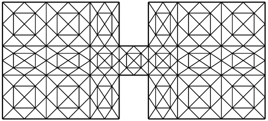
$$\eta_K^2 = \|\nabla u_{h,i} - \mathbf{q}_{h,i}\|^2_{L^2(K)} + \frac{1}{\gamma}\|\lambda_{h,i} u_{h,i} + \operatorname{div} \mathbf{q}_{h,i}\|^2_{L^2(K)}$$

Note: Good for both Weinstein and Lehmann–Goerisch method:

$$\mathbf{q}_{h,i} = (\lambda_{h,i} + \gamma)\boldsymbol{\sigma}_{h,i}$$

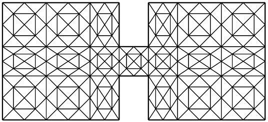
Example: dumbbell

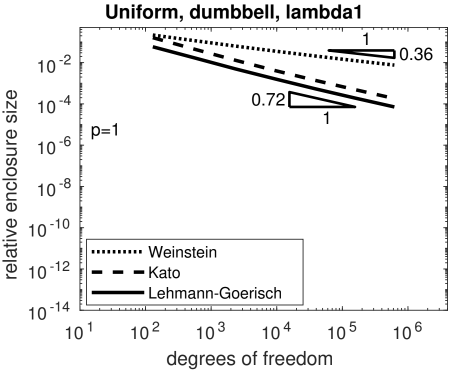
$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i \quad && \text{in } \Omega \\
u_i &= 0 && \text{on } \partial\Omega
\end{aligned}
$$



Example: dumbbell

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$

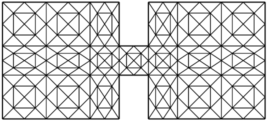


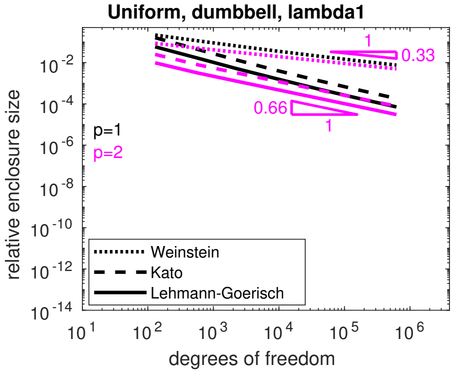


- relative enclosure size: $(\lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}$, $\rho := \ell_{11}^{\mathrm{Wein}} \approx 10.0017 \leq \lambda_{11}$

Example: dumbbell

$$
\begin{aligned}
-\Delta u_i &= \lambda_i u_i \quad \text{in } \Omega \\
u_i &= 0 \qquad\;\; \text{on } \partial\Omega
\end{aligned}
$$

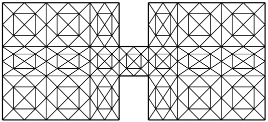


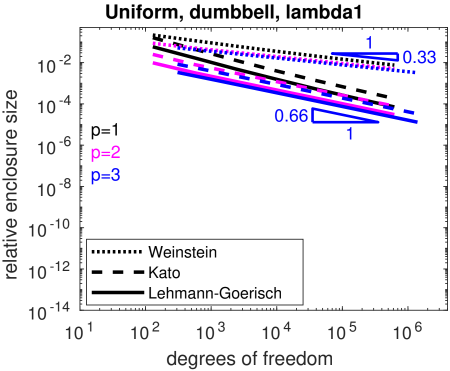


- relative enclosure size: $(\lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}, \quad \rho := \ell_{11}^{\mathrm{Wein}} \approx 10.0017 \leq \lambda_{11}$

Example: dumbbell

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$





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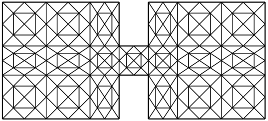

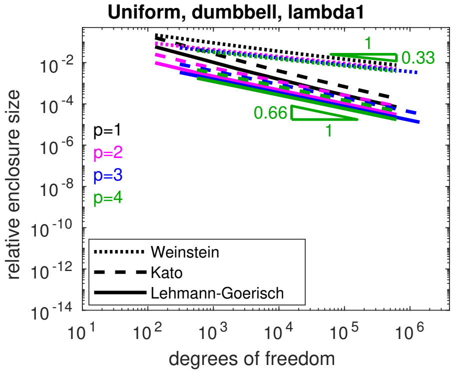

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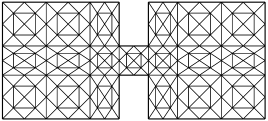

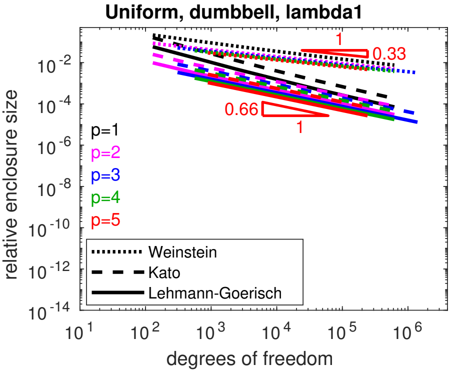

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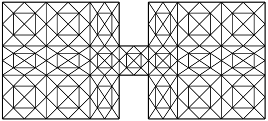

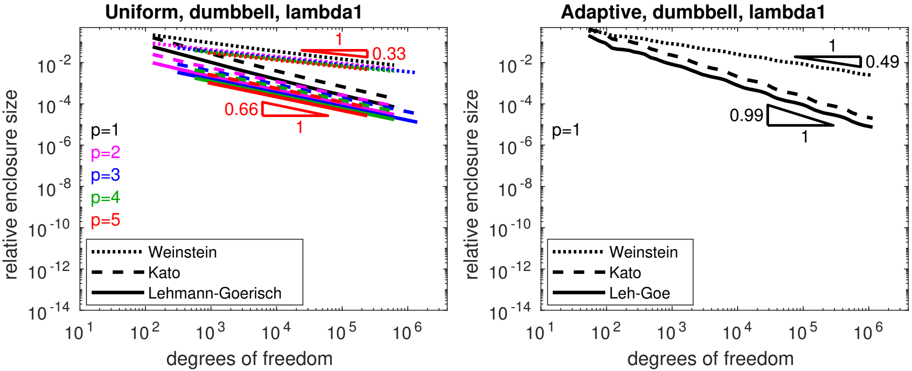

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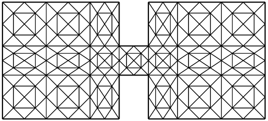

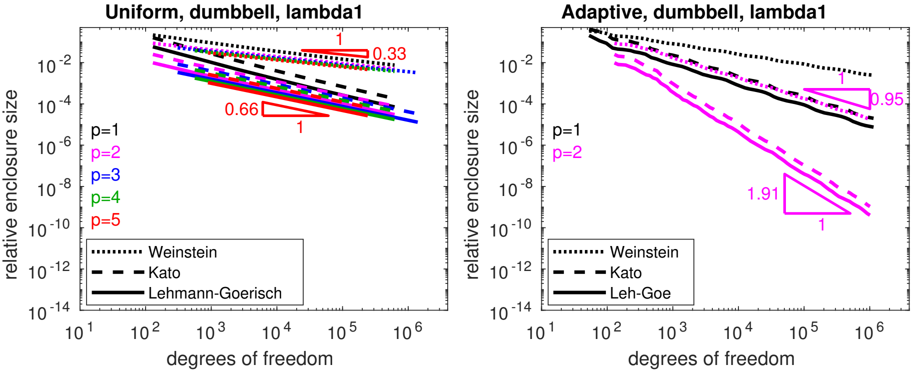

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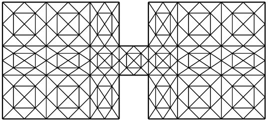

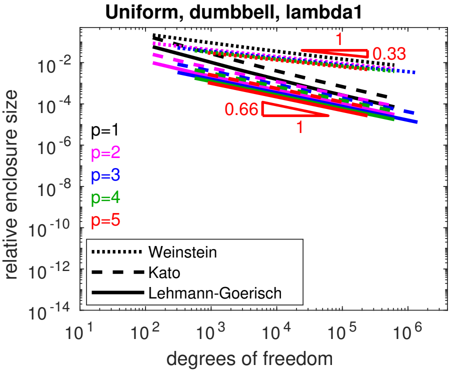

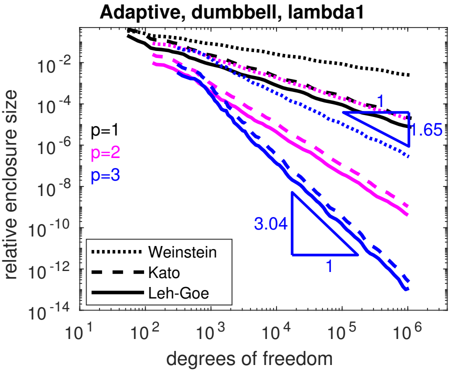

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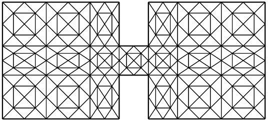

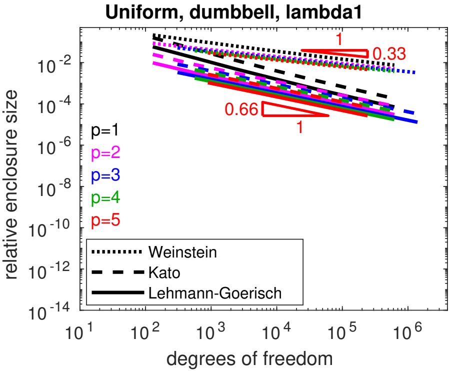

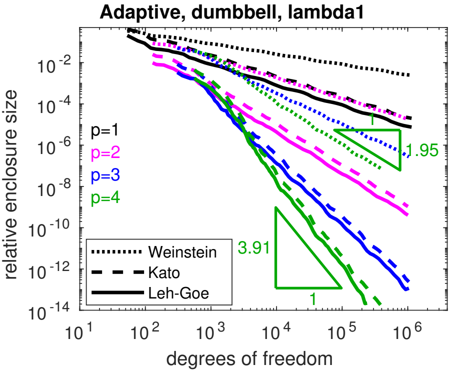

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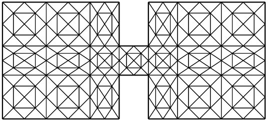

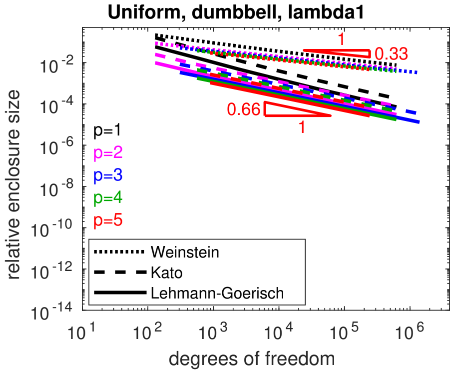

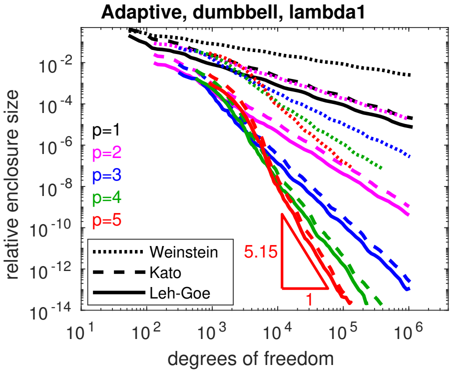

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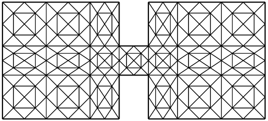
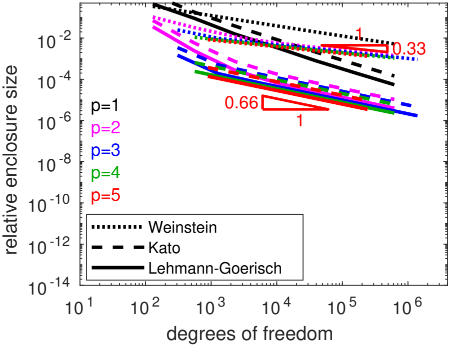
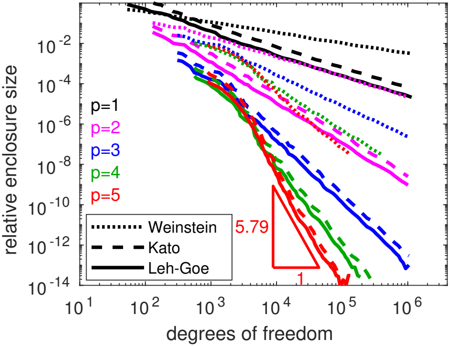


Uniform, dumbbell, lambda5



Adaptive, dumbbell, lambda5



- relative enclosure size: $(\lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}, \quad \rho := \ell_{11}^{\mathrm{Wein}} \approx 10.0017 \leq \lambda_{11}$

Example: dumbbell

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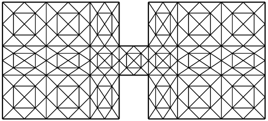
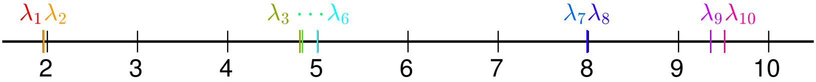




Computed bounds ($p = 5$, adaptive):

$$1.9557937945883 \le \lambda_1 \le 1.9557937945884$$
$$1.9606830315950 \le \lambda_2 \le 1.9606830315951$$
$$4.8007611240339 \le \lambda_3 \le 4.8007611240345$$
$$4.8298952545005 \le \lambda_4 \le 4.8298952545010$$
$$4.9968370972489 \le \lambda_5 \le 4.9968370972490$$
$$4.9968509041015 \le \lambda_6 \le 4.9968509041016$$
$$7.9869672921028 \le \lambda_7 \le 7.9869672921038$$
$$7.9870343068216 \le \lambda_8 \le 7.9870343068227$$

Lehmann–Goerisch method – summary

- optimal speed of convergence
- implementation based on standard FEM
- adaptivity for free
- naturally generalize to higher orders
- good for a wide class of problems
- an a priori lower bound on some eigenvalue is needed

4. Lower bounds on eigenvalues

4.3 Method based on Crouzeix–Raviart elements

[Carstensen, Gallistl, Gedicke 2014], [Liu 2015]

Nonconforming approximation

Eigenvalue problem: Find λ_n and $u_n \in V \setminus \{0\}$ such that

$$a(u_n, v) = \lambda_n b(u_n, v) \quad \forall v \in V$$

Finite dimensional space: $\dim V_h = N < \infty$, but it can be $V_h \not\subset V$.
Discrete eigenvalue problem: Find $\lambda_{h,n} \in \mathbb{R}$, $u_{h,n} \in V_h \setminus \{0\}$:

$$a(u_{h,n}, v_h) = \lambda_{h,n} b(u_{h,n}, v_h) \quad \forall v_h \in V_h$$

Definition:
$V(h) = V \oplus V_h = \{v + v_h : v \in V,\ v_h \in V_h\}$

Extensions of bilinear forms:
$a_h, b_h : V(h) \times V(h) \to \mathbb{R}$
$a_h(u, v) = a(u, v)$ and $b_h(u, v) = b(u, v) \quad \forall u, v \in V$
$a_h(\cdot, \cdot)$ is symmetric and $V(h)$-elliptic
$b_h(\cdot, \cdot)$ is symmetric and positive semidefinite on $V(h)$
Notation: $a = a_h$ and $b = b_h$

Lemmas

Lemma 1 (Discrete Friedrichs inequality).

$$|v_h|_b \le \lambda_{h,1}^{-1/2} \|v_h\|_a \quad \forall v_h \in V_h$$

Proof. $\lambda_{h,1} = \min\limits_{w_h \in V_h} \dfrac{\|w_h\|_a^2}{|w_h|_b^2} \le \dfrac{\|v_h\|_a^2}{|v_h|_b^2}$ □

Elliptic projection: $P_h : V(h) \to V_h$

$$a(u - P_h u, v_h) = 0 \quad \forall v_h \in V_h$$

Lemma 2.

$$\|v\|_a^2 = \|P_h v\|_a^2 + \|v - P_h v\|_a^2$$

Proof.

$$\|v - P_h v\|_a^2 = \|v\|_a^2 - 2a(v, P_h v) + \|P_h v\|_a^2$$

$$a(v, P_h v) = a(P_h v, P_h v) = \|P_h v\|_a^2$$ □

Lower bound

Theorem. Let $|u - P_h u|_b \le C_h \|u - P_h u\|_a$. Then

$$\frac{\lambda_{h,n}}{1 + \lambda_{h,n} C_h^2} \le \lambda_n, \quad n = 1, 2, \dots, N.$$

Proof (for λ_1 only). Let $v \in V$.

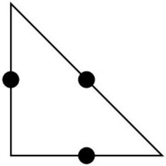
$$\begin{aligned}
|v|_b &\le |P_h v|_b + |v - P_h v|_b \\
&\le \lambda_{h,1}^{-1/2} \|P_h v\|_a + C_h \|v - P_h v\|_a \\
&\le \left(\lambda_{h,1}^{-1} + C_h^2\right)^{1/2} \left(\|P_h v\|_a^2 + \|v - P_h v\|_a^2\right)^{1/2} \\
&= \left(\frac{1 + \lambda_{h,1} C_h^2}{\lambda_{h,1}}\right)^{1/2} \|v\|_a
\end{aligned}$$

$$\lambda_1 = \min_{v \in V} \frac{\|v\|_a^2}{|v|_b^2} \ge \frac{\lambda_{h,1}}{1 + \lambda_{h,1} C_h^2}$$

Crouzeix–Raviart (CR) elements

Laplace eigenvalue problem: Find $\lambda_n \in \mathbb{R}$, $u_n \in H_0^1(\Omega) \setminus \{0\}$:

$$(\nabla u_n, \nabla v) = \lambda_n (u_n, v) \quad \forall v \in H_0^1(\Omega)$$



CR space: $v_h \in V_h^{\mathrm{CR}}$ if

- $v_h|_K \in \mathbb{P}^1(K)$
- v_h is continuous at midpoints of interior edges
- $v_h = 0$ at midpoints of boundary edges

CR eigenvalue problem: Find $\lambda_{h,i}^{\mathrm{CR}} \in \mathbb{R}$, $u_{h,i}^{\mathrm{CR}} \in V_h^{\mathrm{CR}} \setminus \{0\}$:

$$(\nabla u_{h,i}^{\mathrm{CR}}, \nabla v_h) = \lambda_{h,i}^{\mathrm{CR}} (u_{h,i}^{\mathrm{CR}}, v_h) \quad \forall v_h \in V_h^{\mathrm{CR}}.$$

Crouzeix–Raviart interpolation

Let e_i, $i = 1, 2, 3$, be edges of triangle K.
Definition: $\Pi_h : H^1(K) \to \mathbb{P}^1(K)$ such that

$$\int_{e_i} u - \Pi_h u \,\mathrm{d}s = 0 \quad \forall i = 1, 2, 3.$$

Note: If m_i is a midpoint of e_i then $\Pi_h(m_i) = \frac{1}{|e_i|} \int_{e_i} u \,\mathrm{d}s$.

Lemma. $\Pi_h = P_h$
Proof.
Let $u \in H^1(\Omega) \oplus V_h^{\mathrm{CR}}$ and $v_h \in V_h^{\mathrm{CR}}$.

$$a(u - \Pi_h u, v_h) = \sum_{K \in \mathcal{T}_h} \int_K \nabla(u - \Pi_h u) \cdot \nabla v_h$$

$$= \sum_{K \in \mathcal{T}_h} \left(\sum_{i=1}^{3} \int_{e_i} (u - \Pi_h u) \underbrace{\frac{\partial v_h}{\partial \mathbf{n}}}_{=\text{const.}} \,\mathrm{d}s - \int_K (u - \Pi_h u) \underbrace{\Delta v_h}_{=0} \,\mathrm{d}x \right) = 0$$

The value of C_h

Interpolation error estimate:

$$\|u - \Pi_h u\|_{L^2(\Omega)} \leq C_h \|\nabla u - \nabla \Pi_h u\|_{L^2(\Omega)}$$

Local interpolation error estimate:

$$\|u - \Pi_h u\|_{L^2(K)} \leq C_h(K) \|\nabla u - \nabla \Pi_h u\|_{L^2(K)}$$

Lemma.

$$C_h \leq \max_{K \in \mathcal{T}_h} C_h(K)$$

Proof.

$$\|u - \Pi_h u\|^2_{L^2(\Omega)} = \sum_{K \in \mathcal{T}_h} \|u - \Pi_h u\|^2_{L^2(K)} \leq \sum_{K \in \mathcal{T}_h} C_h^2(K) \|\nabla u - \nabla \Pi_h u\|^2_{L^2(K)}$$

$$\leq \max_{K \in \mathcal{T}_h} C_h^2(K) \|\nabla u - \nabla \Pi_h u\|^2_{L^2(\Omega)}$$

$\square$

Explicit estimates of C_h

Interval

- $C_h = h/\pi$

Triangle

- $C_h = 0.4396h$ [Carstensen, Gedicke 2014]
- $C_h = 0.2983h$ [Carstensen, Gallistl 2014]
- $C_h = 0.1893h$ [Liu 2015]

Tetrahedron

- $C_h = 0.3804h$ [Liu 2015]

Explicit estimate of C_h for an interval

Setting: $\Omega = (\alpha, \beta), \quad V = H_0^1(\alpha, \beta),$

$a(u, v) = \int_\alpha^\beta u'v' \,\mathrm{d}x, \; b(u, v) = \int_\alpha^\beta uv \,\mathrm{d}x$

Partition: $\alpha = z_0 < z_1 < \cdots < z_N = \beta$

Elements: $K_i = [z_{i-1}, z_i], \; i = 1, 2, \ldots, N,$

$h_i = z_i - z_{i-1}, \; h = \max_{i=1,\ldots,N} h_i$

CR space: $V_h = \{v \in H_0^1(\alpha, \beta) : v|_{K_i} \in \mathbb{P}^1(K_i), \;\; i = 1, 2, \ldots, N\}$

Interpolation: $\Pi_h : H_0^1(\alpha, \beta) \to V_h$

$(\Pi_h u)(x_i) = u(x_i), \; i = 0, \ldots, N$

Lemma.

$$\|u - \Pi_h u\|_{L^2(\Omega)} \leq \frac{h}{\pi} \|u' - (\Pi_h u)'\|_{L^2(\Omega)}$$

Proof.

$$\min_{v \in H^1(K_i)} R(v - \Pi_h v) = \min_{w \in H_0^1(K_i)} R(w) = R\left(\sin \frac{\pi(x - z_i)}{h_i}\right) = \pi^2 / h_i^2$$

Upper bound

Interpolation to continuous functions: $\mathcal{I} : V_h^{\mathrm{CR}} \to \widetilde{V}_h \subset H^1(\Omega)$
Examples:

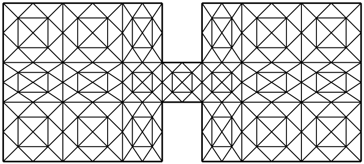
- Oswald quasi-interpolation [Oswald 1994]
- Interpolation to refined mesh [Carstensen, Merdon 2013]

Upper bound

- $\mathcal{T}_h^*$ is the red refinement of $\mathcal{T}_h$
- $u_{h,i}^* = \mathcal{I}_{\mathrm{CM}} \tilde{u}_{h,i}^{\mathrm{CR}}$ for $i = 1, 2, \dots, m$
- $\mathbf{S}, \mathbf{Q} \in \mathbb{R}^{m \times m}$ with entries $\mathbf{S}_{j,k} = (\nabla u_{h,j}^*, \nabla u_{h,k}^*)$ and $\mathbf{Q}_{j,k} = (u_{h,j}^*, u_{h,k}^*)$
- $\mathbf{S}\mathbf{y}_i = \Lambda_i^* \mathbf{Q}\mathbf{y}_i, \quad i = 1, 2, \dots, m$
- $\Lambda_1^* \le \Lambda_2^* \le \dots \le \Lambda_m^*$
- $\lambda_i \le \Lambda_i^* \quad$ for $i = 1, 2, \dots, m$

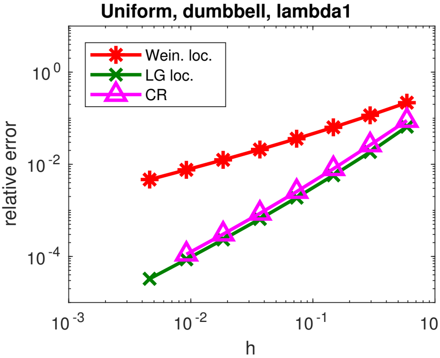
Example: dumbbell

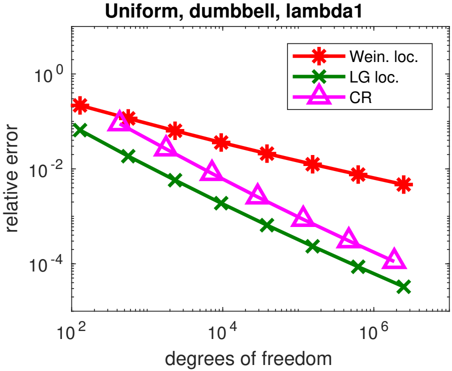
$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = \text{dumbbell}$$
$$u_n = 0 \qquad \text{on } \partial\Omega$$



$$\text{rel\_err} = \frac{|\lambda_n - \lambda_{h,n}|}{\lambda_n} \le \frac{\lambda_{h,n} - \ell_n}{\ell_n}$$

$\gamma = 10^{-6}$





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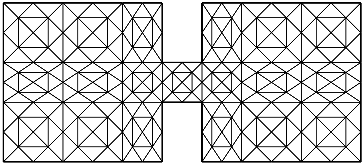

$$\text{rel\_err} = \frac{|\lambda_n - \lambda_{h,n}|}{\lambda_n} \le \frac{\lambda_{h,n} - \ell_n}{\ell_n}$$

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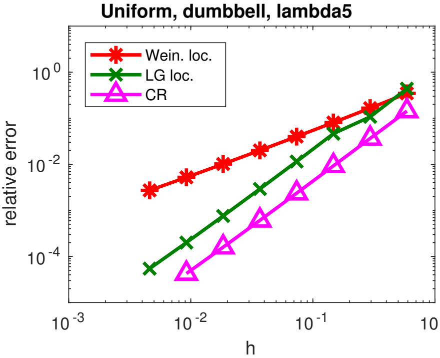

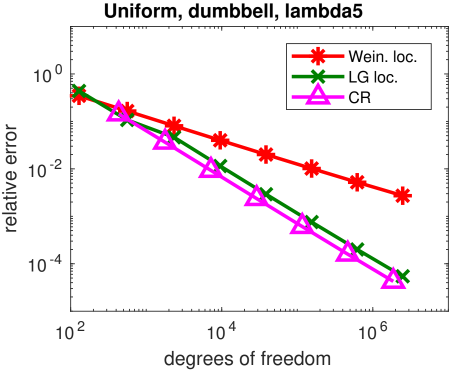

CR method – summary

- no a priori information needed
- optimal speed of convergence
- easy to implement
- interpolation constant known in special cases only
- adaptivity is not for free
- higher order variant is not available

5. Literature

Literature (very incomplete)

Books and chapters

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Appendices

1. Sensitivity of eigenfunctions

Laplace eigenvalue problem in a rectangle

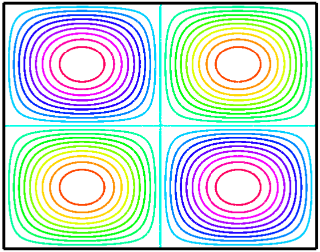
$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega = (0, \alpha\pi) \times (0, \pi)$$
$$u_n = 0 \quad \text{on } \partial\Omega$$

Exact solution

$$\lambda_{k,\ell} = \frac{k^2}{\alpha^2} + \ell^2$$
$$u_{k,\ell} = \sin \frac{kx}{\alpha} \sin \ell y$$

1. Sensitivity of eigenfunctions

$\alpha = 1.27$, $\lambda_4 = 6.4800$



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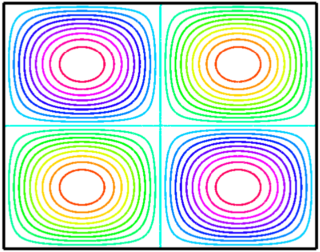
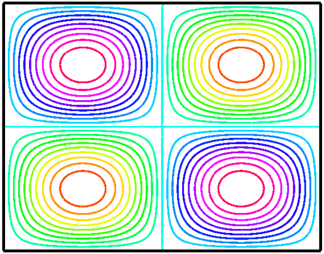


$\alpha = 1.28,\ \lambda_4 = 6.4414$



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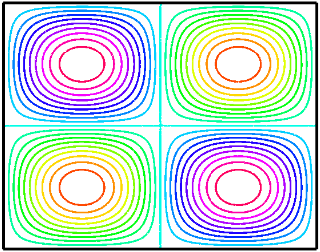

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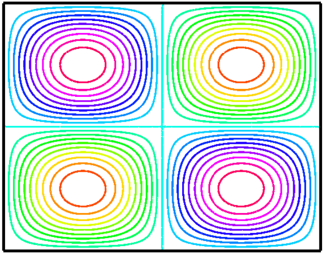
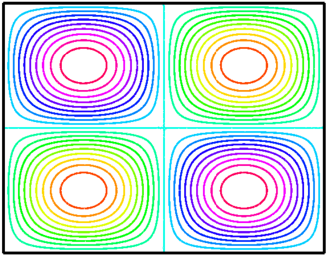


$\alpha = 1.29,\ \lambda_4 = 6.4037$



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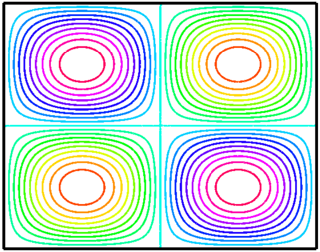

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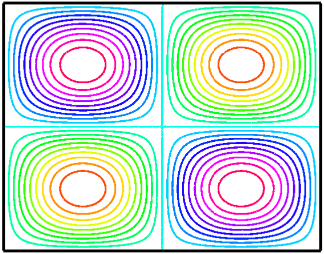

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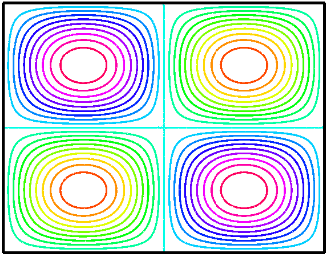
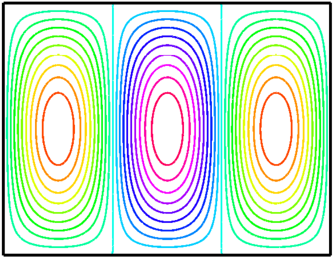


$\alpha = 1.30,\ \lambda_4 = 6.3254$



1. Sensitivity of eigenfunctions

Laplace eigenvalue problem in a rectangle

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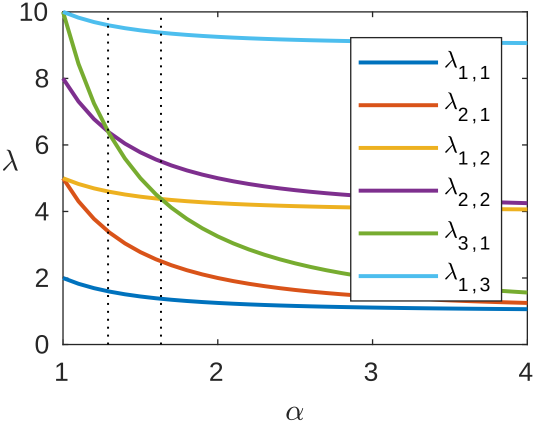
Dependence of eigenvalues on α



$$\alpha^* = \sqrt{5/3} \approx 1.2910$$

[Trefethen, Betcke 2006]

2. Interval arithmetic

Weinstein bound:

- λ_*, u_*, $\mathbf{q}$ can be arbitrary
- $\eta^2 = \|\nabla u_* - \mathbf{q}\|_0^2 + \frac{1}{\gamma}\|\lambda_* u_* + \operatorname{div}\mathbf{q}\|_0^2$

  must be evaluated exactly $(*)$

Lehmann–Goerisch method:

- $\tilde{u}_i$, $\boldsymbol{\sigma}_i$ can be arbitrary
- $(A_0 - \rho A_1)\hat{\mathbf{x}} = \hat{\mu}(A_0 - 2\rho A_1 + \rho^2 \hat{A}_2)\hat{\mathbf{x}}$

  must be solved exactly $(*)$

CR method:

- $\lambda_{h,i}^{\mathrm{CR}}$ must be computed exactly $(*)$

Interval arithmetic enables guaranteed computation of $(*)$.

Thank you for your attention

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