$\Omega \subset \mathbb{R}^N$ domain – an open connected subset of the Euclidean space $\mathbb{R}^N$

$Q \subset \mathbb{R}^N$

$C(\overline{Q})$ ... the Banach space of functions continuous on the closure $\overline{Q}$

$$\|g\|_{C(\overline{Q})} = \sup_{y \in \overline{Q}} |g(y)|$$

$C_{weak}(\overline{Q}, X)$ = the space of ~~ve~~ vector-valued functions on $\overline{Q}$ ranging in a Banach space X continuous with respect to the weak topology

$g \in C_{weak}(\overline{Q}, X)$ if the mapping

$y \longmapsto \|g(y)\|_X$ is bounded
and

$y \longmapsto \langle f; g(y) \rangle_{X^*, X}$

is continuous on $\overline{Q}$ for any linear form f belonging to the dual space X^*.

We shall say

$g_n \to g$ in $C_{weak}(\bar{Q}, X)$ if

$\langle f, g_n \rangle_{X^*; X} \to \langle f, g \rangle_{X^*, X}$

in $C(\bar{Q})$ for all $f \in X^*$

$C^k(\bar{Q})$, $Q \subset \mathbb{R}^N$, k is a non-negative integer, denotes the space of functions on $\bar{Q}$ that are restrictions of k-times continuously differentiable functions on $\mathbb{R}^N$

$C^{k,\nu}(\bar{Q})$, $\nu \in (0,1)$ is the subspace of $C^k(\bar{Q})$ of functions having their k-th derivatives ν-Hölder continuous in $\bar{Q}$

$C^{k,1}(\bar{Q})$ is subspace of $C^k(\bar{Q})$ of functions whose k-th derivatives are Lipschitz on $\bar{Q}$.

$$\|u\|_{C^k(\bar{Q})} = \max_{|\alpha| \le k} \sup_{x \in Q} |D^\alpha u(x)|$$

$$\|u\|_{C^{k,\nu}(\bar{Q})} = \|u\|_{C^k(\bar{Q})} + \max_{|\alpha| = k} \sup_{\substack{x,y \in Q \\ x \neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\nu}$$

$$C^\infty = \bigcap_{k=0}^{\infty} C^k$$

Theorem (Arzelà - Ascoli)

Let $Q \subset \mathbb{R}^M$ be compact and X a compact topological metric space endowed with a metric d_X. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of functions in $C(Q;X)$ that is equi-continuous, meaning for any $\varepsilon > 0$ there is $\delta > 0$ such that

$d_X[v_n(y), v_n(z)] \leq \varepsilon$ provided

$|y - z| < \delta$ independently of $n = 1, 2 \ldots$

then $\{v_n\}_{n=1}^{\infty}$ is precompact in $C(Q, X)$, that is there exists a subsequence and a function $v \in C(Q, X)$ such that

$$\sup_{y \in Q} d_X[v_n(y), v(y)] \to 0 \text{ as } n \to \infty.$$

$Q \subset \mathbb{R}^N$ an open set, $g: Q \to \mathbb{R}$

$supp[g]$ ... support of g in Q

$supp[g] = closure[\{y \in Q \mid g(y) \neq 0\}]$

$C_c^k(Q; \mathbb{R}^M)$ $k \in \{0, 1, \dots \infty\}$ = vector space of functions belonging to $C^k(Q; \mathbb{R}^M)$ and having compact support in Q

$D(Q, \mathbb{R}^M) = C_c^\infty(Q; \mathbb{R}^M)$ with topology induced by the convergence:

$\varphi_n \to \varphi \in \mathcal{D}(Q)$

if

$supp[\varphi_n] \subset K$, $K \subset Q$ a compact set,

$\varphi_n \to \varphi$ in $C^k(K)$ for any $k = 0, 1 \dots$

the dual space $C_c(Q)$ is the space $\underline{M(Q)}$ of Radon measures on an open set Q

$\mathcal{D}'(Q; \mathbb{R}^N)$ = space of distribution on Q with value in $\mathbb{R}^N$

Integrable functions

$\int_\Omega v \, dx$ ... the Lebesgue integral of a measurable function $v = v(x)$ over a measurable set $\Omega \subset \mathbb{R}^3$

The Lebesgue space $L^p(Q; X)$ are spaces of (Bochner) measurable functions v ranging in a Banach space X

$$\|v\|^p_{L^p(Q;X)} = \int_Q \|v\|^p_X \, dy \text{ is finite } \quad 1 \leq p < \infty$$

$$v \in L^\infty(Q; X) \qquad \|v\|_{L^\infty(Q;X)} = \operatorname*{ess\,sup}_{y \in Q} \|v(y)\|_X < \infty$$

$L^p_{loc}(Q, X)$ = locally L^p-integrable functions

$v \in L^p_{loc}(Q; X)$ if $v \in L^p(K; X)$ for any compact set K in Q.

Let $f \in L^1_{loc}(Q)$, Q is an open set

A Lebesgue point $a \in Q$ of f in Q

$$\lim_{r \to 0^+} \frac{1}{|B(a,r)|} \int_{B(a,r)} f(x)\,dx = f(a)$$

For $f \in L^1(Q)$ the set of all Lebesgue points is of <u>full measure</u>, meaning its complement in Q is of <u>zero</u> Lebesg. measure.

<u>Theorem</u>

Let $Q \subset \mathbb{R}^N$ be a measurable set, X a Banach space that is reflexive and separable, $1 \le p < \infty$. then any continuous linear form $\xi \in [L^p(Q;X)]^*$ admits a unique representation $w_\xi \in L^{p'}(Q;X^*)$ by

$$\langle \xi; v \rangle_{L^{p'}(Q;X^*); L^p(Q,X)} = \int_Q \langle w_\xi(y); v(y) \rangle_{X^*;X}$$

$$\forall v \in L^p(Q;X)$$

where
$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Moreover the norm on the dual space is given as

$$\|\xi\|_{[L^p(\Omega;X)]^*} = \|w_\xi\|_{L^{p'}(\Omega;X^*)}.$$

Accordingly, the space $L^p(\Omega;X)$ are reflexive for $1<p<\infty$ as soon as X is reflexive and separable.
[Gajewski, Gröger, Zacharias]

Identifying ξ with w_ξ
$$[L^p(\Omega;\mathbb{R}^d)]^* = L^{p'}(\Omega;\mathbb{R}^d)$$

$$\|\xi\|_{[L^p(\Omega;\mathbb{R}^d)]^*} = \|\xi\|_{L^{p'}(\Omega;\mathbb{R}^d)}, \quad 1<p<\infty$$

Riesz representation formula

Hölder's inequality

$$\|uv\|_{L^r(\Omega)} \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$$

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 1 \le p, q, r \le \infty$$

$\forall u \in L^p(\Omega), v \in L^q(\Omega), \Omega \subset \mathbb{R}^n$

Interpolation inequality for L^p

$$\|u\|_{L^r(\Omega)} \le \|u\|^{\lambda}_{L^p(\Omega)} \|u\|^{1-\lambda}_{L^q(\Omega)} \quad \frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$$

$1 \le p < r < q \le \infty$

$\lambda \in (0,1)$

$u \in L^p \cap L^q$

Jensen inequality

$$\phi\left(\int_\Omega v \, dy\right) \le \int_\Omega \phi(v) \, dy$$

ϕ is convex on the range of v

Gronwall's lemma

Lemma:

Let $a \in L^1(0,T)$, $a \ge 0$, $\beta \in L^1(0,T)$, $b_0 \in \mathbb{R}$ and

$b(\tau) = b_0 + \int_0^\tau \beta(t) \, dt$ be given.

Let $r \in L^{\infty}(0,T)$ satisfy

$r(\tau) \le b(\tau) + \int_0^{\tau} a(t) r(t)$ for a.a. $\tau \in [0,T]$.

Then

$r(\tau) \le b_0 \exp\left(\int_0^{\tau} a(t)\,dt\right) + \int_0^{\tau} B(t) \exp\left(\int_t^{\tau} a(s)\,ds\right) dt$

for a.a. $\tau \in [0,T]$.

Sobolev spaces

$\Omega \subset \mathbb{R}^N$ is of class C = if for each point $x \in \partial\Omega$ there exists $r > 0$ and a mapping $\mu : \mathbb{R}^{N-1} \to \mathbb{R}$ belonging to a function class C such that upon rotating and relabelling the coordinate axes we have

$$\left.\begin{aligned} \Omega \cap B(x;r) &= \{y \mid \mu(y') < y_N\} \cap B(x,r) \\ \partial\Omega \cap B(x;r) &= \{y \mid \mu(y') = y_N\} \cap B(x,r) \end{aligned}\right\}$$

$$y' = (y_1, \dots, y_{N-1})$$

Ω is Lipschitz domain if μ is Lipschitz.

Lipschitz domain Ω admits the outer normal vector $n(x)$ for a.e. $x \in \partial\Omega$

* differential operator ∂^α of order $|\alpha|$ can be identified with a distribution

$$\langle \partial^\alpha v, \varphi \rangle_{\mathcal{D}'(Q); \mathcal{D}(Q)} = (-1)^{|\alpha|} \int_Q v \, \partial^\alpha \varphi \, dy$$

for any locally integrable function v.

<u>the Sobolev spaces</u> $W^{l,p}(Q; \mathbb{R}^M)$,

$1 \le p \le \infty$, l a positive integer $=$ space of functions having all distributional derivatives up to order l in $L^p(Q; \mathbb{R}^M)$

$$\|v\|_{W^{l,p}(Q; \mathbb{R}^M)} = \begin{cases} \left(\sum_{i=1}^{M} \sum_{|\alpha| \le l} \|\partial^\alpha v_i\|^p_{L^p(Q)} \right)^{1/p}, & 1 \le p < \infty \\ \max_{1 \le i \le M} \max_{|\alpha| \le l} \{ \|\partial^\alpha v_i\|_{L^\infty(Q)} \} & \text{if } p = \infty \end{cases}$$

If Q is a bounded domain of class $C^{l-1,1}$ then there exists a continuous linear operator $W^{l,p}(Q) \to W^{l,p}(\mathbb{R}^M)$ .. <u>extension operator</u>

$1 \le p < \infty$ $W^{k,p}(\Omega)$ is separable and $C^k(\bar{\Omega})$ is its dense subspace

$W^{1,\infty}(\Omega)$ is isometrically isomorphic $C^{0,1}(\bar{\Omega})$ of Lipschitz functions on $\bar{\Omega}$

$\overline{C_c^\infty(\Omega; \mathbb{R}^M)} = W_0^{k,p}(\Omega; \mathbb{R}^N)$

$\dot{L}^p(\Omega) = \{u \in L^p(\Omega);\ \int_\Omega u\,dx = 0\}$

Dual spaces to Sob. spaces

$$\langle f; v \rangle_{[W_0^{k,p}(\Omega)]^*; W_0^{k,p}(\Omega)} = \sum_{|\alpha| \le l} \int_\Omega (-1)^\alpha u_\alpha \partial^\alpha v$$

Embeddings

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.

(i) if $kp < N$, $p \ge 1$ $W^{k,p}(\Omega)$ is continuously emb. in $L^q(\Omega)$

$$1 \le q \le p^* = \frac{Np}{N - kp}$$

Moreover, the embedding is <u>compact</u> if $k>0$ and $q<p^*$.

(ii) if $kp = N$, the space $W^{k,p}(\Omega)$ is <u>compactly embedded</u> in L^q for any $q \in [1, \infty)$.

(iii) If $kp > N$ then $W^{k,p}(\Omega)$ is continuously embed. in $C^{k-[N/p]-1,\nu}(\bar{\Omega})$

$[\cdot]$ .. integer part

$$\nu = \begin{cases} [\frac{N}{p}]+1-\frac{N}{p} & \text{if } \frac{N}{p} \notin \mathbb{Z} \\ \text{arbitrary positive number in } (0,1) & \text{if } \frac{N}{p} \in \mathbb{Z}. \end{cases}$$

Moreover, the embedding is <u>compact</u> if $0<\nu<[\frac{N}{p}]+1-\frac{N}{p}$.

<u>Sobolev - Slobodeckii spaces</u>, $Q \subset \mathbb{R}^L$, $L \in \mathbb{N}$

$W^{k+\beta,p}(Q)$ $1 \le p < \infty$, $0<\beta<1$, $k=0,1,\ldots$

$$\|u\|_{W^{k+\beta,p}(Q)} = \left(\|u\|^p_{W^{k,p}(Q)} + \sum_{|\alpha|=k} \int_Q \int_Q \frac{|\partial^\alpha u(y) - \partial^\alpha u(z)|^p}{|y-z|^{L+\beta p}} \right)^{1/p}$$

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then there exists a linear operator γ_0 with the following properties:

$$[\gamma_0(v)](x) = v(x) \text{ for } x \in \partial\Omega \text{ provided } v \in C^\infty(\bar\Omega)$$

$$\|\gamma_0(v)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq c\|v\|_{W^{1,p}(\Omega)} \quad \forall v \in W^{1,p}(\Omega)$$

$$\ker[\gamma_0] = W_0^{1,p}(\Omega)$$

$1 < p < \infty$.

Conversely, there exists a continuous linear operator

$$l : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$$

such that

$$\gamma_0(l(v)) = v \quad \forall v \in W^{1-\frac{1}{p},p}(\partial\Omega)$$

$1 < p < \infty$.

The following formula holds

$$\int_\Omega \partial_{x_i} u v \, dx + \int_\Omega u \partial_{x_i} v \, dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v) n_i \, dS_x \quad i = 1, \dots, N$$

for any $u \in W^{1,p}(\Omega)$, $v \in W^{1,p'}(\Omega)$.

Poincaré inequality

Theorem

Let $1 \le p < \infty$, $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then

(i) For any $A \subset \partial\Omega$ with the non-zero surface measure there exists a positive constant $C = C(p, N, A, \Omega)$ such that

$$\|v\|_{L^p(\Omega)} \le C\Big(\|\nabla v\|_{L^p(\Omega;\mathbb{R}^N)} + \int_A |v| \, dS_x\Big)$$

$$\forall v \in W^{1,p}(\Omega)$$

(ii) There exists a positive constant $C = C(p, \Omega)$ such that

$$\Big\| v - \frac{1}{|\Omega|}\int_\Omega v \, dx \Big\|_{L^p(\Omega)} \le C \|\nabla v\|_{L^p(\Omega;\mathbb{R}^N)}$$

for any $v \in W^{1,p}(\Omega)$.