The transport theorem

Motion of fluid occupying a domain Ω_t at instant t, $t \in (T_1, T_2)$

$F = F(x,t) : M \longrightarrow \mathbb{R}^1$: Eulerian representation of physical quantity transported by fluid particles

$\sigma(t) \subset \Omega_t \Rightarrow$ system of fluids filling $\sigma(t)$ (trans. d.)

The total amount of the quantity by function F that is contained in the volume $\sigma(t)$ at time t

$$\mathcal{F}(t) = \int_{\sigma(t)} F(x,t)\,dx$$

Need: change of quantity F

$$\frac{d\tilde{f}(t)}{dt} = \frac{d}{dt} \int_{\sigma(t)} F(x,t)\, dx$$

$v \in C^1(M)$

$\varphi = \varphi(X, t_0; t)$ C^1 with resp. to space

C to time

$t_0 \in (T_1, T_2)$ fixed ~~space~~ time

$\sigma(t_0) \subset \Omega_0$

$\sigma(t) = \{ \varphi(X; t_0; t) \,,\, X \in \sigma(t_0) \}$

$$\left(\begin{array}{l} \dfrac{\partial \varphi}{\partial t} = v(\varphi(X, t_0; t), t) \\ \varphi(t_0) = X \end{array} \right.$$

<u>Jacobian of the mapping</u>

$X \in \sigma(t_0) \longrightarrow \varphi(X; t_0; t) \in \sigma(t)$

$$J(X,t) = \det \frac{D\varphi(X, t_0; t)}{DX}$$

Lemma

Let $t_0 \in (T_1, T_2)$, $\sigma(t_0)$ be a bounded domain and let $\overline{\sigma(t_0)} \subset \Omega_{t_0}$. Then there exists an interval $(t_1, t_2) \ni t_0$ such that the following conditions are satisfied:

a) the mapping "$t \in (t_1, t_2), X \in \sigma(t_0) \to$ $x = \varphi(X, t_0; t) \in \sigma(t)$" has continuous first order derivatives with respect to t, X_1, X_2, X_3 and continuous second order derivatives $\partial^2 \varphi / \partial t \partial X_i$, $i = 1, 2, 3$

b) the mapping "$X \in \sigma(t_0) \to x = \varphi(X, t_0; t)$ $\in \sigma(t)$" is a continuously differentiable one-to-one mapping of $\sigma(t_0)$ onto $\sigma(t)$ with Jacobian $J(X, t)$ which is continuous, bounded and satisfies the condition

$J(x,t) > 0 \quad \forall x \in \bar{\sigma}(t_0) \quad \forall t \in (t_1, t_2).$

c) the inclusion

$$\{(x,t);\ t \in [t_1, t_2],\ x \in \overline{\bar{\sigma}(t)}\} \subset M$$

holds and thus the mapping φ has continuous and bounded first order derivatives on
$\{(x,t):\ t \in (t_1, t_2),\ x \in \bar{\sigma}(t)\}$

d) $v(\varphi(x, t_0, t), t) = \frac{\partial \varphi}{\partial t}(x, t_0; t)$

$\forall x \in \bar{\sigma}(t_0), \ \forall t \in (t_1, t_2).$

<u>Theorem</u>: Let the previous conditions a)–d) be satisfied and let the function $F = F(x,t)$ have continuous and bounded first order derivatives on the set
$\{(x,t):\ t \in (t_1, t_2),\ x \in \bar{\sigma}(t)\}$.
Then for each $t \in (t_1, t_2)$ there exists a finite derivative

$$\frac{d\mathcal{F}(t)}{dt}(t) = \frac{d}{dt}\int_{\sigma(t)} F(x,t)\,dx =$$

$$= \int_{\sigma(t)} \Big[\frac{\partial F}{\partial t}(x,t) + v(x,t)\cdot \operatorname{grad} F(x,t)$$

$$+ F(x,t)\operatorname{div} v(x,t) \Big] dx.$$

<u>Flux formulation</u>

$$\frac{d\mathcal{F}(t)}{dt} = \int_{\sigma(t)} \Big[\frac{dF}{dt}(x,t) + \operatorname{di}(Fv)(x,t) \Big] dx$$

The continuity equation

The density of fluid: is such a function

$$\rho : \mathcal{M} = \{(x,t);\ t \in (T_1, T_2),\ x \in \Omega_t\} \to (0, +\infty)$$

which allows to determine the mass $m(\sigma, t)$ of the fluid contained in any subdomain $\sigma \subset \Omega_t$

$$m(\sigma, t) = \int_\sigma \rho(x,t)\,dx$$

Assumptions

(i) $\rho \in C^1(\mathcal{M})$.

(ii) $v \in [C^1(\mathcal{M})]^3$

Let us consider an arbitrary time instant $t_0 \in (T_1, T_2)$, moving piece of fluid formed by the same particles at each time and filling at t_0 bounded domain

$\sigma \subset \bar{\sigma} \subset \Omega_{t_0}$ with $\partial\sigma$ Lipschitz continuous boundary =

~~#~~ call = <u>control volume in the domain Ω_{t_0}</u>

$\sigma(t)$.. the domain occupied by this piece of fluid at (identified) time $t \in (t_1, t_2)$ containing t_0. $\Rightarrow$ $\sigma(t_0) = \sigma$

<u>The law of conservation of mass</u>

<u>The mass of the piece of fluid represented by the domain $\sigma(t)$ does not depend on time t.</u>

$$\frac{dm(\sigma(t))}{dt} = 0, \quad t \in (t_1, t_2)$$

$$\Rightarrow \quad m(\sigma(t)) = \int_{\sigma(t)} \rho(x,t)\,dx$$

Using the transport theorem $\Rightarrow$

$$\int_{\sigma(t)} \Big[\frac{\partial \rho}{\partial t}(x,t) + v(x,t)\cdot \operatorname{grad} \rho(x,t) +$$

$$\rho(x,t) \operatorname{div} v(x,t) \Big] dx = 0, \quad t \in (t_1, t_2).$$

Now if $t = t_0$, $\sigma(t_0) = \sigma \Rightarrow$

$$\int_{\sigma} \Big[\frac{\partial \rho}{\partial t}(x,t_0) + \operatorname{div}(\rho v)(x,t_0) \Big] dx = 0$$

for ~~arbitra~~ arbitrary $t_0 \in (T_1, T_2)$

and arbitrary control volume

σ in Ω_{t_0}

$\Downarrow$

$$\frac{\partial \rho}{\partial t}(x,t) + \operatorname{div}(\rho(x,t) v(x,t)) = 0$$

$$t \in (T_1, T_2), \quad x \in \Omega_t$$

The equations of Motion

= The rate of change of the total momentum of a piece of fluid formed by the same particles at each time and occupying the domain $\sigma(t)$ at instant t is equal to the force acting on $\sigma(t)$

The total momentum

$$\underline{H}(\sigma(t)) = \int_{\sigma(t)} \rho(x,t)\, v(x,t)\, dx$$

$\underline{F}(\sigma(t))$ = the force acting on the volume $\sigma(t)$

$$\frac{d\,\underline{H}(\sigma(t))}{dt} = \underline{F}(\sigma(t)) \qquad t \in (t_1, t_2)$$

Using transport theorem

$$\Rightarrow \int_{\sigma(t)} \left[\frac{\partial}{\partial t} (\rho(x,t)\, v_i(x,t)) + \operatorname{div}(\rho(x,t)\, v_i(x,t)\, v(x,t)) \right]$$

$$= F_i(\sigma(t)) \qquad i = 1,2,3$$
$$t \in (t_1, t_2)$$

Taking into account $t_0 \in (T_1, T_2)$ is an arbitrary time instant and $\sigma(t_0) = \sigma \subset \bar{\sigma} \subset \Omega_{t_0}$

$\Rightarrow$

$$\int_{\sigma} \left[\frac{\partial}{\partial t} (\rho(x,t)\, v_i(x,t)) + \operatorname{div}(\rho(x,t)\, v_i(x,t)\, v(x,t)) \right] dx$$

$$= F_i(\sigma, t)$$

$i = 1,2,3$, arbitrary $t \in (T_1, T_2)$
arbitrary control volume σ in Ω_t.

Forces Acting on Fluids

1) **volume forces** = forces acting on volume elements of fluids

density of volume force $f : M \to \mathbb{R}^3$
$f \in [C(M)]^3$

$$\underline{F}_v(\sigma, t) = \int_\sigma \rho(x,t) f(x,t)\, dx$$

gravity force $f = (0, 0, -g)$
potential $f = \nabla U$

2) **Surface forces** =

the action of fluid contained ~~surface forces~~ outside the considered volume σ onto the fluid occupying the domain σ through boundary $\partial\sigma$

$$F_{p,S} = -\int_S p(x,t)\, n(x)\, dS$$

p … pressure $\qquad p \in C^1(M)$

$\Downarrow$

$$\int_\sigma \left[\frac{\partial}{\partial t} (\rho(x,t) v_i(x,t)) + \operatorname{div} (\rho(x,t) v_i(x,t) v(x,t)) \right]$$

$$= \int_\sigma \rho(x,t) f_i(x,t)\, dx - \int_\sigma \frac{\partial p(x,t)}{\partial x_i}\, dx, \; i = 1,2,3$$

<u>Stress tensor</u>

$$F_S = \int_S T(x,t,n(x))\, dS$$

Newton law $\Rightarrow$ $T(x,t,n) = -T(x,t,-n)$

$$F_S(\sigma,t) = \int_{\partial\sigma} T(x,t,n(x))\, dS$$

TD

$$F_i(\sigma, t) = \int_\sigma \rho(x,t) f_i(x,t)\,dx + \int_{\partial\sigma} \underline{T_i(x,t,n(x))}\,ds$$

$T = -pI \rightarrow$ Euler eq.

$T = -pI + T' \xrightarrow{\text{Stokes's postulate}} \underline{N-S}$ $\quad T' = f(D)$

$T_{ji} = T_i(x,t,e_j) \xRightarrow{\text{pos to show}} T_i(x,t,n) = \sum_{j=1}^{3} n_j T_{ji}(x,t)$

$$T = (-p + \lambda\, \mathrm{div}\, v) I + 2\mu D$$

$$\boxed{\int_{\partial\sigma} T_i(x,t,n(x)) = \int_\sigma \frac{\partial T_{ji}}{\partial x_j}}$$ Green's formula

the Law of Conservation of Energy

$$\frac{d}{dt} \mathcal{E}(\sigma(t)) = \int_{\sigma(t)} \rho(x,t) f(x,t) \cdot v(x,t)\,dt + \int_{\partial\sigma(t)} T(x,t,n(x)) \cdot v(x,t)\,ds + Q(\sigma(t))$$

$$\mathcal{E}(\sigma(t)) = \int_{\sigma(t)} \rho(x,t) E(x,t)$$ → energy

$$E = u + \frac{|v|^2}{2}$$

u → internal energy

$\frac{|v|^2}{2}$ → kinetic energy

$$\varphi(\sigma(t)) = \int_{\sigma(t)} \rho(x,t)\, q(x,t)\, dx - \int_{\partial\sigma(t)} q(x,t) \cdot n(x)\, dS$$

The rate of change of the total energy of the fluid particles, occupying the domain $\sigma(t)$ at time t is equal to the sum of powers of the volume forces acting on the volume $\sigma(t)$ and the surface force acting on the surface $\partial\sigma(t)$ and of the amount of heat transmitted to $\sigma(t)$

$$q = -k \operatorname{grad} \theta$$

$$\int_{\partial\sigma(t)} q(x,t) \cdot n(x)\, dS = - \int_{\partial\sigma(t)} k(x,t) \frac{\partial\theta(x,t)}{\partial n}\, dS$$

$$\frac{d}{dt}\int \rho(x,t)\, E(x,t)\, dt =$$

$$\int_{\sigma(t)} \left[\frac{\partial}{\partial t}(\rho E)(x,t) + \operatorname{div}(\rho E v)(x,t)\right] dx$$

$$\int_{\sigma(t)} \left[\frac{\partial}{\partial t}(\rho E)(x,t) + \operatorname{div}(\rho E v)(x,t)\right] dx$$

$$= \int_{\sigma(t)} \rho(x,t)\, f(x,t)\, dx \; +$$

$$\int_{\partial\sigma(t)} \sum_{i,j=1}^{3} T_{ji}(x,t)\, n_j(x)\, v_i(x,t)\, dS \; +$$

$$+ \int_{\sigma(t)} \rho(x,t)\, q(x,t)\, dx - \int_{\partial\sigma(t)} q(x,t) \cdot n(x)\, dS$$

$\Downarrow$ Green's theorem, $t \in (T_1, T_2)$, $\sigma(t) = \sigma$

$$\boxed{\frac{\partial}{\partial t}(\rho E) + \operatorname{div}(\rho E v) = \rho f \cdot v + \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}(T_{ji} v_i) + \rho q - \operatorname{div} q}$$