On weak solution approach to problems in fluid dynamics

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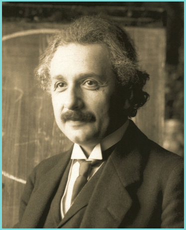
Why “weak” solutions?



Everything Should Be Made as Simple as Possible, But Not Simpler...
Albert Einstein [1874-1965]

Weak solutions

- The *largest possible class* of objects that can be identified as “solution” of the a given problem, here a system of partial differential equations. **The weaker the better**
- Weak solutions are easy to be identified as asymptotic limits of approximate solutions, notably solutions of **numerical schemes**.
- A weak solution coincides with the (unique) strong solution as long as the latter exists. **Weak–strong uniqueness principle**

Nonlinear balance laws

Field equations

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F}(\mathbf{U}) = \mathbf{G}(\mathbf{U}),\ \mathbf{U} = [U_1, \ldots, U_M],\ \mathbf{U} = \mathbf{U}(t, x)$$

$$t \in (0, T),\ x \in R^N$$

physical space: t the time variable, x the spatial variables

Initial state

$$\mathbf{U}(0, \cdot) = \mathbf{U}_0$$

$\mathbf{U}$, $\mathbf{U}_0$ in state space $\mathcal{F} \subset R^M$

Admissibility conditions

$$\partial_t \mathcal{S}(\mathbf{U}) + \mathrm{div}_x F_{\mathcal{S}}(\mathbf{U}) = (\boxed{\geq})\mathbf{G}(\mathbf{U})\nabla_{\mathbf{U}}\mathcal{S}(\mathbf{U})$$

$\mathcal{S}$ - entropy

Approximate problems - numerics

Approximate problem - discretization

$$D_t \mathbf{U}_n + \mathrm{div}_x \mathbb{F}_n(\mathbf{U}_n) = \mathbf{G}_n(\mathbf{U}_n) + \mathrm{error}_n$$

Goal

$$\mathbf{U}_n \to \mathbf{U} \text{ in some sense, } \mathbf{U} \text{ exact solution}$$

Tools

- Make the target space for the limit as large as possible - easy convergence proof
- Show weak–strong uniqueness

What can be weak...

Weak derivatives - functions replaced by their spatial averages

$$v = v(t,x) \ \approx \int v(t,x)\phi(t,x)\ \mathrm{d}x\mathrm{d}t$$

$$Dv \ \approx - \int v(t,x) D\phi(t,x)\ \mathrm{d}x\mathrm{d}t$$

ϕ smooth test function

$$v(t,x) \approx \lim_{|B(t,x)| \to 0} \frac{1}{|B(t,x)|} \int_{B(t,x)} v \ \mathrm{d}x\mathrm{d}t$$

weak convergence = convergence in averages

Measure–valued (MV) solution

$$v = v(t,x) \approx \left\langle \mathcal{V}_{(t,x)}; v \right\rangle, \ \{\mathcal{V}\}_{t,x} \text{ family of probability measures}$$

$$F(v) \approx \left\langle \mathcal{V}_{(t,x)}; F(v) \right\rangle \text{ for any Borel function } F$$

(MV) convergence = convergence in the weak topology of measures

Weak formulation of a general system

Probability (Young) measure

$$\mathcal{V}_{t,x} \in \mathcal{P}(\mathcal{F})$$

Field equations

$$\left[\int_{R^N} \langle \mathcal{V}_{t,x}; \mathbf{U} \rangle \varphi \right]_{t=0}^{t=\tau}$$

$$= \int_0^T \int_{R^N} \langle \mathcal{V}_{t,x}; \mathbf{U} \rangle \partial_t \varphi + \langle \mathcal{V}_{t,x}; \mathbb{F}(\mathbf{U}) \rangle \cdot \nabla_x \varphi - \langle \mathcal{V}_{t,x}; \mathbf{G}(\mathbf{U}) \rangle \varphi \mathrm{d}x \mathrm{d}t$$

φ smooth compactly supported in $[0, T) \times R^N$

Admissibility conditions

$$\left[\int_{R^N} \langle \mathcal{V}_{t,x}; \mathcal{S}(\mathbf{U}) \rangle \varphi \right]_{t=0}^{t=\tau} =$$

$$\int_0^T \int_{R^N} \langle \mathcal{V}_{t,x}; \mathcal{S}(\mathbf{U}) \rangle \partial_t \varphi + \langle \mathcal{V}_{t,x}; \mathbf{F}_{\mathcal{S}}(\mathbf{U}) \rangle \cdot \nabla_x \varphi - \langle \mathcal{V}_{t,x}; \mathbf{G}(\mathbf{U}) \nabla_{\mathbf{U}} \mathcal{S}(\mathbf{U}) \rangle \varphi \mathrm{d}x \, \mathrm{d}t$$

Dynamics of compressible fluids

Phase variables

$$\text{mass density } \varrho = \varrho(t,x),\ \ t \in (0,T),\ \ x \in \Omega \subset R^3$$

$$\text{(absolute) temperature } \vartheta = \vartheta(t,x),\ \ t \in (0,T),\ \ x \in \Omega \subset R^3$$

$$\text{(bulk) velocity field } \ \mathbf{u} = \mathbf{u}(t,x),\ \ t \in (0,T),\ \ x \in \Omega \subset R^3$$

Standard formulation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho,\vartheta) = 0$$

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) + p(\varrho,\vartheta) \right) \mathbf{u} \right] = 0$$

Impermeability condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Complete Euler system in conservative variables

Conservative variables

$$\text{mass density } \varrho = \varrho(t,x),\ t \in (0,T),\ x \in \Omega \subset R^3$$

$$\text{(total energy) } E = E(t,x),\ t \in (0,T),\ x \in \Omega \subset R^3$$

$$\text{momentum } \mathbf{m} = \mathbf{m}(t,x),\ t \in (0,T),\ x \in \Omega \subset R^3$$

$$p = (\gamma - 1)\varrho e,\ p = (\gamma - 1)\left(E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} \right)$$

Field equations

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0$$

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

Entropy

Gibbs' relation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy balance

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) \boxed{\geq} 0$$

Entropy in the polytropic case

$$s = S\left(\frac{p}{\varrho^\gamma}\right) = S\left((\gamma - 1)\frac{E - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}}{\varrho^\gamma}\right)$$

Several concepts of solutions

Classical solutions

The phase variables are smooth (differentiable), the equations are satisfied in the standard sense. Classical solutions are often uniquely determined by the data. The main issue here is global in time existence that may fail for generic initial data

Weak (distributional) solutions

Limits of classical solutions, limits of regularized problems. Equations are satisfies in the distributional sense. Weak solutions may not be uniquely determined by the data.

Viscosity solutions

Limits of the Navier-Stokes-Fourier system for vanishing transport coefficients.

Limits of approximate (numerical) schemes

Zero step limits of numerical schemes. Examples are Lax–Friedrichs and related schemes mimicking certain approximations - e.g. a model proposed by H.Brenner.

Admissible (entropy) weak solutions

Field equations

$$\left[\int_\Omega \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x\mathrm{d}t$$

for all $\varphi \in C^1([0,T] \times \overline{\Omega})$

$$\left[\int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

for all $\boldsymbol{\varphi} \in C^1([0,T] \times \overline{\Omega}; R^3)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$

$$\left[\int_\Omega E \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[E \partial_t \varphi + \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] \cdot \nabla_x \varphi \right] \ \mathrm{d}x\mathrm{d}t$$

for all $\varphi \in C^1([0,T] \times \overline{\Omega})$

Entropy inequality

$$\left[\int_\Omega \varrho s \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} \boxed{\geq} \int_0^\tau \int_\Omega \left[\varrho s \partial_t \varphi + s \mathbf{m} \cdot \nabla_x \varphi \right] \ \mathrm{d}x\mathrm{d}t$$

for all $\varphi \in C^1([0,T] \times \overline{\Omega})$, $\varphi \geq 0$

Infinitely many weak solutions

Initial data

$$\varrho(0,\cdot) = \varrho_0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \ \vartheta(0,\cdot) = \vartheta_0.$$

Existence via convex integration

Let $N = 2, 3$. Let ϱ_0, ϑ_0 be piecewise constant (arbitrary) positive. Then the exists $\mathbf{u}_0 \in L^\infty$ such that the Euler system admits infinitely many admissible weak solution in $(0, T) \times \Omega$.

Dissipative measure–valued (DMV) solutions

Parameterized measure

$$\underbrace{\mathcal{F}}_{\text{phase space}} = \left\{ \varrho \geq 0,\ \mathbf{m} \in R^3,\ E \in [0, \infty) \right\}, \quad \underbrace{Q_T}_{\text{physical space}} = (0, T) \times \Omega$$

$$\{\mathcal{V}_{t,x}\}_{(t,x) \in Q_T}, \quad Y_{t,x} \in \mathcal{P}(\mathcal{F})$$

Field equations

$$\partial_t \left\langle \mathcal{V}_{t,x}; \varrho \right\rangle + \mathrm{div}_x \left\langle \mathcal{V}_{t,x}; \mathbf{m} \right\rangle = 0$$

$$\partial_t \left\langle \mathcal{V}_{t,x}; \mathbf{m} \right\rangle + \mathrm{div}_x \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle + \nabla_x \left\langle \mathcal{V}_{t,x}; p \right\rangle = D_x \mu_C$$

$$\partial_t \int_\Omega \left\langle \mathcal{V}_{t,x}; E \right\rangle \ \mathrm{d}x + \mathcal{D} = 0,\ \partial_t \left\langle \mathcal{V}_{t,x}; \varrho s \right\rangle + \mathrm{div}_x \left\langle \mathcal{V}_{t,x}; s\mathbf{m} \right\rangle \geq 0$$

Compatibility

$$\int_0^T \int_\Omega |\mu_C| \ \mathrm{d}x\mathrm{d}t \leq C \int_0^T \mathcal{D}\mathrm{d}t$$

Why to go measure–valued?

Motto: The larger (class) the better

- Universal limits of *numerical* schemes
- Limits of more complex physical systems - vanishing viscosity/heat conductivity limit
- Singular limits (low Mach etc.)

Weak–strong uniqueness

A (DMV) solution coincides with a smooth solution with the same initial data as long as the latter solution exists

Thermodynamic stability

Thermodynamic stability in the standard variables

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Thermodynamic stability in the conservative variables

$$(\varrho, \mathbf{m}, E) \mapsto \varrho s(\varrho, \mathbf{m}, E)$$

is a (strictly) concave function

Thermodynamic stability in the polytropic case

$$\varrho s = \varrho S \left(\frac{p}{\varrho^\gamma} \right), \quad p = (\gamma - 1)\varrho e$$

$$S'(Z) > 0, \quad (1 - \gamma) S'(Z) - \gamma S''(Z) Z > 0$$

Relative energy

Relative energy in the standard variables

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right)$$
$$= \frac{1}{2}\varrho|\mathbf{u} - \tilde{\mathbf{u}}|^2 + \partial_\varrho H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})(\varrho - \tilde{\varrho}) - H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})$$

$$H_{\tilde{\vartheta}}(\varrho, \vartheta) = \varrho\Big(e(\varrho, \vartheta) - \tilde{\vartheta}s(\varrho, \vartheta)\Big)$$

Relative energy in the conservative variables

$$\mathcal{E}\left(\varrho, \mathbf{m}, E \middle| \tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{E}\right)$$
$$= -\tilde{\vartheta}\Big[\varrho s - \partial_\varrho(\varrho s)(\varrho - \tilde{\varrho}) - \nabla_{\mathbf{m}}(\varrho s) \cdot (\mathbf{m} - \tilde{\mathbf{m}}) - \partial_E(\varrho s)(E - \tilde{E})$$
$$- \tilde{\varrho}\tilde{s}\Big]$$

Relative energy inequality

Relative energy revisited

$$\mathcal{E}\left(\varrho, \mathbf{m}, E \,\middle|\, \tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}}\right) \equiv E - \tilde\vartheta \mathcal{S}(\varrho, \mathbf{m}, E) - \mathbf{m}\cdot\tilde{\mathbf{u}} + \frac{1}{2}\varrho|\tilde{\mathbf{u}}|^2 + p(\tilde\varrho, \tilde\vartheta)$$
$$- \left(e(\tilde\varrho, \tilde\vartheta) - \tilde\vartheta s(\tilde\varrho, \tilde\vartheta) + \frac{p(\tilde\varrho, \tilde\vartheta)}{\tilde\varrho} \right)\varrho$$

Relative energy inequality

$$\left[\int_\Omega \left\langle \mathcal{V}_{t,x}; \mathcal{E}\left(\varrho, \mathbf{m}, E \,\middle|\, \tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}}\right)\right\rangle \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \mathcal{D}(\tau) \le \int_0^\tau \mathcal{R}(t)\ \mathrm{d}t$$

Stability of strong solutions

Measure–valued strong uniqueness

Suppose the thermodynamic functions p, e, and s comply with the hypothesis of thermodynamic stability. Let $(\varrho, \mathbf{m}, E)$ be a smooth (C^1) solution of the Euler system and let $(Y_{t,x}; \mathcal{D})$ be a dissipative measure–valued solution of the same system with the same initial data, meaning

$$Y_{0,x} = \delta_{\varrho_0(x), \mathbf{m}_0(x), E_0(x)} \text{ for a.a. } x \in \Omega.$$

Then

$$\mathcal{D} \equiv 0, \ \ Y_{t,x} = \delta_{\varrho(t,x), \mathbf{m}(t,x), E(t,x)}$$

for a.a. $(t,x) \in (0,T) \times \Omega$.

Maximal dissipation principle

Entropy production rate

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho \mathbf{m}) = \boxed{\sigma} \geq 0$$

Dissipative ordering

$$\mathcal{V}^1_{t,x} \succeq \mathcal{V}^2_{t,x} \text{ iff } \sigma_1 \geq \sigma_2 \text{ in } [0,T) \times \Omega$$

$$\int_0^T \int_\Omega \left[\left\langle \mathcal{V}^1_{t,x}; \mathcal{S}(\varrho, \mathbf{m}, E) \right\rangle \partial_t \varphi + \left\langle \mathcal{V}^1_{t,x}; \mathcal{S}(\varrho, \mathbf{m}, E) \frac{\mathbf{m}}{\varrho} \right\rangle \cdot \nabla_x \varphi \right] \, \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^T \int_\Omega \left[\left\langle \mathcal{V}^2_{t,x}; \mathcal{S}(\varrho, \mathbf{m}, E) \right\rangle \partial_t \varphi + \left\langle \mathcal{V}^2_{t,x}; \mathcal{S}(\varrho, \mathbf{m}, E) \frac{\mathbf{m}}{\varrho} \right\rangle \cdot \nabla_x \varphi \right] \, \mathrm{d}x\mathrm{d}t$$

Maximal dissipation principle

A (DMV) solution is admissible if it is *maximal* with respect to the ordering $\succeq$. A maximal (DMV) solution exists.

Generating MV solutions - zero viscosity limit

Navier–Stokes–Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \varepsilon \mathrm{div}_x \mathbb{S}$$

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \varepsilon \nabla_x \mathbf{q} = \varepsilon \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}$$

Physical dissipation

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right),$$

$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta$$

Generating MV solutions - artificial viscosity

Lax–Friedrichs numerical scheme

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = \varepsilon \mathrm{div}_x(\lambda \nabla_x \varrho)$$
$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x\left(\varrho \mathbf{u} \otimes \mathbf{u}\right) + \nabla_x p = \varepsilon \mathrm{div}_x(\lambda \nabla_x(\varrho \mathbf{u}))$$
$$\partial_t E + \mathrm{div}_x((E + p)\mathbf{u}) = \varepsilon \mathrm{div}_x(\lambda \nabla_x E)$$

Entropy preserving

$$\partial(\varrho S) + \mathrm{div}_x(\varrho S \mathbf{u}) \boxed{\geq} \varepsilon \mathrm{div}_x(\lambda \varrho \nabla_x S)) + \text{“}\textit{defect}\text{''}$$

Brenner's model

Two velocities principle

$$\mathbf{u} - \mathbf{u}_m = \varepsilon K \nabla_x \log(\varrho)$$

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}_m) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x\left(\varrho \mathbf{u} \otimes \mathbf{u}_m\right) + \nabla_x p(\varrho, \vartheta) = \varepsilon \mathrm{div}_x \mathbb{S}$$

$$\partial_t \left(\varrho(\frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta)) \right) + \mathrm{div}_x \left(\varrho(\frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta))\mathbf{u}_m \right) + \mathrm{div}_x(p(\varrho, \vartheta)\mathbf{u})$$
$$+ \varepsilon \mathrm{div}_x \mathbf{q} = \varepsilon \mathrm{div}_x(\mathbb{S}\mathbf{u})$$

Constitutive relations

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right), \ \mathbf{q} = -\kappa \nabla_x \vartheta,$$

$$K = \frac{\kappa}{c_v \varrho}$$