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Brinkman-Oseen transmission problem

$\Omega_B = R^n \setminus \overline{\Omega}_O,\ n = 2, 3$
$\overline{\omega} \subset \Omega_B,\ S \subset \partial\omega$

$$\mathbf{\Delta u}_O - \lambda\partial_1 \mathbf{u}_O - \nabla\pi_O = 0,\ \nabla \cdot \mathbf{u}_O = 0 \quad \text{in} \quad \Omega_O,$$

$$\mathbf{\Delta u}_B - \alpha \mathbf{u}_B - \nabla\pi_B = \mathbf{0},\ \nabla \cdot \mathbf{u}_B = 0 \quad \text{in} \quad \Omega_B \setminus S,$$

$$\mathbf{u}_B - \mathbf{u}_O = \mathbf{g} \quad \text{on} \quad \partial\Omega_O,$$

$$\partial^0_\nu(\mathbf{u}_B, \pi_B) - c_O \partial^\lambda_\nu(\mathbf{u}_O, \pi_O) + h\mathbf{u}_B = \mathbf{f} \quad \text{on} \quad \partial\Omega_O,$$

$$[\mathbf{u}_B]_+ - [\mathbf{u}_B]_- = \tilde{\mathbf{g}} \text{ on } S,$$

$$[\partial^0_\nu(\mathbf{u}_B, \pi_B)]_+ - [\partial^0_\nu(\mathbf{u}_B, \pi_B)]_- + \tilde{h}[\mathbf{u}_B]_+ = \tilde{\mathbf{f}} \text{ on } S.$$

where

$$\partial^\beta_\nu(\mathbf{u}, \pi) := [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\nu - \pi\nu - \frac{\beta}{2}\nu_1 \mathbf{u}$$

$\mathbf{\Phi} \in W^{1,q}(\partial\Omega_O; R^m)$, $\mathbf{\Psi} \in L^q(\partial\Omega_O; R^m)$,
$\mathbf{\Theta} \in L^q(S; R^m)$, $\mathbf{\Theta} = 0 = \tilde{\mathbf{g}}$ on $\partial\omega \setminus S$

$$\mathbf{u}_O = W^O_{\Omega_O}\mathbf{\Phi} + V^O_{\Omega_O}\mathbf{\Psi}$$

$$\pi_O = P^O_{\Omega_O}\mathbf{\Phi} + Q^O_{\Omega_O}\mathbf{\Psi}$$

$$\mathbf{u}_B = W^B_{\Omega_B}\mathbf{\Phi} + V^B_{\Omega_B}\mathbf{\Psi} + W^B_\omega\tilde{\mathbf{g}} + V^B_\omega\mathbf{\Theta}$$

$$\pi_B = P^B_{\Omega_B}\mathbf{\Phi} + Q^B_{\Omega_B}\mathbf{\Psi} + P^B_\omega\tilde{\mathbf{g}} + Q^B_\omega\mathbf{\Theta}$$

Integral equation

$$\tau(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) = (\mathbf{g}, \mathbf{f}, \tilde{\mathbf{f}})$$

$q \leq 2$ or $\partial\Omega_0 \in C^1$

1) τ is a Fredholm operator on $W^{1,q}(\partial\Omega_O; R^m)\times L^q(\partial\Omega_O; R^m) \times L^q(S; R^m)$ - **easy**.
M. Mitrea, M. Wright: Boundary value problems for the Stokes system in arbitrary Lipschitz domains. Astérisque 344, Paris 2012
V. Maz'ya, M. Mitrea, T. Shaposhnikova: The inhomogenous Dirichlet problem for the Stokes system in Lipschitz domains with unit normal-close to VMO^*. Funct. Anal. Appl. 43 (2009), No. 3, 217–235

2) uniqueness of the transmission problem - **difficult**

3) solvability of $\tau(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) = (\mathbf{g}, \mathbf{f}, \tilde{\mathbf{f}})$ - **very difficult**

Theorem. Let $q \le 2$ or $\partial\Omega_0 \in C^1$. Suppose that Ω_O is unbounded.

- If $(\mathbf{u}_B, \pi_B, \mathbf{u}_O, \pi_O)$ is an L^q-solution of the Brinkman-Oseen transmission problem then $\mathbf{u}_O(x) \to \mathbf{u}_\infty$, $\pi_O(x) \to \pi_\infty$ as $|x| \to \infty$.

- If $\mathbf{u}_\infty \in R^m$, $\pi_\infty \in R^1$, $\mathbf{g} \in W^{1,q}(\partial\Omega_O; R^m)$, $\mathbf{f} \in L^q(\partial\Omega_O; R^m)$, $\tilde{\mathbf{g}} \in W^{1,q}(\partial\omega; R^m)$, $\tilde{\mathbf{f}} \in L^q(\partial\omega, R^m)$, and $|\tilde{\mathbf{g}}| + |\tilde{\mathbf{f}}| = 0$ outside S, then there exists a unique L^q-solution of the Brinkman-Oseen transmission problem such that $\mathbf{u}_O(x) \to \mathbf{u}_\infty$, $\pi_O(x) \to \pi_\infty$ as $|x| \to \infty$.

Theorem. Let $q \le 2$ or $\partial\Omega_0 \in C^1$. Suppose that Ω_B is unbounded. If $(\mathbf{u}_B, \pi_B, \mathbf{u}_O, \pi_O)$ is an L^q-solution of the Brinkman-Oseen transmission problem then $\mathbf{u}_B(x) \to \mathbf{u}_\infty$, $\pi_B(x) \to \pi_\infty$ as $|x| \to \infty$. Suppose that $\mathbf{u}_\infty \in R^m$, $\pi_\infty \in R^1$, $\mathbf{g} \in W^{1,q}(\partial\Omega_O; R^m)$, $\mathbf{f} \in L^q(\partial\Omega_O; R^m)$, $\tilde{\mathbf{g}} \in W^{1,q}(\partial\omega; R^m)$, $\tilde{\mathbf{f}} \in L^q(\partial\omega, R^m)$, and $|\tilde{\mathbf{g}}| + |\tilde{\mathbf{f}}| = 0$ outside S.

- If $m = 3$ then there exists a unique L^q-solution of the Brinkman-Oseen transmission problem such that $\mathbf{u}_O(x) \to \mathbf{u}_\infty$, $\pi_O(x) \to \pi_\infty$ as $|x| \to \infty$.

- If $m = 2$ then there exists an L^q-solution of the Brinkman-Oseen transmission problem such that $\mathbf{u}_O(x) \to \mathbf{u}_\infty$, $\pi_O(x) \to \pi_\infty$ as $|x| \to \infty$ if and only if

$$\int_{\partial\Omega_B} \mathbf{g} \cdot \mathbf{n}^{\Omega_B} \, \mathrm{d}\sigma + \int_S \tilde{\mathbf{g}} \cdot \mathbf{n}^\omega \, \mathrm{d}\sigma = 0.$$

  This solution is unique.