Mathematics of fluids in motion

Eduard Feireisl

May 24, 2018

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, CZ-115 67 Praha 1, Czech Republic
and
Institute of Mathematics, TU Berlin
Strasse des 17.Juni, Berlin

Abstract

The goal of this lecture series is to show examples of "synergy" effect between mathematical analysis and numerics in fluid dynamics. We show how certain purely theoretical results may shed some light on convergence of some numerical schemes. Then we undertake a short excursion in the mathematical theory, highlighting some recent rather negative results obtained via the method of convex integration.

Keywords: Navier–Stokes system, Euler system, weak–strong uniqueness, measure–valued solutions, finite volume method

Contents

1 Analysis can be useful

Consider the following problem that appears in modeling of compressible viscous fluid:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mu \Delta_x \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}. \tag{1.2}$$

Here $\varrho = \varrho(t,x)$ is the mass density and $\mathbf{u} = \mathbf{u}(t,x)$ the macroscopic (bulk) velocity of the fluid expressed as numerical functions of the time $t \in (0,T)$ and the spatial coordinate $x \in \Omega \subset R^N$, $N = 1,2,3$. Equation (1.1) is the mathematical formulation of the *mass conservation*, while (1.2) corresponds to Newton's second law. Both are written in the *Eulerian reference frame*, where x denotes coordinates of points in the physical domain Ω. The fluid in question is both compressible and viscous, the symbol $p = p(\varrho)$ denotes the (barotropic) pressure, the expression on the right–hand side of (1.2) represents viscous forces. The effect of external forces has been deliberately omitted for the sake of simplicity. The system of equations (1.1), (1.2) is supplemented by the no–slip boundary condition,

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.3}$$

and the initial conditions

$$\varrho(0,\cdot) = \varrho_0, \ \ \varrho\mathbf{u}(0,\cdot) = (\varrho\mathbf{u})_0. \tag{1.4}$$

1.1 Numerical methods

Suppose we want to find an approximate solution to (1.1–1.4) via a *numerical method*.

1.1.1 Weak formulation

We start with the so-called weak formulation of problem (1.1–1.4). Multiplying equation (1.1) on a smooth *test function* $\varphi = \varphi(x)$ and integrating the resulting expression yields

$$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x\right]_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\varphi \ \mathrm{d}x\,\mathrm{d}t. \tag{1.5}$$

Applying a similar treatment to (1.2) we obtain

$$\begin{aligned}&\left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi} \ \mathrm{d}x\right]_{t_1}^{t_2}\\ &= \int_{t_1}^{t_2}\int_\Omega \Big(\varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\Big) \ \mathrm{d}x\,\mathrm{d}t\\ &- \int_{t_1}^{t_2}\int_\Omega \Big(\mu\nabla_x\mathbf{u} : \nabla_x\boldsymbol{\varphi} + \lambda\mathrm{div}_x\mathbf{u}\mathrm{div}_x\boldsymbol{\varphi}\Big) \ \mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{1.6}$$

The integral identities (1.5), (1.6) should hold for any sufficiently smooth test function φ, $\boldsymbol{\varphi}$, respectively, where, in addition, $\boldsymbol{\varphi}$ should comply with the no-slip boundary condition (1.3).

1.1.2 Finite volume numerical scheme

Rather unexpectedly, it is the weak formulation (1.5), (1.6) rather than the classical one (1.1–1.4) that is at heart of a numerical method based on *finite volume approximation*. To begin, we discretize the time variable,

$$t_0 = 0, \ t_{n+1} = t_n + \Delta t.$$

Denoting by ϱ_n, $\mathbf{u}_n$ the value of the solution at the time level t_n we get

$$\int_\Omega \Big(\varrho_n - \varrho_{n-1}\Big)\varphi \ \mathrm{d}x \approx \Delta t\int_\Omega \varrho_n\mathbf{u}_n\cdot\nabla_x\varphi \ \mathrm{d}x, \tag{1.7}$$

$$\begin{aligned}&\int_\Omega \Big(\varrho_n\mathbf{u}_n - \varrho_{n-1}\mathbf{u}_{n-1}\Big)\cdot\boldsymbol{\varphi} \ \mathrm{d}x\\ &\approx \Delta t\int_\Omega \Big(\varrho_n\mathbf{u}_n\otimes\mathbf{u}_n : \nabla_x\boldsymbol{\varphi} + p(\varrho_n)\mathrm{div}_x\boldsymbol{\varphi}\Big) \ \mathrm{d}x\,\mathrm{d}t\\ &- \Delta t\int_\Omega \Big(\mu\nabla_x\mathbf{u}_n : \nabla_x\boldsymbol{\varphi} + \lambda\mathrm{div}_x\mathbf{u}_n\mathrm{div}_x\boldsymbol{\varphi}\Big) \ \mathrm{d}x.\end{aligned} \tag{1.8}$$

To perform the space discretization, we suppose that the spatial domain Ω is a polygon that can be written as a union of small control volumes - polygonal compact sets of a common diameter h:

$$\Omega \approx \Omega_h = \cup_{K \in K_h} K.$$

Typical examples of the sets K are cubes or tetrahedra; it is also assumed that if $K_1 \cap K_2$ is either void, or a common face ($N = 3$), or a common edge ($N = 2, 3$), or a common vertex ($N = 1, 2, 3$) for any couple K_1, $K_2 \in K_h$.

We look for approximate solutions ϱ_n^h, $\mathbf{u}_n^h$ that are constant on each element $K \in K_h$; similarly, the family of test functions is no longer smooth but piecewise constant. Denote $Q_h(\Omega_h)$ the space of functions that are constant on each $K \in K_h$. Formally, $\nabla_x \phi$ for $\phi \in Q_h$ is a distribution vector "sitting" on the faces $\sigma \in \Sigma_h$, its direction coincides with the normal vector $\mathbf{n}$ to the face σ, its amplitude proportional to the jump

$$[[\phi]]_\sigma = \lim_{s \to 0+} \phi(x + s\mathbf{n}) - \lim_{s \to 0+} \phi(x - s\mathbf{n}).$$

Note carefully that

$$\nabla_x \varphi \approx [[\phi]]_\sigma \mathbf{n}$$

is independent of the choice of the direction of the normal $\mathbf{n}$.

Consequently, denoting

$$\overline{r} = \frac{\lim_{s \to 0+} r(x + s\mathbf{n}) + \lim_{s \to 0+} r(x - s\mathbf{n})}{2}$$

the average of $r \in Q_h$ on the face σ, we may rewrite the system (1.7), (1.8) as

$$\int_{\Omega_h} \frac{\varrho_n^h - \varrho_{n-1}^h}{\Delta t} \varphi \ \mathrm{d}x = \sum_{\sigma \in \Sigma_h} \int_{\sigma_h} \overline{\varrho_n^h \mathbf{u}_n^h} \cdot \mathbf{n} [[\sigma]]_\sigma \mathrm{d}S_h, \tag{1.9}$$

$$\begin{aligned} \int_{\Omega_h} \frac{\varrho_n^h \mathbf{u}_n^h - \varrho_{n-1}^h \mathbf{u}_{n-1}^h}{\Delta t} \cdot \boldsymbol{\varphi} \ \mathrm{d}x &= \sum_{\sigma \in \Sigma_h} \int_{\sigma_h} \Big(\overline{\varrho_n^h \mathbf{u}_n^h \otimes \mathbf{u}_n^h} \cdot \mathbf{n} \cdot [[\boldsymbol{\varphi}]]_\sigma + \overline{p(\varrho_n^h)} \mathbf{n} \cdot [[\boldsymbol{\varphi}]]_\sigma \Big) \mathrm{d}S_h \\ &- \sum_{\sigma \in \Sigma_h} \int_{\sigma_h} \Big(\mu [[\mathbf{u}_n^h]]_\sigma : [[\boldsymbol{\varphi}]]_\sigma + \lambda [[\mathbf{u}_n^h]]_\sigma \cdot \mathbf{n} [[\boldsymbol{\varphi}]]_\sigma \cdot \mathbf{n} \Big) \mathrm{d}S_h. \end{aligned} \tag{1.10}$$

Equations (1.9), (1.10) are to be satisfied for any piecewise constant test functions $\varphi \in Q_h$, $\boldsymbol{\varphi} \in Q_{0,h}(\Omega_h; R^N)$ and therefore represent a finite system of nonlinear algebraic equations.

To obtain the final finite volume numerical scheme, a "numerical viscosity" is usually added to (1.9), (1.10), specifically,

$$\int_{\Omega_h} \frac{\varrho_n^h - \varrho_{n-1}^h}{\Delta t} \varphi \ \mathrm{d}x = \sum_{\sigma \in \Sigma_h} \int_{\sigma_h} \Big(\overline{\varrho_n^h \mathbf{u}_n^h} \cdot \mathbf{n} [[\varphi]]_\sigma - \lambda_h [[\varrho^h]]_\sigma [[\varphi]]_\sigma \Big) \mathrm{d}S_h, \tag{1.11}$$

$$\int_{\Omega_h} \frac{\varrho_n^h \mathbf{u}_n - \varrho_{n-1}^h \mathbf{u}_{n-1}^h}{\Delta t} \cdot \boldsymbol{\varphi} \ \mathrm{d}x = \sum_{\sigma\in\Sigma_h} \int_{\sigma_h} \Big(\overline{\varrho_n^h \mathbf{u}_n^h \otimes \mathbf{u}_n^h} \cdot \mathbf{n} \cdot [[\boldsymbol{\varphi}]]_\sigma + \overline{p(\varrho_n^h)} \mathbf{n} \cdot [[\boldsymbol{\varphi}]]_\sigma \Big) \mathrm{d}S_h$$
$$- \sum_{\sigma\in\Sigma_h} \int_{\sigma_h} \Big(\mu [[\mathbf{u}_n^h]]_\sigma : [[\boldsymbol{\varphi}]]_\sigma + \lambda [[\mathbf{u}_n^h]]_\sigma \cdot \mathbf{n} [[\boldsymbol{\varphi}]]_\sigma \cdot \mathbf{n} \Big) \mathrm{d}S_h \tag{1.12}$$
$$- \sum_{\sigma\in\Sigma_h} \int_{\sigma_h} \lambda_h [[\varrho_n^h \mathbf{u}_n^h]]_\sigma \cdot [[\boldsymbol{\varphi}]]_\sigma \mathrm{d}S_h.$$

Keeping in mind our convention concerning representation of the gradient, the extra terms added,

$$\sum_{\sigma\in\Sigma_h} \int_{\sigma_h} \lambda_h [[\varrho^h]]_\sigma [[\varphi]]_\sigma \Big) \mathrm{d}S_h \approx -h \mathrm{div}_x (\lambda_h \nabla_x \varrho), \ \sum_{\sigma\in\Sigma_h} \int_{\sigma_h} \lambda_h [[\varrho_n^h \mathbf{u}_n^h]]_\sigma \cdot [[\boldsymbol{\varphi}]]_\sigma \mathrm{d}S_h \approx -h \mathrm{div}_x (\lambda_h \nabla_x (\varrho \mathbf{u})),$$

correspond indeed to artificial viscosity.

The approximate problem (1.11), (1.12) can be seen as a numerical scheme for solving the original system (1.1–1.4).

1.2 Convergence of the approximate solutions

A fundamental question is to which extent the solutions ϱ_n^h, $\mathbf{u}_n^h$ resulting form the numerical scheme (1.11), (1.12) approximate solutions to the original problem (1.1–1.4). We claim the following result that can be rigorously justified:

Claim:
Suppose the following:

- The initial data ϱ_0, $(\varrho \mathbf{u})_0$ are smooth $\varrho_0 > 0$, and $(\varrho \mathbf{u})_0$ satisfies relevant compatibility conditions.
- The domains Ω_h approximate a domain Ω, where the latter has smooth boundary.
- The sequence of family of approximate densities $\{\varrho_n^h\}_{h>0, n\geq 1}$, is bounded uniformly for $\Delta t, h \to 0$.

Conclusion:

$$\varrho_n^h \to \varrho, \ \mathbf{u}_n^h \to \mathbf{u} \text{ in } L^1((0,T)\times\Omega),$$

where ϱ, $\mathbf{u}$ is a smooth solution of the problem (1.1–1.4).

In the remaining part of this section, we outline the proof of **Claim**. We point out that the *existence* of the smooth solution to the limit problem *is not a priori* assumed; whence the proof of **Claim** shows a rather nice synthesis of purely analytical and numerical methods.

1.2.1 Step 1 - generating measure–valued solutions

The approximate scheme generates a dissipative measure–valued solution to the limit system (1.1–1.4). Such a result has been shown in [13]. Dissipative measure valued solutions represent a rather large class of objects that accommodates asymptotic limits of various approximate problems. The convergence to a dissipative measure–valued solution is a relatively easy task. At the level of numerical analysis, it requires only *stability* and *consistency* properties of the scheme. The dissipative measure–valued solutions to problems in fluid mechanics will be introduced and discussed in Section 3 below.

1.2.2 Step 2 - weak–strong uniqueness principle

The weak–strong uniqueness principle asserts that any dissipative measure valued solution coincides with the strong solution emenating from the same initial data as long as the latter solution exists. This property for the problem (1.1–1.4) was shown in [11].

1.2.3 Step 3 - local existence of smooth solutions for the limit problem

The data ϱ_0, $(\varrho \mathbf{u})_0$ as well as the spatial domain Ω being smooth, the problem (1.1–1.4) admits a smooth solution ϱ, $\mathbf{u}$ defined on a maximal time interval $(0, T_{\max})$. This is nowadays a standard result, the relevant references concerning (1.1–1.4) are e.g. [16], [15], [20], [19], [21].

1.2.4 Step 4 - a blow up criterion for smooth solutions

A remarkable result of Sun, Wang, and Zhang [18] asserts that the local smooth solution remains smooth as long as the density ϱ remains bounded. In other words,

$$\limsup_{t \to T_{\max}-} \| \varrho(t, \cdot) \|_{L^\infty(\Omega)} = \infty \text{ whenever } T_{\max} > 0.$$

1.2.5 Step 5 - synthesis

The approximate solution generate a dissipative measure–valued solution. The latter coincides with the smooth solution at least on compact subintervals of $[0, T_{\max})$. Thus the approximate solutions converge to the smooth solution on compact subintervals of $[0, T_{\max})$. As the approximate densities are uniformly bounded and convergence takes place in the strong topology of L^1, the density ϱ remains bounded up to $T_{\max}$. By the blow-up criterion, we get $T_{\max} \geq T$, which yields the desired conclusion.

2 Analysis can be awful

In this section, we review certain rather negative results concerning well–posedness of the inviscid variant of the problem (1.1–1.4). Specifically, we consider the barotropic *Euler system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$
$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0.$$

Introducing the absolute temperature ϑ, we may even consider the physically more relevant *complete Euler system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.1}$$
$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0, \tag{2.2}$$

with the total energy balance,

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \mathbf{u} \right] + \mathrm{div}_x(p\mathbf{u}) = 0, \tag{2.3}$$

where e is the specific internal energy. The system can be supplemented by the impermeability condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{2.4}$$

and the initial conditions

$$\varrho(0, \cdot) = \varrho_0, \ (\varrho \mathbf{u})(0, \cdot) = (\varrho \mathbf{u})_0, \ \varrho e(\varrho, \vartheta)(0, \cdot) = (\varrho e)_0. \tag{2.5}$$

The iconic example of the state equation is the Boyle-Marriot law for the pressure,

$$p = \varrho \vartheta, \text{ supplemented with } e = c_v \vartheta, \ c_v > 0 \text{ a positive constant.}$$

2.1 Entropy

The constitutive relations imposed on the pressure $p = p(\varrho, \vartheta)$ and the internal energy $e = e(\varrho, \vartheta)$ are not completely arbitrary. They should comply with the Gibbs equation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho} \right). \tag{2.6}$$

The new quantity $s = s(\varrho, \vartheta)$ is the specific entropy. Thanks to (2.6), (smooth) solutions of the Euler system (2.1–2.3) satisfy the entropy balance equation,

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) = 0. \tag{2.7}$$

Unfortunately, smooth solutions to the Euler system (2.1–2.3) do not, in general, exist globally in time for a fairly general class of initial data, see e.g. the monographs by Dafermos [6], Smoller [17].

Discontinuities may appear in a finite time no matter how smooth the initial data are. If one still believes in physical relevance of the Euler system, a new concept of solutions must be developed, where smoothness of solutions is relaxed. These are the so–called *weak (distributional) solutions* introduced in the following section. At this moment, we just point out that for the weak solutions, the entropy balance (2.7) is replaced by *inequality*,

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) \geq 0. \tag{2.8}$$

2.2 Weak solutions

The weak formulation of the Euler system (2.1–2.3) reads

$$\left[\int_\Omega \varrho \varphi(t, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t \tag{2.9}$$

for any test function $\varphi \in C^1([0,T] \times \overline{\Omega})$;

$$\left[\int_\Omega \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(t, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \, \mathrm{d}t \tag{2.10}$$

for any test function $\boldsymbol{\varphi} \in C^1([0,T] \times \overline{\Omega}; R^N)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$,

$$\begin{aligned} &\left[\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \varphi(t, \cdot) \ \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_\Omega \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \partial_t \varphi + \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + p \right) \mathbf{u} \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{2.11}$$

for any test function $\varphi \in C^1([0,T] \times \overline{\Omega})$. It is customary to append the system (2.9–2.11) by the weak formulation of the entropy inequality (2.8),

$$\left[\int_\Omega \varrho s \varphi(t, \cdot) \ \mathrm{d}x \right]_{t=0}^{t \to \tau -} \geq \int_0^T \int_\Omega \left[\varrho s \partial_t \varphi + \varrho s \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \tag{2.12}$$

for any $\varphi \in C^1([0,T] \times \overline{\Omega})$, $\varphi \geq 0$.

Note that (2.9–2.11) is slightly different from the weak formulation introduced in (1.5), (1.6), however both forms are equivalent. Having time dependent test functions as in (2.9–2.11) is rather convenient and will be used in the future.

2.3 Well posedness

It is known that the system (2.9–2.11) is not well-posed in the class of weak solutions. There are examples of infinitely many weak solutions emanating from the same initial data, see e.g.

Dafermos [7]. There has been a common believe that imposing the entropy inequality (2.12) as an *admissibility criterion* will rule out the unphysical solutions. Unfortunately, this is not the case at least in the physically relevant multidimensional case $N = 2, 3$.

Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded Lipschitz domain. We consider piece–wise constant initial distribution of the density and the temperature. A function r is piecewise constant if Ω admits a decomposition

$$\overline{\Omega} = \cup_{i=1}^{M} \overline{K}_i, \ K_i \text{ Lipschitz domains}, \ K_i \cap K_j = \emptyset \text{ for } i \neq j$$

such that $r|_{K_i} = r_i$-a constant for each $i = 1, \dots, M$. We report the following result proved in [12, Theorem 2.6].

Theorem 2.1. *Let $\Omega \subset R^N$ be a bounded Lipschitz domain. Let the initial data $\varrho_0 > 0$, $\vartheta_0 > 0$ be piece-wise constant functions.*

Then there exists a vector field $\mathbf{u}_0 \in L^\infty(\Omega; R^N)$ such that the problem (2.9–2.12) admits infinitely many solutions starting from ϱ_0, ϑ_0, $\mathbf{u}_0$. In addition, the entropy balance (2.12) holds as an equality, meaning for all test functions $\varphi \in C^1([0, T] \times \overline{\Omega})$ (not necessarily non–negative).

The proof of this rather striking result is based on the method of convex integration, recently adapted to problems in fluid mechanics by De Lellis and Székelyhidi et al [10], [9], [8].

2.3.1 Convex integration for incompressible fluid flows

We show that Theorem 2.1 follows from rather innocently looking statement related to the incompressible Euler system. Consider the following problem:

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathrm{div}_x \mathbf{v} = 0, \tag{2.13}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I} \right) = 0 \tag{2.14}$$

supplemented with the "no–flux" boundary conditions specified below. We consider the weak solutions of (2.13), (2.14) satisfying

$$\left[\int_\Omega \mathbf{v} \cdot \boldsymbol{\varphi} \ \mathrm{d}\mathbf{x} \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{v} \otimes \mathbf{v} : \nabla_x \boldsymbol{\varphi} - \frac{1}{N} |\mathbf{v}|^2 \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \, \mathrm{d}t \tag{2.15}$$

for any $\boldsymbol{\varphi} \in C^1([0, T] \times R^N; R^N)$,

$$\int_0^\tau \int_\Omega \mathbf{v} \cdot \nabla_x \varphi \ \mathrm{d}x \, \mathrm{d}t = 0 \tag{2.16}$$

for any $\varphi \in C^1([0, T] \times R^N)$. The fact that $\boldsymbol{\varphi}$, φ behave arbitrarily on $\partial\Omega$ enforces the no–flux boundary conditions mentioned above.

Despite the fact that problem (2.15)–(2.16) is apparently overdetermined, we report the following result proved by De Lellis and Székelyhidi [9].

Theorem 2.2. *Let $N = 2, 3$ and let*

$$E = \frac{1}{2}|\mathbf{v}|^2$$

be the kinetic energy associated to the field $\mathbf{v}$. There exists $\Lambda_0 \geq 0$ such that for any $\Lambda \geq \Lambda_0$, there is $\mathbf{v}_0 \in L^\infty(\Omega; R^N)$ such that the problem (2.15), (2.16) admits a solution $\mathbf{v}$ in the class

$$\mathbf{v} \in C_{\rm weak}([0,T]; L^2(\Omega; R^N)) \cap L^\infty((0,T) \times \Omega; R^N)$$

such that

$$E = \frac{1}{2}|\mathbf{v}|^2 = \Lambda \ \textit{for any } t \in [0,T].$$

In the remaining part of this section we show how Theorem 2.1 follows from Theorem 2.2. Let

$$\overline{\Omega} = \cup_{i=1} \overline{K}_i$$

be the decomposition of Ω associated to the piecewise constant initial data, meaning

$$\varrho = \varrho_i > 0, \ \vartheta = \vartheta_i > 0 \text{ on each } K_i.$$

Applying Theorem 2.2 on each K_i and performing a simple rescaling $t \approx \varrho_i$ in the time variable, we obtain the existence of the initial data $\mathbf{v}_{0,i}$ and the associated solutions $\mathbf{v}_i$ satisfying

$$\left[\int_{K_i} \mathbf{v}_i \cdot \boldsymbol{\varphi} \ {\rm d}\mathbf{x} \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{K_i} \left[\mathbf{v}_i \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{v}_i \otimes \mathbf{v}_i}{\varrho_i} : \nabla_x \boldsymbol{\varphi} - \frac{1}{N} \frac{|\mathbf{v}_i|^2}{\varrho_i} {\rm div}_x \boldsymbol{\varphi} \right] \ {\rm d}x \, {\rm d}t \tag{2.17}$$

for any $\boldsymbol{\varphi} \in C^1([0,T] \times R^N; R^N)$,

$$\int_0^\tau \int_{K_i} \mathbf{v}_i \cdot \nabla_x \varphi \ {\rm d}x \, {\rm d}t = 0 \tag{2.18}$$

for any $\varphi \in C^1([0,T] \times R^N)$,

$$\frac{1}{2} \frac{|\mathbf{v}_i|^2}{\varrho_i} = \Lambda - \frac{N}{2} p(\varrho_i, \vartheta_i) \tag{2.19}$$

for some $\Lambda > 0$ that may be taken *the same* on all K_i. Consequently, introducing the velocity field $\mathbf{u}_i$,

$$\mathbf{u}_i = \frac{1}{\varrho_i} \mathbf{v}_i,$$

we easily deduce from (2.19) the weak formulation of the continuity equation (2.9), namely

$$\left[\int_{K_i} \varrho_i \varphi \ {\rm d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{K_i} \left[\varrho_i \partial_t \varphi + \varrho_i \mathbf{u}_i \cdot \nabla_x \varphi \right] \ {\rm d}x \, {\rm d}t \tag{2.20}$$

for any $\varphi \in C^1([0,T] \times R^N)$. Similarly, equation (2.17) together with relation (2.19), give rise to

$$\left[\int_{K_i} \varrho_i \mathbf{u}_i \cdot \boldsymbol{\varphi} \ \mathrm{d}\mathbf{x} \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{K_i} \left[\varrho_i \mathbf{u}_i \cdot \partial_t \boldsymbol{\varphi} + \varrho_i \mathbf{u}_i \otimes \mathbf{u}_i : \nabla_x \boldsymbol{\varphi} + p(\varrho_i, \vartheta_i) \mathrm{div}_x \boldsymbol{\varphi} - 2\Lambda \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \, \mathrm{d}t \tag{2.21}$$

for any $\boldsymbol{\varphi} \in C^1([0,T] \times R^N; R^N)$.

Finally, as the total energy as well as the pressure are constant on K_i, we deduce from (2.16) that

$$\begin{aligned} &\left[\int_{K_i} \left(\frac{1}{2} \varrho_i |\mathbf{u}_i|^2 + \varrho_i e(\varrho_i, \vartheta_i) \right) \varphi \, \mathrm{d}x \right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_{K_i} \left[\left(\frac{1}{2} \varrho_i |\mathbf{u}_i|^2 + \varrho_i e(\varrho_i, \vartheta_i) \right) \partial_t \varphi + \left(\frac{1}{2} \varrho_i |\mathbf{u}_i|^2 + \varrho_i e(\varrho_i, \vartheta_i) + p(\varrho_i, \vartheta_i) \right) \mathbf{u}_i \cdot \nabla_x \varphi \right] \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{2.22}$$

and

$$\left[\int_{K_i} \varrho_i s(\varrho_i, \vartheta_i) \varphi \, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{K_i} \left[\varrho_i s(\varrho_i, \vartheta_i) \partial_t \varphi + \varrho_i s(\varrho_i, \vartheta_i) \mathbf{u}_i \cdot \nabla_x \varphi \right] \mathrm{d}x \, \mathrm{d}t \tag{2.23}$$

for any $\varphi \in C^1([0,T] \times R^N; R^N)$.

Now, relations (2.21–2.22) can be summed up over $i = 1, \ldots, M$ to yield the desired conclusion. Note carefully this is possible thanks to our choice of the boundary conditions. Finally, the

$$\sum_{i=1}^M \int_{K_i} 2\Lambda \mathrm{div}_x \boldsymbol{\varphi} \ \mathrm{d}x = \int_\Omega 2\Lambda \mathrm{div}_x \boldsymbol{\varphi} \ \mathrm{d}x$$

in (2.21) drops out provided $\boldsymbol{\varphi} \cdot \mathbf{n}$ vanishes on the boundary $\partial\Omega$. Thus we have shown Theorem 2.1.

2.4 Conclusion

Theorem 2.1 shows that the complete Euler system in the dimension two and higher is *ill–posed* in the class of L^∞ weak solutions even if the entropy inequality (2.12) is appended as an admissibility criterion. This is in sharp contrast with the simplified monodimensional case, where the entropy criterion is believed to pick up the (unique) physically relevant solution. This fact is related to possible oscillations in the families of weak solutions that may develop at any time. To the present state of the art, it is a challenging open problem if those can be ruled out by more sophisticated but still physically relevant admissibility criteria. In what follows, we introduce a more general class of solutions to the Euler and similar systems in fluid dynamics, where oscillatory behavior is anticipated.

3 Analysis can be beautiful

We introduce a rather general class of objects - dissipative measure valued (DMV) solutions - that may be associated to evolutionary equations in fluid mechanics. For the sake of simplicity, we focus on the barotropic Euler system:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{3.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x\left(\varrho \mathbf{u} \otimes \mathbf{u}\right) + \nabla_x p(\varrho) = 0, \tag{3.2}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{3.3}$$

In addition, the energy of the system,

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho),\ P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z$$

satisfies

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) + \mathrm{div}_x(p\mathbf{u}) = 0. \tag{3.4}$$

3.1 A priori estimates or stability

A priori estimates are natural bounds imposed on the family of solutions to a given system through the data. In the case (3.1–3.3), the date are given by the initial condition

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0. \tag{3.5}$$

They can deduced formally assuming all quantities in (3.1–3.3) are sufficiently smooth. A natural counterpart of *a priori* bounds are stability estimates for numerical methods. It turns out that the available *a priori* bounds for the system (3.1–3.3) are rather poor. Rewriting the equation of continuity as

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho = -\varrho \mathrm{div}_x \mathbf{u}$$

we deduce, integrating along characteristics, that

$$\varrho \geq 0 \text{ provided } \varrho_0 \geq 0.$$

Unfortunately, a stronger bound

$$\varrho(\tau,\cdot) \geq \inf_{x\in\Omega} \varrho_0 \exp\left(-\int_0^\tau \|\mathrm{div}_x \mathbf{u}\|_{L^\infty(\Omega)}\right)$$

seems out of reach due to the lack of estimates on $\mathrm{div}_x\mathbf{u}$. Similar problem occurs even for the "more regular" Navier–Stokes system.

Next, we may integrate (3.1) over Ω and use the boundary condition (3.3) to see that the total mass of the fluid is a conserved quantity,

$$M = \int_\Omega \varrho(t,\cdot)\ \mathrm{d}x = \int_\Omega \varrho_0\ \mathrm{d}x \text{ for all } t \geq 0. \tag{3.6}$$

Furthermore, we may integrate the total energy balance (3.4) to obtain

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right]\ \mathrm{d}x = \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0)\right]\ \mathrm{d}x. \tag{3.7}$$

Seeing that

$$P''(\varrho) = \frac{p'(\varrho)}{\varrho} \text{ for any } \varrho > 0$$

we deduce that P is a (strictly) convex function of ϱ as soon as the pressure p is a (strictly) increasing function of ϱ. We shall therefore suppose that

$$p'(\varrho) > 0 \text{ whenever } \varrho > 0, \tag{3.8}$$

in other words, *compressibility* of the fluid is always positive. Such a stipulation is sometimes called *hypothesis of thermodynamic stability*.

Keeping in mind the total mass conservation (3.6) we choose $\overline{\varrho}$,

$$M = \int_\Omega \varrho\ \mathrm{d}x = \int_\Omega \overline{\varrho}\ \mathrm{d}x = |\Omega|\overline{\varrho},$$

tacitly assuming that Ω is a bounded domain. Consequently,

$$\begin{aligned}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right]\ \mathrm{d}x &= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho})\right]\ \mathrm{d}x \\ &= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho})\right]\ \mathrm{d}x + P(\overline{\varrho})|\Omega|,\end{aligned}$$

where the rightmost integral is non–negative. This yields

$$\int_\Omega \varrho|\mathbf{u}|^2(t,\cdot)\ \mathrm{d}x + \int_\Omega |P(\varrho)(t,\cdot)|\ \mathrm{d}x \stackrel{<}{\sim} 1 \text{ uniformly for } t \geq 0. \tag{3.9}$$

Unfortunately, the energy bound (3.9) seems to be the best one available for the Euler system (3.1–3.3), at least if $N > 1$. Note that this is, in general, not enough to render the pressure $p(\varrho)$ integrable and, in addition, we have basically no bounds on the velocity $\mathbf{u}$ due to the hypothetical possibility of vacuum regions, where ϱ may vanish.

To control the pressure, we make another hypothesis, namely,

$$|p(\varrho)| \stackrel{<}{\sim} 1 + P(\varrho) \text{ for all } \varrho \geq 0. \tag{3.10}$$

Note that (3.10) holds for the iconic example of the isentropic EOS:

$$p(\varrho) = a\varrho^\gamma,\ \gamma \geq 1.$$

To overcome the problem with velocity, we introduce a new variable, the momentum $\mathbf{m} = \varrho\mathbf{u}$, and replace systematically $\mathbf{u}$ by $\mathbf{m}/\varrho$.

3.2 Weak formulation

With the new state variable $\mathbf{m}$, we may write the weak formulation of the Euler system (3.1–3.3) in the form

$$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi\right] \ \mathrm{d}x\,\mathrm{d}t \tag{3.11}$$

for any $\varphi \in C^1([0,T]\times\overline{\Omega})$,

$$\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi} \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right] \ \mathrm{d}x\,\mathrm{d}t \tag{3.12}$$

for any $\boldsymbol{\varphi} \in C^1([0,T]\times R^N; R^N)$, $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega} = 0$. Finally, we append (3.11), (3.12) by the energy inequality

$$-\int_0^T \partial_t\psi \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right](t,\cdot) \ \mathrm{d}x\,\mathrm{d}t \leq \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0)\right] \ \mathrm{d}x \tag{3.13}$$

for any $\psi \in C_c^1[0,T)$, $\psi \geq 0$, $\psi(0) = 1$.

3.3 Weak sequential compactness

We assume that $[\varrho_n, \mathbf{m}_n]$ is a sequence of (weak) solutions satisfying (3.11–3.13) with the initial data $[\varrho_{0,n}, \mathbf{m}_{0,n}]$. Our goal is to study the limit problem for the accummulation points of $\{\varrho_n, \mathbf{m}_n\}_{n=1}^\infty$ as $n \to \infty$ in suitable topologies.

3.3.1 Weak topology

Under the hypothesis (3.10), we control all terms in (3.11–3.13) at least in the L^1 topology. More precisely, seeing that

$$\mathbf{m} = \sqrt{\varrho}\frac{\mathbf{m}}{\sqrt{\varrho}} \leq \frac{1}{2}\varrho + \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}$$

we deduce from the bounds (3.6), (3.9) that

$$\varrho_n, \ \mathbf{m}_n, \ \frac{\mathbf{m}_n\otimes\mathbf{m}_n}{\varrho_n}, \ p(\varrho_n) \text{ are bounded in } L^\infty(0,T;L^1)$$

uniformly for $n \to \infty$.

We pause at this stage to briefly discuss possible behaviour of sequences of functions that are bounded only in the L^1-norm. The first problem that may occur are *oscillations*. A typical examples is the sequence

$$v_n(y) = v(ny), \text{ where } v \text{ is periodic on } R.$$

Apparently, $\{v_n\}_{n=1}^{\infty}$ does not converge pointwise to any function but rather oscillates around its integral mean. Thus the only possibility how to study convergence of $\{v_n\}_{n=1}^{\infty}$ is to consider its integral averages,

$$\int_B v_n(y)\ \mathrm{d}y,$$

where B is a Borel set. It turns out that the integral averages do converge, at least for a suitable subsequence we do not relabel. Specifically, there is a periodic function v such that

$$\int_B v_n(y)\ \mathrm{d}y \to \int_B v(y)\ \mathrm{d}y \tag{3.14}$$

for any Borel set B, or, equivalently,

$$\int v_n(y)\phi(y)\ \mathrm{d}y \to \int v(y)\phi(y)\ \mathrm{d}y \tag{3.15}$$

for any $\phi \in C_c^{\infty}(R)$.

Convergence in integral averages is called *weak-L^1* convergence. It requires *equi-integrability* of the sequence and does not commute with the non–linear compositions. Specifically,

$$\text{if } v_n \to v \text{ weakly in } L^1 \text{ and } F(v_n) \to \overline{F(v)} \text{ weakly in } L^1, \text{ then, in general, } \overline{F(v)} \neq F(v).$$

Another type of singularity that may occur for an L^1- bounded sequence is *concentration*. An iconic example is

$$v_n(y) = nv_n(ny),\ \int_R |v(y)|\ \mathrm{d}y < \infty,\ v \text{ compactly supported in } R.$$

It is easy to see that

$$v_n(y) \to 0 \text{ whenever } y \neq 0,$$

and

$$\int_R v_n(y)\ \mathrm{d}y = \text{const.}$$

Apparently, the asymptotic limit cannot be characterized by integral averages but rather by a measure supported at the origin $y = 0$.

3.4 Measure–valued solutions

Suppose there is a sequence $\{\mathbf{U}_n\}_{n=1}^{\infty}$ of vector–valued functions defined on a domain $Q \subset R^n$ and ranging in a closed set $\mathcal{P} \subset R^m$. In addition, suppose that

$$\int_Q |\mathbf{U}_n|\ \mathrm{d}y \lesssim 1 \text{ uniformly in } n.$$

Finally let $F : \mathcal{P} \to R$ be a continuous function,

$$\int_Q |F(\mathbf{U}_n)| \ \mathrm{d}y \stackrel{<}{\sim} 1 \text{ uniformly in } n.$$

We can interpret $F(\mathbf{U}_n)$ as a family of (signed) measures on the physical space $\mathbf{Q}$, therefore, at least for a suitable subsequence,

$$F(\mathbf{U}_n) \to \widehat{F(\mathbf{U})} \text{ weakly-(*) in } \mathcal{M}(\overline{Q}).$$

On the other hand, we can extract a subsequence such that

$$B(\mathbf{U}_n) \to \overline{B(\mathbf{U})} \text{ for any } B \in C_c(\mathcal{P}).$$

Accordingly, the mapping

$$B \mapsto \overline{B(\mathbf{U})}(y), \ B \in C_c(\mathcal{P})$$

can be seen as a *probability measure* on the phase space $\mathcal{P}$ for a.a. $y \in Q$, see e.g. the standard reference by Ball [1]. This is the Young measure associated to the sequence $\{\mathbf{U}_n\}_{n=1}^{\infty}$. We denote it by $\{\mathcal{V}_y\}_{y \in Q}$. It can be checked that F is integrable with respect to a.a. measures $\mathcal{V}_y$ and the function

$$y \mapsto \langle \mathcal{V}_y; F(\mathbf{U}) \rangle \text{ is integrable in } Q, \ \int_Q |\langle \mathcal{V}_y; F(\mathbf{U}) \rangle| \ \mathrm{d}y < \infty.$$

The signed measure

$$\mu_C = \widehat{F(\mathbf{U})} - \langle \mathcal{V}_y; F(\mathbf{U}) \rangle \ \mathrm{d}y$$

is called *concentration defect* for F. The function $y \to \langle \mathcal{V}_y; F(\mathbf{U}) \rangle$ is the *biting limit* of the sequence $\{F(\mathbf{U}_n)\}_{n=1}^{\infty}$, cf. Ball and Murat [2].

We are ready to introduce the concept of *dissipative measure–valued (DMV) solution for the Euler system* (3.1–3.3). We denote by

$$Q_T = (0, T) \times \Omega$$

the associated *physical space*, and by

$$\mathcal{P} = \left\{ [\varrho, \mathbf{m}] \ \Big| \ \varrho \geq 0, \ \mathbf{m} \in R^N \right\}$$

the *phase space*.

Definition 3.1. A parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x) \in Q_T}$ is *dissipative measure–valued (DMV)-solution* of the problem (3.1–3.3), with the initial conditions $\{\mathcal{V}_{0,x}\}_{x \in \Omega}$, if:

- $$(t, x) \mapsto \mathcal{V}_{t,x} \in L^{\infty}_{\mathrm{weak}}(Q_T; \mathcal{P}(\mathcal{F})); \tag{3.16}$$

- $$\int_0^T \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \rangle \, \partial_t \varphi + \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \varphi \right] \, \mathrm{d}x \, \mathrm{d}t = - \int_\Omega < \mathcal{V}_{0,x}; \varrho > \varphi(0, \cdot) \, \mathrm{d}x + \int_0^T \int_{\overline{\Omega}} \nabla_x \varphi \cdot \mathrm{d}\mu_C^1 \tag{3.17}$$

  for any $\varphi \in C_c^1([0,T) \times \overline{\Omega})$, where $\mu_C^1 \in \mathcal{M}([0,T] \times \overline{\Omega}; R^N)$ is a (signed) vector-valued measure;

- $$\begin{aligned} \int_0^T \int_\Omega \left[\langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \, \mathrm{d}t \\ = - \int_\Omega < \mathcal{V}_{0,x}; \mathbf{m} > \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x + \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} \cdot \mathrm{d}\mu_C^2 \end{aligned} \tag{3.18}$$

  for any $\varphi \in C_c^1([0,T) \times \overline{\Omega}; R^N)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$, where $\mu_C^2 \in \mathcal{M}([0,T] \times \overline{\Omega}; R^{N \times N})$ is a (signed) tensor-valued measure;

- $$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \, \mathrm{d}x + \mathcal{D}(\tau) \leq \int_\Omega \left\langle \mathcal{V}_{0,x}; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \, \mathrm{d}x \tag{3.19}$$

  for a.a. $\tau \in [0,T)$, where $\mathcal{D} \in L^\infty(0,T)$, $\mathcal{D} \geq 0$;

- there exists a constant $d > 0$ such that

  $$\int_0^T \psi \int_{\overline{\Omega}} d|\mu_C^1| + \int_0^T \psi \int_{\overline{\Omega}} d|\mu_C^2| \leq d \int_0^T \psi \mathcal{D} \, \mathrm{d}t \tag{3.20}$$

  for any $\psi \in C_c^1[0,T)$, $\psi \geq 0$.

Some comments are in order. Although we try to avoid postulating the existence of a "generating sequence", the measures $\mathcal{V}_{t,x}$ are clearly associated to a Young measure generated by some family of approximate solutions that may result from a numerical scheme. The measures μ_C^1, μ_C^2 are the corresponding concentration measures. Note that there is *a priori* nothing known concerning the size of μ_C^1, μ_C^2, the only required and crucial restriction is (3.20). The quantity $\mathcal{D}$ is the *dissipation defect* of the total energy. Accordingly, a measure–valued solution is dissipative if (3.20) holds, specifically, if the concentration measures are controlled by the dissipation of the total energy. It is exactly the condition (3.20) that plays a crucial role in the proof of the *weak–strong uniqueness principle* discussed below.

It follows from (3.17), (3.18), and (3.20) that the quantities

$$t \mapsto \int_\Omega \langle \mathcal{V}_{t,x}; \varrho \rangle \, \phi \, \mathrm{d}x, \ \phi \in C^1(\overline{\Omega}), \ t \mapsto \int_\Omega \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \boldsymbol{\varphi} \, \mathrm{d}x, \ \boldsymbol{\varphi} \in C^1(\overline{\Omega}; R^N), \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

are continuous, at least on compact subintervals of $[0,T)$. Accordingly, we may rewrite (3.17), (3.1) in a more suitable form:

$$\left[\int_\Omega < \mathcal{V}_{t,x}; \varrho > \varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho\rangle \, \partial_t \varphi + \langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \nabla_x \varphi\right] \ \mathrm{d}x \, \mathrm{d}t - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \varphi \cdot \mathrm{d}\mu_C^1 \quad (3.21)$$

for any $0 \leq \tau < T$, $\varphi \in C^1([0,T] \times \overline{\Omega})$;

$$\begin{aligned} &\left[\int_\Omega < \mathcal{V}_{t,x}; \mathbf{m} > \cdot \boldsymbol{\varphi} \ \mathrm{d}x\right]_{t=0}^{t=\tau} \\ &= \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \mathbf{m}\rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}_{t,x}; p(\varrho)\rangle \, \mathrm{div}_x \boldsymbol{\varphi}\right] \ \mathrm{d}x \, \mathrm{d}t \\ &- \int_0^\tau \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} \cdot \mathrm{d}\mu_C^2 \end{aligned} \quad (3.22)$$

for any $\varphi \in C^1([0,T] \times \overline{\Omega}; R^N)$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$.

3.5 Relative energy inequality

The relative energy functional is a crucial tool for comparing the measure–valued solutions with calssical once. More precisely, to evaluate the momentum of the corresponding deviatoric measure. We consider

$$E\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right) = \frac{1}{2}\varrho \left| \frac{\mathbf{m}}{\varrho} - \frac{\tilde{\mathbf{m}}}{\tilde{\varrho}} \right|^2 + P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho}).$$

As the function P is strictly convex, it is easy to check that $\mathcal{E} \geq 0$, and that $\varrho = \tilde{\varrho}$, $\mathbf{m} = \tilde{\mathbf{m}}$ whenever $E\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right) = 0$, $\tilde{\varrho} > 0$. We define the *relative energy* for a (DMV) solution as

$$\begin{aligned} &\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) \\ &= \int_\Omega \left\langle \mathcal{V}_{\tau,x}; \frac{\varrho}{2} \left| \frac{\mathbf{m}}{\varrho} - \frac{\tilde{\mathbf{m}}(\tau,x)}{\tilde{\varrho}(\tau,x)} \right|^2 + P(\varrho) - P'(\tilde{\varrho}(\tau,x))(\varrho - \tilde{\varrho}(\tau,x)) - P(\tilde{\varrho}(\tau,x)) \right\rangle \ \mathrm{d}x, \ \tau \geq 0. \end{aligned}$$

Note carefully that

$$\begin{aligned} &\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) \\ &= \int_\Omega \left\langle \mathcal{V}_{\tau,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \ \mathrm{d}x - \int_\Omega \langle \mathcal{V}_{\tau,x}; \mathbf{m}\rangle \cdot \frac{\tilde{\mathbf{m}}(\tau,x)}{\tilde{\varrho}(\tau,x)} \ \mathrm{d}x \\ &+ \int_\Omega \langle \mathcal{V}_{\tau,x}; \varrho\rangle \left(\frac{1}{2} \left| \frac{\tilde{\mathbf{m}}(\tau,x)}{\tilde{\varrho}(\tau,x)} \right|^2 - P'(\tilde{\varrho}(\tau,x)) \right) \ \mathrm{d}x + \int_\Omega p(\tilde{\varrho}(\tau,x)) \ \mathrm{d}x. \end{aligned}$$

In particular, the time evolution of all the above integrals can be expressed by means of the weak formulation (3.17–3.19) as long as

- $\{\mathcal{V}_{t,x}\}_{(t,x)\in Q_T}$ is a (DMV) solution of the Euler system;
- $\tilde{\varrho}$, $\tilde{\mathbf{m}}$ are continuously differentiable, and $\tilde{\varrho}$ is bounded below away from zero on Q_T;
- $\tilde{\mathbf{m}} \cdot \mathbf{n}|_{\partial\Omega} = 0$.

Introducing $\tilde{\mathbf{u}} = \frac{\tilde{\mathbf{m}}}{\tilde{\varrho}}$ we therefore obtain:

$$
\begin{aligned}
&\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \leq \mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(0) \\
&+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \varrho\tilde{\mathbf{u}} - \mathbf{m} \rangle \cdot \partial_t \tilde{\mathbf{u}} \ \mathrm{d}x \,\mathrm{d}t + \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathbf{m} \otimes \left(\tilde{\mathbf{u}} - \frac{\mathbf{m}}{\varrho} \right) \right\rangle : \nabla_x \tilde{\mathbf{u}} \ \mathrm{d}x \,\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \tilde{\mathbf{u}} \ \mathrm{d}x \,\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \rangle \, \partial_t P'(\tilde{\varrho}) + \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \nabla_x P'(\tilde{\varrho}) - \tilde{\varrho} \partial_t P'(\tilde{\varrho}) \right] \ \mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^\tau \int_{\overline{\Omega}} |\nabla_x \tilde{\mathbf{u}}| \ \mathrm{d}|\mu_C^1| + \int_0^\tau \int_{\overline{\Omega}} |\nabla_x \tilde{\mathbf{u}}| \ \mathrm{d}|\mu_C^2|.
\end{aligned}
\tag{3.23}
$$

Although (3.23) might seem a bit mysterious at the first glance, it can be derived by direct manipulation, the reader may consult [11] for details. It is worth noting that the "test" functions $\tilde{\varrho}$, $\tilde{\mathbf{u}}$ are quite arbitrary required only to satisfy the obvious restrictions.

3.6 Weak strong uniqueness

Our ultimate goal is to show that a (DMV) solution to the Euler system coincides with the strong solution emanating from the same initial data as long as the latter exists. Here "emanating from the same initial data" means that the initial measure $\mathcal{V}_{0,x}$ is given as

$$\mathcal{V}_{0,x} = \delta_{\varrho_0(x), \mathbf{m}_0(x)},$$

where ϱ_0, $\mathbf{m}_0$ are the initial data of the strong solution. With the relative energy inequality (3.23) at hand, the plan is to plug in the strong solution $\tilde{\varrho}$, $\tilde{\mathbf{m}} = \tilde{\varrho}\tilde{\mathbf{u}}$ as test functions in (3.23).

3.6.1 Step 1 - concentration measures

As the initial data coincide, the relative energy inequality (3.23) gives rise to

$$
\begin{aligned}
&\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \\
&\leq \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \varrho\tilde{\mathbf{u}} - \mathbf{m} \rangle \cdot \partial_t \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathbf{m} \otimes \left(\tilde{\mathbf{u}} - \frac{\mathbf{m}}{\varrho} \right) \right\rangle : \nabla_x \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \rangle \, \partial_t P'(\tilde{\varrho}) + \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \nabla_x P'(\tilde{\varrho}) - \tilde{\varrho} \partial_t P'(\tilde{\varrho}) \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_{\overline{\Omega}} |\nabla_x \tilde{\mathbf{u}}| \ \mathrm{d}|\mu_C^1| + \int_0^\tau \int_{\overline{\Omega}} |\nabla_x \tilde{\mathbf{u}}| \ \mathrm{d}|\mu_C^2|.
\end{aligned}
$$

Our goal is to show that all integrals on the right–hand side of the above inequality are bounded above (modulo a multiplication constant) by

$$
\int_0^\tau \left[\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \right] \mathrm{d}t
$$

and to use the Gronwall lemma. The first observation is the the this is true for the last two integrals containing the concentration measures as a direct consequence of (3.20). We therefore deduce

$$
\begin{aligned}
&\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \\
&\lesssim \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \varrho\tilde{\mathbf{u}} - \mathbf{m} \rangle \cdot \partial_t \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathbf{m} \otimes \left(\tilde{\mathbf{u}} - \frac{\mathbf{m}}{\varrho} \right) \right\rangle : \nabla_x \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \rangle \, \partial_t P'(\tilde{\varrho}) + \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \nabla_x P'(\tilde{\varrho}) - \tilde{\varrho} \partial_t P'(\tilde{\varrho}) \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \left[\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \right] \mathrm{d}t.
\end{aligned}
$$

3.6.2 Step 2 - convective term

Next, we write

$$
\begin{aligned}
\int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathbf{m} \otimes \left(\tilde{\mathbf{u}} - \frac{\mathbf{m}}{\varrho} \right) \right\rangle : \nabla_x \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t &= \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; (\mathbf{m} - \varrho\tilde{\mathbf{u}}) \otimes \left(\tilde{\mathbf{u}} - \frac{\mathbf{m}}{\varrho} \right) \right\rangle : \nabla_x \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; (\varrho\tilde{\mathbf{u}} - \mathbf{m}) \rangle \cdot \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\mathbf{u}} \ \mathrm{d}x.
\end{aligned}
$$

Moreover, we use the equation

$$\partial_t \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\mathbf{u}} = -\frac{1}{\tilde{\varrho}} \nabla_x p(\tilde{\varrho})$$

concluding

$$\begin{aligned}
&\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \\
&\lesssim \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; \mathbf{m} - \varrho \tilde{\mathbf{u}} \rangle \cdot \frac{1}{\tilde{\varrho}} \nabla_x p(\tilde{\varrho}) \ \mathrm{d}x \, \mathrm{d}t - \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \rangle \, \partial_t P'(\tilde{\varrho}) + \langle \mathcal{V}_{t,x}; \mathbf{m} \rangle \cdot \nabla_x P'(\tilde{\varrho}) - \tilde{\varrho} \partial_t P'(\tilde{\varrho}) \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \left[\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \right] \mathrm{d}t.
\end{aligned}$$

3.6.3 Step 3 - pressure terms and conclusion

Finally, seeing that

$$P''(\tilde{\varrho}) = \frac{p'(\tilde{\varrho})}{\tilde{\varrho}}; \text{ whence } \frac{1}{\tilde{\varrho}} \nabla_x p(\tilde{\varrho}) = \nabla_x P'(\tilde{\varrho}),$$

and

$$\begin{aligned}
&\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \\
&\lesssim - \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \tilde{\mathbf{u}} \rangle \cdot \frac{1}{\tilde{\varrho}} \nabla_x p(\tilde{\varrho}) + \langle \mathcal{V}_{t,x}; \varrho \rangle \, \frac{1}{\tilde{\varrho}} \partial_t p(\tilde{\varrho}) \right] \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega \partial_t p(\tilde{\varrho}) \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \left[\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \right] \mathrm{d}t.
\end{aligned}$$

Next,

$$\partial_t p(\tilde{\varrho}) = -\mathrm{div}_x (p(\tilde{\varrho}) \tilde{\mathbf{u}}) + (p(\tilde{\varrho}) - p'(\tilde{\varrho}) \tilde{\varrho}) \, \mathrm{div}_x \tilde{\mathbf{u}}, \tag{3.24}$$

and therefore

$$\begin{aligned}
&\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \\
&\lesssim - \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; p(\varrho) - p'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - p(\tilde{\varrho}) \rangle \, \mathrm{div}_x \tilde{\mathbf{u}} \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left[\langle \mathcal{V}_{t,x}; \varrho \rangle \, \tilde{\mathbf{u}} \cdot \frac{1}{\tilde{\varrho}} \nabla_x p(\tilde{\varrho}) + \langle \mathcal{V}_{t,x}; \varrho \rangle \, \frac{1}{\tilde{\varrho}} \partial_t p(\tilde{\varrho}) + \langle \mathcal{V}_{t,x}; \varrho \rangle \, p'(\tilde{\varrho}) \mathrm{div}_x \tilde{\mathbf{u}} \right] \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \left[\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \right] \mathrm{d}t.
\end{aligned}$$

Thus using (3.24) again, we conclude

$$\begin{aligned}
&\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \\
&\lesssim -\int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; p(\varrho) - p'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - p(\tilde{\varrho}) \rangle \, \mathrm{div}_x \tilde{\mathbf{u}} \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \left[\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, \tilde{\varrho}, \tilde{\mathbf{m}}\right)(\tau) + \mathcal{D}(\tau) \right] \mathrm{d}t.
\end{aligned}$$

It remains to impose an extra hypothesis on the pressure function p so that

$$|p(\varrho) - p'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - p(\tilde{\varrho})| \lesssim P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho}). \tag{3.25}$$

Note that (3.25) is not automatically satisfied if the pressure p is merely strictly monotone, however, it holds for the iconic examples of the isentropic and barotropic pressure law:

$$p(\varrho) = a\varrho^\gamma, \ \gamma \geq 1,$$

and, in the more general case

$$p \in C^1[0, \infty), \ p'(\varrho) > 0 \text{ for } \varrho > 0, \ p(\varrho) \lesssim (1 + P(\varrho)) \text{ for all } \varrho. \tag{3.26}$$

It is also easy to see that the above discussion extends easily to the class of Lipschitz continuous strong solutions (instead of C^1). We have shown the following rather remarkable result, cf. also Gwiazda, Swierczewska–Gwiazda, Wiedemann [14].

Theorem 3.2. *Let the pressure $p = p(\varrho)$ comply with the hypothesis* (3.26). *Let $\Omega \subset R^N$, $N = 1, 2, 3$ be a bounded Lipschitz domain. Suppose that $[\tilde{\varrho}, \tilde{\mathbf{m}}]$ is a strong Lipschitz (in $\overline{Q}_T$) solution of the Euler system (3.1–3.3) starting from the initial data*

$$\varrho(0, \cdot) = \varrho_0, \ \tilde{\mathbf{m}}(0, \cdot) = \mathbf{m}_0, \ \varrho_0 \geq \underline{\varrho} > 0 \ \textit{in } \overline{\Omega}.$$

Let $\{\mathcal{V}_{t,x}\}_{(t,x) \in Q_T}$ be a (DMV) solution of the same problem such that

$$\mathcal{V}_{0,x} = \delta_{\varrho_0(x), \mathbf{m}_0(x)} \ \textit{for a.a. } x \in \Omega.$$

Then

$$\mathcal{V}_{t,x} = \delta_{\tilde{\varrho}(t,x); \tilde{\mathbf{m}}(t,x)} \ \textit{for a.a. } (t, x) \in Q_T.$$

Similar results hold for the complete Euler system including the thermal effect, [3], [4] as well as the compressible Navier–Stokes system [11] and the Navier–Stokes–Fourier system [5].

References

[1] J.M. Ball. A version of the fundamental theorem for Young measures. *In Lect. Notes in Physics 344, Springer-Verlag*, pages 207–215, 1989.

[2] J.M. Ball and F. Murat. Remarks on Chacons biting lemma. *Proc. Amer. Math. Soc.*, **107**:655–663, 1989.

[3] J. Březina and E. Feireisl. Measure-valued solutions to the complete Euler system. 2017. arxiv preprint No. 1702.04878, to appear in J. Math. Soc. Japan.

[4] J. Březina and E. Feireisl. Measure-valued solutions to the complete Euler system revisited. 2017. arxiv preprint No. 1710.10751.

[5] J. Březina, E. Feireisl, and A. Novotný. Stability of strong solutions to the Navier–Stokes–Fourier system. 2018. Preprint IM Prague 2018-7.

[6] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, fourth edition, 2016.

[7] C.M. Dafermos. The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.*, **70**:167–179, 1979.

[8] C. De Lellis and L. Székelyhidi, Jr. The Euler equations as a differential inclusion. *Ann. of Math. (2)*, **170**(3):1417–1436, 2009.

[9] C. De Lellis and L. Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, **195**(1):225–260, 2010.

[10] C. De Lellis and L. Székelyhidi, Jr. The h-principle and the equations of fluid dynamics. *Bull. Amer. Math. Soc. (N.S.)*, **49**(3):347–375, 2012.

[11] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann. Dissipative measure-valued solutions to the compressible Navier–Stokes system. *Calc. Var. Partial Differential Equations*, 55(6):55:141, 2016.

[12] E. Feireisl, C. Klingenberg, O. Kreml, and S. Markfelder. On oscillatory solutions to the complete Euler system. 2017. arxiv preprint No. 1710.10918.

[13] E. Feireisl and M. Lukáčová-Medviďová. Convergence of a mixed finite element finite volume scheme for the isentropic Navier–Stokes system via dissipative measure-valued solutions. *Foundations of Computational Mathematics*, 2018. To appear.

[14] P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann. Weak-strong uniqueness for measure-valued solutions of some compressible fluid models. *Nonlinearity*, **28**(11):3873–3890, 2015.

[15] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, **20**:67–104, 1980.

[16] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of compressible and heat conductive fluids. *Comm. Math. Phys.*, **89**:445–464, 1983.

[17] J. Smoller. *Shock waves and reaction-diffusion equations*. Springer-Verlag, New York, 1967.

[18] Y. Sun, C. Wang, and Z. Zhang. A Beale-Kato-Majda criterion for three dimensional compressible viscous heat-conductive flows. *Arch. Ration. Mech. Anal.*, **201**(2):727–742, 2011.

[19] A. Valli. A correction to the paper: "An existence theorem for compressible viscous fluids" [Ann. Mat. Pura Appl. (4) **130** (1982), 197–213; MR 83h:35112]. *Ann. Mat. Pura Appl. (4)*, **132**:399–400 (1983), 1982.

[20] A. Valli. An existence theorem for compressible viscous fluids. *Ann. Mat. Pura Appl. (4)*, **130**:197–213, 1982.

[21] A. Valli and M. Zajaczkowski. Navier-Stokes equations for compressible fluids: Global existence and qualitative properties of the solutions in the general case. *Commun. Math. Phys.*, **103**:259–296, 1986.