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of bounded functions with countable
support**

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EMBEDDING BANACH SPACES INTO THE SPACE OF BOUNDED FUNCTIONS WITH COUNTABLE SUPPORT

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ABSTRACT. We prove that a WLD subspace of the space $\ell_\infty^c(\Gamma)$ consisting of all bounded, countably supported functions on a set Γ embeds isomorphically into ℓ_∞ if and only if it does not contain isometric copies of $c_0(\omega_1)$. Moreover, a subspace of $\ell_\infty^c(\omega_1)$ is constructed that has an unconditional basis, does not embed into ℓ_∞ , and whose every weakly compact subset is separable (in particular, it cannot contain any isomorphic copies of $c_0(\omega_1)$).

1. INTRODUCTION

It is classical that every separable Banach space is isometrically isomorphic to a subspace of ℓ_∞ , the space of bounded sequences with the supremum norm. Since every weakly compact subset of ℓ_∞ is separable, any weakly compactly generated space; in particular, any reflexive space; that admits an injective bounded linear operator into ℓ_∞ must be separable. (A Banach space is *weakly compactly generated*, WCG for short, when it contains a weakly compact subset whose linear span is dense.) For this reason, there is no bounded linear injection from $c_0(\Gamma)$ into ℓ_∞ when the set Γ is uncountable. Nevertheless, $c_0(\omega_1)$ sits naturally as a subspace of ℓ_∞ 's close cousin, the space $\ell_\infty^c(\Gamma)$, which consists of all bounded scalar-valued functions on Γ that are non-zero on at most countably many points in Γ .

The aim of this note is to study Banach spaces that embed into $\ell_\infty^c(\omega_1)$ but do not embed into ℓ_∞ and their relation to containment of isomorphic, or even isometric, copies of $c_0(\omega_1)$. In particular, we prove that a non-separable weakly Lindelöf determined (WLD) subspace of $\ell_\infty^c(\Gamma)$ contains an isometric copy of $c_0(\omega_1)$. (A Banach space X is WLD provided for some set Γ there exists an injective linear operator $T: X^* \rightarrow \ell_\infty^c(\Gamma)$ that is continuous as a map from X^* with the weak* topology to $\ell_\infty^c(\Gamma)$ with the topology of pointwise convergence.)

The notation is standard. We just mention that all operators are assumed to be bounded and linear, and an isomorphism is a bounded linear operator that is bounded below on the unit sphere of its domain. We consider cardinal numbers as initial ordinal numbers. A cardinal number λ is *regular* whenever a set of cardinality λ cannot be expressed as a union of fewer than λ sets that have cardinality less than λ .

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2. THE RESULTS

For a WLD space X , the density character of X^* endowed with the weak* topology is the same as of X with the norm topology ([3, Proposition 5.40]), hence the existence of a bounded linear injection $T: X \rightarrow \ell_\infty$ implies that X is separable. Indeed, by Goldstine's theorem, $\ell_\infty^* = \ell_1^{**}$ is weak*-separable. By injectivity of T , the adjoint map $T^*: \ell_\infty^* \rightarrow X^*$ has dense range, and so X^* has a weak*-separable dense subspace, therefore it must be weak*-separable. As X is WLD, also X must be separable. We shall employ this fact to construct copies of $c_0(\omega_1)$ in non-separable WLD subspaces of $\ell_\infty^c(\Gamma)$. (In the case of $X = c_0(\Gamma)$ the result was already recorded in [7] and [4, Lemma 6]).

Theorem 2.1. *Let Γ be a set and let X be a WLD subspace of $\ell_\infty^c(\Gamma)$. Then the following assertions are equivalent:*

- (i) X is separable,
- (ii) X embeds into ℓ_∞ ,
- (iii) there exists a bounded linear injection from X into ℓ_∞ ,
- (iv) X does not contain a subspace that is isomorphic to $c_0(\omega_1)$,
- (v) X does not contain a subspace that is isometrically isomorphic to $c_0(\omega_1)$.

In particular, every reflexive subspace of $\ell_\infty^c(\Gamma)$ is separable.

First we introduce some notation. Let Γ be a set and let $\Lambda \subseteq \Gamma$. Consider the contractive projection $P_\Lambda: \ell_\infty^c(\Gamma) \rightarrow \ell_\infty^c(\Gamma)$ given by

$$(P_\Lambda f)(\gamma) = \begin{cases} f(\gamma) & \gamma \in \Lambda, \\ 0, & \gamma \in \Gamma \setminus \Lambda \end{cases} \quad (f \in \ell_\infty^c(\Gamma)).$$

We identify the range of P_Γ with the space $\ell_\infty^c(\Gamma)$. Certainly, when Γ is countably infinite, the range of P_Γ is isometrically isomorphic to ℓ_∞ .

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) are clear. We have already observed in the introduction that for WLD spaces (iii) \Rightarrow (i). We shall prove now that (v) \Rightarrow (iii) by contraposition.

Suppose that there is no bounded linear injection from X into ℓ_∞ . In particular, for every countable set $\Lambda \subset \Gamma$, the restriction operator $P_\Lambda|_X$ is not injective, which means that $P_\Lambda f_\Lambda = 0$ for some unit vector f_Λ in the range of P_Λ . Consequently, it is possible to choose by transfinite recursion an uncountable family of pairwise disjoint countable subsets $(\Lambda_\alpha)_{\alpha < \omega_1}$ and unit vectors $f_\alpha \in \ell_\infty(\Lambda_\alpha) \cap X$. Then, the closed linear span of $\{f_\alpha: \alpha < \omega_1\}$ is isometric to $c_0(\omega_1)$. \square

Remark 2.2. Theorem A implies that the unit sphere of a non-separable WLD subspace of $\ell_\infty^c(\Gamma)$ contains an uncountable symmetrically (1+)-separated subset, that is, a set A such that $\|x \pm y\| > 1$ for distinct $x, y \in A$; this is because $c_0(\omega_1)$ has this property. This observation complements [2, Corollary 3.6], where it was proved that WLD spaces of density greater than the continuum contain such sets. It should be noted however that not every renorming of $c_0(\omega_1)$ embeds isometrically into $\ell_\infty^c(\Gamma)$, as at least under the Continuum

Hypothesis, there exists a renorming of $c_0(\omega_1)$ that does not contain isometric copies of itself ([2, Theorem 5.9]).

The hypothesis of being WLD cannot be removed completely from the statement of Theorem A. Before we give a relevant example, we prove a simple lemma.

Lemma 2.3. *Let X be a subspace of $\ell_\infty^c(\omega_1)$. If X embeds into ℓ_∞ , then there is a $\alpha < \omega_1$ such that the operator $P_{[0,\alpha]}|_X$ is bounded below; that is, bounded below on the unit sphere of X .*

Proof. Since ℓ_∞ is injective, there is an operator $J: \ell_\infty^c(\omega_1) \rightarrow \ell_\infty$ so that the restriction of J to X is bounded below (indeed, J is any extension of an embedding of X into ℓ_∞ to $\ell_\infty^c(\omega_1)$). It is therefore enough to observe that for any operator $J: \ell_\infty^c(\omega_1) \rightarrow \ell_\infty$ there is a countable ordinal α such that J vanishes on $\ell_\infty^c([\alpha, \omega_1])$ (because then J factors through the quotient $\ell_\infty^c(\omega_1)/\ell_\infty^c([\alpha, \omega_1])$, which is isomorphic to $\ell_\infty^c([0, \alpha]) \cong \ell_\infty$). \square

Theorem 2.4. *There exists a subspace Z of $\ell_\infty^c(\omega_1)$ with an unconditional basis such that Z does not embed into ℓ_∞ and $c_0(\omega_1)$ does not embed into Z . Moreover, Z contains an isomorphic copy of $\ell_1(\omega_1)$ and there is an injective operator from Z into ℓ_∞ .*

Proof. Since ℓ_∞ contains an isometric copy of $\ell_1(\mathfrak{c})$, we may fix a countable set Γ_0 in ω_1 and a family of unit vectors $(f_\alpha)_{\alpha < \omega_1}$ in the range of P_{Γ_0} that is isometrically equivalent to the unit vector basis of $\ell_1(\omega_1)$. Let $\alpha = \gamma(\alpha) + n(\alpha)$ be a countable ordinal, where $\gamma(\alpha)$ is a (possibly zero) limit ordinal and $n(\alpha)$ is a finite ordinal. We set

$$z_\alpha = e_\alpha + \frac{1}{n(\alpha)+1} z_\alpha \quad (\alpha < \omega_1),$$

where $(e_\alpha)_{\alpha < \omega_1}$ is the standard unit vector basis of $c_0(\omega_1) \subset \ell_\infty^c(\omega_1)$. Let Z be the closed linear span of z_α ($\alpha < \omega_1$) in $\ell_\infty^c(\omega_1)$. Then $(z_\alpha)_{\alpha < \omega_1}$ is a 1-unconditional basis for Z . Moreover the operator $P_{\Gamma_0}|_Z$ is injective, so Z cannot contain non-separable weakly compact sets as every weakly compact subset of ℓ_∞ is separable. In particular, X does not contain any isomorphic copies of $c_0(\omega_1)$.

By Lemma 2.3, Z does not embed into ℓ_∞ because the operator $P_{[0,\alpha]}|_X$ is not bounded below for any countable ordinal α .

Finally, we remark that P_{Γ_0} is an isomorphism when restricted to the copy of $\ell_1(\omega_1)$ spanned by the family $\{z_\alpha: \alpha < \omega_1, n(\alpha) = 0\}$. \square

Remark 2.5. As was noted in the proof of Theorem 2.4, the example Z does not embed into ℓ_∞ but there is an injective operator from Z into ℓ_∞ . The first space having these properties was constructed in [6], but that space does not have an unconditional basis.

2.1. An extension to higher densities. For every cardinal λ , there is a natural generalisation of the space $\ell_\infty^c(\Gamma)$; namely, $\ell_\infty^\lambda(\Gamma)$, the subspace of $\ell_\infty(\Gamma)$ that comprises functions whose supports have cardinality strictly less than λ . In this notation, $\ell_\infty^c(\Gamma) = \ell_\infty^{\omega_1}(\Gamma)$. We note that Theorem 2.1 has a natural counterpart for spaces $\ell_\infty^\lambda(\Gamma)$, whenever λ is a regular cardinal.

Theorem 2.6. *Let Γ be a set, λ a regular cardinal number, and let X be a subspace of $\ell_\infty^\lambda(\Gamma)$. Then the following assertions are equivalent:*

- (i) $w^*\text{-dens } X^* < \lambda$,
- (ii) X embeds into $\ell_\infty(\kappa)$ for some $\kappa < \lambda$,
- (iii) there exists a bounded linear injection from X into $\ell_\infty(\kappa)$ for some $\kappa < \lambda$,
- (iv) X does not contain a subspace that is isomorphic to $c_0(\lambda)$,
- (v) X does not contain a subspace that is isometrically isomorphic to $c_0(\lambda)$.

Proof. Note that (iii) \Rightarrow (i). Indeed, if there is a bounded linear injection T from X into $\ell_\infty(\kappa)$, then T^* has weak*-dense range. As (by Goldstine's theorem) the weak* density of $\ell_\infty(\kappa)^*$ is κ , the conclusion follows. The implication (iii) \Rightarrow (iv) follows from the fact that the weak* density of $c_0(\lambda)^*$ is λ and thus there is no bounded linear injection from $c_0(\lambda)$ to $\ell_\infty(\kappa)$ for $\kappa < \lambda$ (see, e.g., [3, Fact 4.10]). (We remark in passing that the implication (ii) \Rightarrow (iv) was proved directly in [5, Proposition 3.4].) As previously, it is enough to prove that (v) \Rightarrow (iii).

Assume contrapositively that for all $\kappa < \lambda$ there is no bounded linear injection from X into $\ell_\infty(\kappa)$. In particular, $|\Gamma| \geq \lambda$ as otherwise $\ell_\infty^\lambda(\Gamma) = \ell_\infty(\Gamma)$ but X is a subspace of $\ell_\infty^\lambda(\Gamma)$.

Let \mathcal{A} be a family of non-zero vectors in X that is maximal with respect to the property that the vectors have pairwise disjoint supports. If \mathcal{A} has cardinality λ , the conclusion follows as \mathcal{A} spans an isometric copy of $c_0(\lambda)$. So assume that $|\mathcal{A}| < \lambda$. Let

$$\Lambda = \bigcup_{f \in \mathcal{A}} \text{supp } f.$$

As λ is regular (and $|\Gamma| \geq \lambda$), $|\Lambda| < \lambda$. Consequently, by maximality of \mathcal{A} , the contractive projection $P_\Lambda : \ell_\infty^\lambda(\Gamma) \rightarrow \ell_\infty^\lambda(\Lambda)$ maps X injectively into $\ell_\infty(\Lambda)$; a contradiction \square

The conclusion of Theorem 2.6 fails for singular cardinal numbers. Indeed, let us take $\lambda = \omega_\omega = \lim_{n \rightarrow \infty} \omega_n$. The space $\ell_\infty(\omega_n)$ contains an isometric copy of the Hilbert space $\ell_2(\omega_n)$. In particular, the c_0 -direct sum of $\ell_2(\omega_n)$ ($n \in \mathbb{N}$) embeds isometrically into $\ell_\infty^\lambda(\lambda)$, has density λ , and is WCG (and even Asplund). On the other hand, it does not contain $c_0(\lambda)$.

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