THE CONVERSE PROBLEM FOR A GENERALIZED DHOMBRES FUNCTIONAL EQUATION

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Abstract. We consider the functional equation $f(xf(x)) = \varphi(f(x))$ where $\varphi\colon\ J \to J$ is a given homeomorphism of an open interval $J \subset (0, \infty)$ and $f\colon\ (0, \infty) \to J$ is an unknown continuous function. A characterization of the class $\mathcal{S}(J, \varphi)$ of continuous solutions f is given in a series of papers by Kahlig and Smítal 1998–2002, and in a recent paper by Reich et al. 2004, in the case when φ is increasing. In the present paper we solve the converse problem, for which continuous maps $f\colon\ (0, \infty) \to J$, where J is an interval, there is an increasing homeomorphism φ of J such that $f \in \mathcal{S}(J, \varphi)$. We also show why the similar problem for decreasing φ is difficult.

Keywords: iterative functional equation, equation of invariant curves, general continuous solution, converse problem

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1. Introduction

If not specified, by function we always mean a *continuous* function. We consider the functional equation

$$f(xf(x)) = \varphi(f(x)), \ x \in (0, \infty) \tag{1.1}$$

where $\varphi\colon\ J \to J$ is a given (surjective) homeomorphism of an interval $J \subset (0, \infty)$ onto itself, and $f\colon\ (0, \infty) \to J$ is an unknown function. Denote by $\mathcal{S}(J, \varphi)$ the class of solutions f of (1.1) with the range J.

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This equation is a special case of *equations of invariant curves.* A survey of general results can be found in [6] and [7]. Solutions of (1.1) with increasing φ have been studied, e.g., in [1]–[5], where another references can be found. While [4] contains characterization of the equations which have only monotone solutions, our last paper [5] contains a characterization of the class of continuous solutions of (1.1). We recall the main results.

1.1 Theorem (Cf. [3], [4].). *Let R_f denote the range of f. Assume φ is increasing, and f is a nonconstant solution of* (1.1).

(i) *If $1 \notin R_f$ then $R_f = (p, q)$ is an open interval, φ has no fixed point in (p, q), and the case $p > 1$ can be reduced to $q < 1$ by a suitable transformation. Moreover, if $q = 1$ then f is monotone.*

(ii) *If $1 \in R_f$ then f is monotone, 1 is a fixed point of φ, and $R_f = (p, q)$, $R_f = (p, 1]$ or $R_f = [1, q)$. Hence, in either of the last two cases, f must be constant on an interval $(0, a]$ or $[a, \infty)$. Moreover, the case $R_f = [1, q)$ can be reduced to $R_f = (p, 1]$.*

Thus, in view of the previous theorem, the case $1 \in R_f = (p, q)$ splits into two separate cases $R_f = (p, 1]$ and $R_f = [1, q)$, and f splits into two solutions $f_p = \min\{f, 1\}$ and $f_q = \max\{f, 1\}$. Consequently, the class $\mathcal{S}(J, \varphi)$ of solutions of (1.1) with $R_f = J$ an arbitrary interval is determined by the classes $\mathcal{S}(J, \varphi)$ with J, φ satisfying the conditions

$$J = (p, q), \ 0 \leqslant p < q \leqslant 1, \text{ and } \varphi(y) \neq y \text{ for } y \in J. \tag{1.2}$$

If not specified we assume (1.2) throughout the remainder of the paper.

The main result concerning monotone solutions is the following one.

1.2 Theorem (Cf. [4].). *If $q < 1$ then any continuous solution of* (1.1) *is monotone if and only if*

$$\varphi(y) < y \ \text{ in } \ J \ \text{ and } \ \prod_{k=0}^{\infty} \frac{\varphi^k(u)}{\varphi^k(v)} = \infty, \quad \text{for any } \ u > v \ \text{ in } \ J, \tag{1.3}$$

or

$$\varphi(y) > y \ \text{ in } \ J \ \text{ and } \ \prod_{k=1}^{\infty} \frac{\varphi^{-k}(u)}{\varphi^{-k}(v)} = \infty, \quad \text{for any } \ u > v \ \text{ in } \ J. \tag{1.4}$$

1.3 Remark. Assume (1.2). Then neither (1.3) nor (1.4) can be satisfied if $p > 0$, cf. [4]. Thus, by the above theorems, non-monotone solutions of (1.1) do exist if and only if one of the following three conditions is satisfied: (i) $0 < p < q < 1$; (ii) $0 = p < q < 1$, $\varphi(y) < y$ in J, and (1.3) is not true; (iii) $0 = p < q < 1$, $\varphi(y) > y$ in J, and (1.4) is not true.

Recall that by a piecewise monotone function defined on an interval we always mean a function with finite number of monotone pieces.

1.4 Theorem (Cf. [5].). *Assume $q < 1$. Then for any solution f of* (1.1) *there is a sequence $\{f_n\}_{n=1}^{\infty}$ of solutions converging uniformly on every compact set to f, and such that any f_n is piecewise monotone on every compact interval.*

1.5 Theorem (Cf. [5].). *Assume* (1.1) *has a solution which is not monotone. Then there is a solution f of* (1.1) *and a compact interval $I \subset (0,\infty)$ such that f is monotone on no subinterval of I.*

1.6 Remark. It is easy to see that the space $\mathcal{S}(J,\varphi)$ is closed with respect to the almost uniform convergence, i.e., convergence which is uniform on any compact subset of $(0,\infty)$. Consequently, by Theorem 1.5, the space $\mathcal{S}(J,\varphi)$ is the almost uniform closure of the set of solutions piecewise monotone on every compact interval. Theorems 1.4 and 1.5 imply that, for $q < 1$, this is a non-trivial statement.

The next section contains solution of the converse problem. Theorem 2.7 characterizes monotone functions which are solutions of (1.1) for a suitable φ, while Theorem 2.8 gives a characterization of the continuous solutions. Theorem 2.9 then shows that a typical nondecreasing function is a solution of (1.1) while a typical nonincreasing function and a typical continuous function fail to be solutions. By "typical" function we mean a function from a residual subset of the space of functions under consideration. Finally, in Section 3 we show that, for decreasing φ, the class $\mathcal{S}(J,\varphi)$ can be empty even in the case $J = (0,1)$; note that for increasing φ there are always nonconstant monotone solutions [3].

2. The converse problem for increasing φ

Throughout this section we assume (1.2). For simplicity we shall say that a homeomorphism ψ of J is *regular* if it is increasing and has no fixed points in J. For a function $f\colon (0,\infty) \to J$, we let τ_f, or simply τ, denote the function given by $\tau_f(x) = xf(x)$, for $x > 0$.

The next two results follow immediately from (1.1). Recall that $\mathcal{S}(J,\varphi)$ is the set of continuous solutions f of (1.1) with $R_f = J$. Obviously, for distinct homeomorphisms φ and ψ of J, $\mathcal{S}(J,\varphi) \cap \mathcal{S}(J,\psi) = \emptyset$.

2.1 Lemma. *Let $f \in \mathcal{S}(J,\varphi)$, and let φ be regular. Then τ_f is increasing.*

Proof. Assume that $\tau_f(x_1) < \tau_f(x_2)$ and $\tau_f(x_3) < \tau_f(x_2)$, for some $x_1 < x_2 < x_3$. Since τ_f is continuous, there are u in (x_1,x_2) and v in (x_2,x_3) such that $\alpha := \tau_f(u) = \tau_f(v)$. Then $f(u) > f(v)$ and consequently, $f(\alpha) = f(uf(u)) = \varphi(f(u)) >$

$\varphi(f(v)) = f(vf(v)) = f(\alpha)$ which is impossible. Similarly if $\tau_f(x_1), \tau_f(x_3) < \tau_f(x_2)$. Thus, τ_f is strictly monotone. To finish the argument assume that τ_f is decreasing. Then, for $u < v$, $u/v > f(v)/f(u)$, and since $f < 1$, letting $u \to \infty$ we obtain $0 > f(v) > 0$. $\square$

The next lemma follows directly from (1.1).

2.2 Lemma. *Let f be a continuous increasing map $(0,\infty)$, with $R_f = J$, and let $\varphi(y) := f(yf^{-1}(y))$, for $y \in J$. Then φ is a regular homeomorphism of J such that $f \in \mathcal{S}(J,\varphi)$.*

2.3 Lemma. *Let f be a decreasing continuous function on $(0,\infty)$, with $R_f = J$. Then there is a regular homeomorphism φ of J such that $f \in \mathcal{S}(J,\varphi)$ if and only if τ_f is strictly increasing.*

Proof. One implication follows since if τ is strictly increasing then, similarly as in Lemma 2.2 it suffices to take $\varphi(y) := f(yf^{-1}(y))$, for $y \in J$. The other implication follows by Lemma 2.1. $\square$

2.4 Definition. Let $\mathcal{L}$ be a family of level sets of a function $f\colon (0,\infty) \to J$. Thus, $\mathcal{L}$ consists of sets $f^{-1}(\{y\})$, for y in an $A \subset J$. Then $\mathcal{L}$ is said to be *τ_f-consistent* if it has a decomposition $\mathcal{L} = \bigcup_{t\in T} \mathcal{L}_t$ into τ_f-orbits $\mathcal{L}_t = \{\tau_f^n(f^{-1}(\{y\}))\}_{n=-\infty}^{\infty}$, for any $t \in T$.

2.5 Lemma. *Let f be a continuous function on $(0,\infty)$, with $R_f = J$. If $f \in \mathcal{S}(J,\varphi)$ for a regular homeomorphism φ of J then the system $\mathcal{L}$ of level sets of f is τ_f-consistent.*

Proof. Assume that $f \in \mathcal{S}(J,\varphi)$. Since, by Lemma 2.1, τ is increasing, f is constant on $\tau(K)$, for any $K \in \mathcal{L}$. Since τ is invertible, (1.1) implies $f(z) = \varphi^{-1}(f(\tau^{-1}(\{z\})))$ and consequently, f is constant on the τ-preimage of any level set K. Hence, $\mathcal{L}$ is τ-consistent. $\square$

2.6 Lemma. *Let f be a continuous function on $(0,\infty)$ with $R_f = J$, and such that*

$$\lim_{x\to 0} f(x) \in \{p,q\}. \tag{2.1}$$

Assume that τ is strictly increasing, and that the system $\mathcal{L}$ of the level sets of f is τ-consistent. Then there is a regular homeomorphism φ of J such that $f \in \mathcal{S}(J,\varphi)$.

Proof. Put $B = \min\{f^{-1}(\{y\}); y \in J\}$. Since $f|B$ is a bijection $B \to J$, it has the inverse $g\colon J \to B$, not necessarily continuous. Let

$$\varphi(y) = f(yg(y)), \quad \text{for} \quad y \in J. \tag{2.2}$$

For any $y \in J$ let $x = g(y)$. Then (2.2) can be rewritten to

$$\varphi(f(x)) = f(\tau(x)), \quad \text{for} \quad x \in B. \tag{2.3}$$

For an $x \in B$, let $K(x) \in \mathcal{L}$ be the level set containing x. Since $\mathcal{L}$ is τ-consistent and τ is continuous and increasing, (2.3) implies

$$f(\tau(K(x))) = \varphi(f(K(x))), \quad \text{for any} \quad x \in B. \tag{2.4}$$

Since any $K \in \mathcal{L}$ is of the form $K = K(x)$, for some $x \in B$, (2.3) and (2.4) imply that (1.1) is satisfied for any $x > 0$.

Since $\mathcal{L}$ is τ-consistent we have $\tau(B) = B$ whence, by (2.3), $R_\varphi = J$. Hence to show that φ is a homeomorphism it suffices to show that it is increasing. For, $f|B$ being strictly monotone on B, it is increasing if the limit in (2.1) equals p, and it is decreasing if the limit equals q. Since τ is increasing, (2.3) implies that φ in either case is increasing. The regularity of φ (or rather, the fact that φ has no fixed point) now follows by Theorem 1.1 since $f \in \mathcal{S}(J, \varphi)$. □

2.7 Theorem. *Let f be a continuous monotone function on $(0, \infty)$, with $R_f = J$. Then $f \in \mathcal{S}(J, \varphi)$, for a regular homeomorphism φ of J if and only if one of the following conditions is satisfied:*

(i) *f is increasing;*

(ii) *f is decreasing and τ is increasing;*

(iii) *f is nondecreasing and the system of maximal intervals of constancy of f is τ-consistent;*

(iv) *f is nonincreasing, τ is increasing, and the system of intervals of constancy of f is τ-consistent.*

Proof. The first two conditions are given in Lemmas 2.2 and 2.3. Condition (iii) follows by Lemmas 2.5 and 2.6 since, for a monotone function, the level sets are τ-consistent if and only if the maximal intervals of constancy are consistent. Similarly, (iv) follows by Lemmas 2.1, 2.5 and 2.6. □

The condition (iv) of Theorem 2.7 can be easily modified to an arbitrary continuous function f satisfying the necessary condition that $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$ are distinct points in $\{p, q\}$.

2.8 Theorem. *Let f be a continuous function on $(0, \infty)$, with $R_f = J$. Then $f \in \mathcal{S}(J, \varphi)$, for a regular homeomorphism φ of J if and only if $\lim_{x\to\infty} f(x) \in \{p, q\}$, τ is increasing, and the system of level sets of f is τ-consistent.*

Proof. Condition $\lim_{x\to\infty} f(x) \in \{p, q\}$ is necessary whenever φ is a regular homeomorphism; this can be easily verified (see also [3]). The necessity of the other two conditions follows by Lemmas 2.1 and 2.5. Finally, Lemma 2.6 gives the sufficient condition. $\square$

In the sequel let, for an open interval J, $\mathcal{M}_+(J)$ and $\mathcal{M}_-(J)$ denote the class of continuous nondecreasing, resp. nonincreasing functions from $(0,\infty)$ onto the closure of J. Let $\mathcal{G}(J)$ be the class of continuous functions f from $(0,\infty)$ onto the closure of J such that $\lim\limits_{x\to\infty} f(x) \in \{p,q\}$. Let $\mathcal{S}(J) = \bigcup\limits_{\varphi} \mathcal{S}(J,\varphi)$, where the union is taken over all regular homeomorphisms φ of J. Finally, let $\mathcal{S}_+(J) = \mathcal{S}(J) \cap \mathcal{M}_+(J)$, and similarly define $\mathcal{S}_-(J)$. Obviously, $\mathcal{M}_+(J)$ and $\mathcal{M}_-(J)$ are complete metric spaces with respect to the uniform metric. The following result states that, roughly speaking, a typical nondecreasing function is a solution of (1.1) for some φ, but a typical nonincreasing function as well as a typical continuous "globally" monotone function is not a solution.

2.9 Theorem.
(i) $\mathcal{S}_+(J)$ *is residual in* $\mathcal{M}_+(J)$*;*
(ii) $\mathcal{S}_-(J)$ *is nowhere dense in* $\mathcal{M}_-(J)$*;*
(iii) $\mathcal{S}(J)$ *is nowhere dense in* $\mathcal{G}(J)$*.*

P r o o f. It is well-known that the class of strictly increasing functions from $(0,\infty)$ onto J is residual in $\mathcal{M}_+(J)$. This follows by the fact that the set $\mathcal{M}_+^n(J)$ consisting of f in $\mathcal{M}_+(J)$ which have no interval of constancy $K \subset (0,n]$ of length greater than $1/n$, is nowhere dense in $\mathcal{M}_+(J)$. This proves (i), by Lemma 2.2.

To prove (ii), let G be an open (in the uniform topology) neighborhood of an $f \in \mathcal{M}_-(J)$. It is easy to see that there is a function $g \in \mathcal{M}_-(J) \cap G$ such that $ug(u) > vg(v)$, for some $u < v$. But then τ_g is not increasing and, by the continuity, the same is true for any h belonging to an open neighborhood $H \subset G$ of g in $\mathcal{M}_-(J)$. Consequently, by Lemma 2.1, $H \cap \mathcal{S}(J) = \emptyset$.

Proof of (iii) is similar and we omit it. $\square$

3. The converse problem for decreasing φ

While the solutions of (1.1) in the regular case (i.e., with φ an increasing homeomorphism) are completely characterized, it seems to be difficult to obtain similar results as, e.g., in Theorems 1.1, 1.2, 1.4 and 1.5 for φ decreasing. On the other hand, we conjecture that the converse problem for φ decreasing is solvable and characterization as in Theorems 2.7 and 2.8 would be possible.

The essential difference between this and the regular case is that for certain decreasing homeomorphisms φ there are no nonconstant continuous solutions at all. This can be indicated by the following two examples. They exhibit decreasing homeomorphisms of an open interval J such that any point is periodic with period 1 or 2.

In the first case there is a φ and an uncountable nested family of compact subintervals of J, each being the range of a continuous solution of (1.1); note that by Theorem 1.1, this is impossible if φ is increasing. In the second case, (1.1) has only a constant solution.

3.1 Example. Let $\alpha \in (0,1)$, and let $\varphi(y) = \alpha/y$, for $y \in J = (0,\infty)$. Thus, $\sqrt{\alpha}$ is a fixed point and any $y \neq \alpha$ in J is a periodic point of φ of period 2. However, for any β such that $0 < \beta < \alpha/\beta < 1$ there is a continuous solution f of (1.1) with $R_f = [\beta, \alpha/\beta]$.

Proof. It is easy to see that both mappings

$$\Phi(x,y) = \left(xy, \frac{\alpha}{y}\right) \quad \text{and} \quad \Phi^{-1}(x,y) = \left(\frac{xy}{\alpha}, \frac{\alpha}{y}\right) \tag{3.1}$$

are continuous bijections of the strip $H_\beta = (0,\infty) \times [\beta, \alpha/\beta]$ onto itself. Thus, Φ is a homeomorphism of H_β. It is easy to see that a not necessarily surjective function $f\colon (0,\infty) \to J$ is a solution of (1.1) if $\Phi(f) \subset f$ (cf. also [3]); here we identify a function with its graph. Moreover, the second iterates of the functions (3.1) are given by

$$\Phi^2(x,y) = (x\alpha, y), \quad \text{and} \quad \Phi^{-2}(x,y) = (x/\alpha, y). \tag{3.2}$$

To prove the theorem it suffices to find a continuous map f with range $[\beta, \alpha/\beta]$ such that $\Phi(f) = f$. Fix an $x_0 > 0$, put $y_0 = \alpha/\beta$, and let $\{(x_n, y_n)\}_{n=-\infty}^{\infty}$ be the full orbit of (x_0, y_0), i.e., the sequence such that $\Phi(x_n, y_n) = (x_{n+1}, y_{n+1})$, for any integer n. Then $\{y_n\}_{n=-\infty}^{\infty}$ is an alternating sequence with terms β and α/β, and $\{x_n\}_{n=0}^{\infty}$ is a decreasing sequence tending to 0. Let f_0 be any increasing continuous function on $[x_1, x_0]$, with values $y_1 = f_0(x_1)$, and $y_0 = f_0(x_0)$ at the endpoints. For any integer n define f_n as $\Phi^n(f_0)$. Since $x \mapsto xy$ is increasing on $[x_1, x_0]$ and maps this interval onto $[x_2, x_1]$, $f_1\colon [x_2, x_1] \to [\beta, \alpha/\beta]$ is continuous and decreasing. Similarly, $f_{-1}\colon [x_0, x_{-1}] \to [\beta, \alpha/\beta]$ is continuous and decreasing. By (3.2), f_2 and f_{-2} are increasing, etc. An induction argument yields that for any integer n, $f_n = \Phi^n(f_0)$ is a continuous strictly monotone map $[x_{n+1}, x_n] \to [\beta, \alpha/\beta]$, attaining the values β, α/β only at the endpoints x_{n+1} and x_n. To finish the construction put $f = \bigcup\limits_{n=-\infty}^{\infty} f_n$. It is easy to see that $\Phi(f) = f$. $\square$

3.2 Example. Let $\varphi(y) = 1 - y$ in $J = (0,1)$. Then $\frac{1}{2}$ is a fixed point and any $y \neq \frac{1}{2}$ in J is a periodic point of φ of period 2, but (1.1) has no continuous solution different from $f \equiv \frac{1}{2}$.

Proof. Assume there is a nonconstant solution f. Then there is an $\alpha > \frac{1}{2}$ in J such that $f(x_0) = \alpha$, for some x_0. Since the range R_f of f is an interval, and $\varphi(R_f) \subset R_f$, we have $\frac{1}{2} \in R_f$ (note that $\alpha = \varphi^2(\alpha)$ is a periodic point of φ). Let Φ be a homeomorphism of the strip $H = (0,\infty) \times (0,1)$ onto itself, given by $\Phi(x,y) = (xy, 1-y)$. Then $\Phi^2(x,y) = (xy(1-y), y)$, and since f is a solution, $\Phi(f) \subset f$ whence, $\Phi^2(f) \subset f$. By induction,

$$\Phi^{2k}(x,y) = (x[y(1-y)]^k, y) \quad \text{and} \quad \Phi^{2k}(f) \subset f. \tag{3.3}$$

Let $\{x_n, y_n\}_{n=-\infty}^{\infty}$ be the Φ-orbit of (x_0, y_0), with $y_0 = \alpha$. Let $z_0 < x_0$ be the maximal point such that $f(z_0) = \frac{1}{2}$. By (3.3), $z_{2k} = z_0 4^{-k}$ and $x_{2k} = x_0[\alpha(1-\alpha)]^k$, and since $\alpha(1-\alpha) < \frac{1}{4}$, we have $\lim\limits_{k\to\infty} z_{2k}/x_{2k} = +\infty$. Thus, there is an n such that $x_{2n} < x_{2n-2} < z_{2n}$. Then the range of $f_{2n} = f|(x_{2n}, z_{2n})$ contains the fixed point $\frac{1}{2}$, while the range $(\frac{1}{2}, \alpha)$ of $\Phi^{2n}(f|(z_0, x_0))$ does not (note that $X = (\frac{1}{2}, \alpha)$). But $\Phi^{2n}(f|(z_0, x_0))$ is a (graph of) function $g_{2n}\colon (x_{2n}, z_{2n}) \to J$. This contradicts the fact that $\Phi^{2n}(f) \subset f$ since f_{2n} and g_{2n} are continuous. $\square$

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