ON DISCRETENESS OF SPECTRUM OF A FUNCTIONAL DIFFERENTIAL OPERATOR

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Abstract. We study conditions of discreteness of spectrum of the functional-differential operator

$$\mathcal{L}u = -u'' + p(x)u(x) + \int_{-\infty}^{\infty} (u(x) - u(s))\,\mathrm{d}_s r(x,s)$$

on $(-\infty,\infty)$. In the absence of the integral term this operator is a one-dimensional Schrödinger operator. In this paper we consider a symmetric operator with real spectrum. Conditions of discreteness are obtained in terms of the first eigenvalue of a truncated operator. We also obtain one simple condition for discreteness of spectrum.

Keywords: spectrum; functional differential operator

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1. The problem

1.1. Introduction. The first result about discreteness of the spectrum for the Schrödinger operator

$$\mathcal{L}_0 u = -u'' + pu \tag{1.1}$$

where $u(x)$ is defined on the whole axis $\mathbb{R} = (-\infty,\infty)$ and $p(x)$ assumed to be continuous (and its n-dimensional variant) was obtained by K. Friedrichs [4], [5]. The spectrum is discrete and bounded from below if $\lim\limits_{x\to\infty} p(x) = +\infty$. A necessary and sufficient condition of discreteness of spectrum for the differential operator (1.1) was obtained by A. M. Molchanov [14]. The spectrum is discrete and bounded from below if and only if for any $a > 0$

$$\lim_{x\to\infty} \int_x^{x+a} p(t)\,\mathrm{d}t = +\infty.$$

Note the result of R. S. Ismagilov [8]: let $\lambda(\Delta)$ be the minimal eigenvalue of the operator $-u'' + pu$ considered on the segment Δ with Dirichlet conditions on Δ. For discreteness and boundedness from below of the spectrum of the operator $\mathcal{L}_0$ a necessary and sufficient condition is that $\lambda(\Delta) \to \infty$ when Δ moves to ∞ conserving its length. But the same result can be seen in the article of A. M. Molchanov [14]. Molchanov called this the *principle of localization*.

For further generalizations see for example [13] and references therein.

Here we study the functional differential operator

$$\mathcal{L}u(x) = -u''(x) + p(x)u(x) + \int_{-\infty}^{\infty} (u(x) - u(s))\,\mathrm{d}_s r(x,s) \tag{1.2}$$

on $x \in (-\infty, \infty)$. This expression contains an expression with deviating argument as a special case:

$$-u'' + p(x)u(x) + \sum_{i=1}^{n} q_i(x)(u(x) - u(h_i(x))).$$

Expression (1.2) is not only a generalization but may perhaps also have applications in quantum mechanics. In the case of finite interval $[0, l]$ this operator describes the behavior of a loaded string. The singular problem

$$-(pu')' + qu + \int_0^l (u(x) - u(s))\,\mathrm{d}_s r(x,s) = \lambda \varrho u$$

with Sturm-Liouville boundary conditions is studied in [11], [12]. A particular case

$$\mathcal{L}_1 u = -u'' + p(x)u(x) + q(x)(u(x) - u(x-\delta)) + q(x+\delta)(u(x) - u(x+\delta))$$

of (1.2) is considered in [7].

Our aim is to generalize the principle of localization. However, for the operator (1.2) it cannot be obtained directly. This is a special feature of an ordinary differential operator. We introduce a pseudo eigenvalue $\widetilde{\mu}(\Delta)$, and use it to compare it with the eigenvalues of the *truncated* problem.

1.2. Results. This subsection summarizes the main results of the paper. Assume that the function p in (1.2) is locally integrable (Lebesgue integrable on any segment), and essentially bounded from below. We can assume that $p(x) \geqslant 1$. The function $r(x,s)$ is nondecreasing in s on $\mathbb{R}$ for almost all $x \in \mathbb{R}$, measurable and locally integrable in x for any $s \in \mathbb{R}$. We also assume that the function $\xi(x,s) = \int_0^x r(t,s)\,\mathrm{d}t$ is symmetric: $\xi(x,s) = \xi(s,x)$, $x, s \in \mathbb{R}$. Denote $q(x) = r(x,\infty) - r(x,-\infty)$.

Let $\Delta = [a,b] \subset (-\infty, \infty)$, and

$$\mathcal{L}_\Delta u = -u'' + p(x)u(x) + \int_a^b (u(x) - u(s))\,\mathrm{d}_s r(x,s). \tag{1.3}$$

It may be called a *truncated* operator. Consider two eigenvalue problems

$$\mathcal{L}_\Delta u = \lambda u, \quad u(a) = u(b) = 0 \tag{1.4}$$

and

$$\mathcal{L}_\Delta u = \mu u, \quad u'(a) = u'(b) = 0. \tag{1.5}$$

Let $\lambda(\Delta)$ be the minimal eigenvalue of the problem (1.4), and $\mu(\Delta)$ the minimal eigenvalue of the problem (1.5).

Theorem 1.1. *For discreteness of the spectrum of $\mathcal{L}$ it is sufficient that one of the following conditions holds:*
▷ *spectrum of $\mathcal{L}_0$ is discrete,*
▷ *for any sequence of segments Δ_n of fixed length that tend to infinity,*

$$\lim \mu(\Delta_n) = \infty. \tag{1.6}$$

Thus, if $\lim\limits_{x\to\infty} \int_x^{x+a} p(t)\,\mathrm{d}t = \infty$ for any $a > 0$, then the spectrum of operator (1.2) is discrete.

Let us introduce the following condition:

$$M = \operatorname*{ess\,sup}_{x\in\mathbb{R}} \frac{q(x)}{p(x)} < \infty. \tag{1.7}$$

Theorem 1.2. *Suppose* (1.7) *holds. For discreteness of the spectrum of* (1.2) *it is necessary that the relation*

$$\lim_{n\to\infty} \lambda(\Delta_n) = \infty \tag{1.8}$$

holds for any sequence of segments Δ_n of fixed length that tend to infinity.

Theorem 1.3. *Suppose the condition* (1.7) *holds, then the spectra of both the operators $\mathcal{L}$ and $\mathcal{L}_0$ are discrete or neither of them is discrete.*

2. Abstract scheme

We use a simple scheme, sufficient for our purpose. In contrast to the general spectral theory [1], [2], we avoid the use of unbounded operators. But actually this scheme is the same as that in [2], Chapter 10, except for notation. We also find it convenient explicitly use the *embedding* T from W to H (see below). This scheme is also used in [10], [11], [12].

Let W and H be Hilbert spaces with inner products $[u, v]$ and (f, g), respectively. Let $T\colon W \to H$ be a linear bounded operator. The equation

$$[u, v] = (f, Tv), \quad \forall v \in W, \tag{2.1}$$

has a unique solution $u = T^* f$ for any $f \in H$, where T^* is the adjoint operator. Let $D_{\mathcal{L}} = T^*(H)$. Assume that

(1) the image $T(W)$ of the operator T is dense in H,

(2) $\dim \ker T = 0$.

Lemma 2.1. *If the image $T(W)$ of the operator T is dense in H, then T^* is an injection.*

Proof. Suppose $T^* f = 0$ for a $f \in H$. Then for any $g \in T(W)$

$$(f, g) = (f, Tu) = [T^* f, u] = 0.$$

Since $T(W)$ is dense in H, $f = 0$. □

Corollary 2.1 (Euler equation). *The operator T^* has an inverse $\mathcal{L}$ defined on the set $D_{\mathcal{L}}$. The equation* (2.1) *is equivalent to*

$$\mathcal{L}u = f. \tag{2.2}$$

The spectral problem for the operator $\mathcal{L}$ we write in the form

$$\mathcal{L}u = \lambda T u. \tag{2.3}$$

Let λ_0 be the greatest lower bound of the spectrum of $\mathcal{L}$. It is well known (see for example [2], Chapter 6) that

$$\lambda_0 = \inf_{u \neq 0} \frac{(\mathcal{L}u, Tu)}{(Tu, Tu)}.$$

Since $(\mathcal{L}u, Tu) = [T^*\mathcal{L}u, u] = [u, u]$,

$$\lambda_0 = \inf_{u \neq 0} \frac{[u,u]}{(Tu, Tu)} = \|T\|^{-2}. \tag{2.4}$$

Since the equation (2.3) is equivalent to $u = \lambda T^* T u$, discreteness of the spectrum of the problem (2.3) is equivalent to compactness of T^*T. However, both the operators T^*T and T^* are compact [2], Chapter 10. Thus the following theorem holds.

Theorem 2.1. *The spectrum of $\mathcal{L}$ is discrete if and only if T is compact.*

Theorem 2.2. *Suppose T is compact. Then the equation* (2.3) *has a nonzero solution u_n only in the case of $\lambda = \lambda_n$, $n = 0, 1, 2, \ldots$, i.e.*

$$\mathcal{L}u_n = \lambda_n T u_n, \quad n = 1, 2, \ldots$$

The system u_n forms an orthogonal basis in W. The sequence λ_n forms a nondecreasing sequence of positive numbers

$$0 < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots$$

and $\lim \lambda_n = \infty$.

Remark 2.1. The minimal eigenvalue satisfies the equality (2.4).

3. Notation and important relations

According to the scheme in Section 2, we introduce two spaces W and H.

3.1. Basic notation. Let $L_2(S, p)$ be the space[1] of square integrable on S with the weight p functions, $L_2(S) = L_2(S, 1)$. Let $\mathbb{R} = (-\infty, \infty)$, let $L_2 = L_2(\mathbb{R})$ be the Hilbert space of funcions measurable and square integrable on $\mathbb{R}$ with scalar product

$$(f, g) = \int_{\mathbb{R}} f(x) g(x) \, \mathrm{d}x. \tag{3.1}$$

Let us consider real functions having in view complex functions involved in the spectral problem. Let

$$[u, v] = \int_{-\infty}^{\infty} (u'v' + puv) \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}\times\mathbb{R}} (u(x) - u(s))(v(x) - v(s)) \, \mathrm{d}\xi, \tag{3.2}$$

[1] where S is a measurable space; we accept also the measure, instead of the weight

where the function $\xi(x,s) = \int_0^x r(t,s)\,\mathrm{d}t$ defines a measure on $\mathbb{R}\times\mathbb{R}$. It is easy to see that this form is symmetric independently of the symmetry of ξ.

Let W be the set of all functions u absolutely continuous on any segment $[a,b] \subset \mathbb{R}$ such that $[u,u] < \infty$. Then W is a Hilbert space with inner product $[u,v]$ (Lemma 5.1). Let $T\colon W \to L_2$ be the operator defined by the equality $Tu(x) = u(x)$, $x \in \mathbb{R}$. This operator is continuous (Lemma 5.2).

We can now use the scheme from Section 2. Lemma 5.5 asserts that the operator $\mathcal{L}$ (see (1.2)) is associated with the form (3.2):

$$\boxed{\text{form (3.2)}} \to \boxed{\text{operator (1.2)}}.$$

Thus from Theorem 2.1 we have

Theorem 3.1. *The spectrum of $\mathcal{L}$ is discrete if and only if the operator T is compact.*

3.2. More notation. We need the analogous notation for a finite interval. Let $\Delta \subset \mathbb{R}$ be a measurable subset (we will use mainly a segment $[a,b] \subset \mathbb{R}$), and

$$(f,g)_\Delta = \int_\Delta f(x)g(x)\,\mathrm{d}x.$$

Introduce two *truncated* forms. For $u, v \in W$

$$[u,v]_\Delta = \int_\Delta (u'v' + puv)\,\mathrm{d}x + \frac{1}{2}\int_{\Delta\times\mathbb{R}} (u(x)-u(s))(v(x)-v(s))\,\mathrm{d}\xi.$$

Integration on $\Delta\times\mathbb{R}$ signifies that one variable is in Δ but the other is in $\mathbb{R}$ (for example, $x \in \Delta$, $s \in \mathbb{R}$). Note that if $\Delta = \Delta_1 \cup \Delta_2$, $\Delta_1 \cap \Delta_2 = \emptyset$, then

$$[u,u]_\Delta = [u,u]_{\Delta_1} + [u,u]_{\Delta_2}. \tag{3.3}$$

The second *truncated* form is only for functions defined on a segment $\Delta = [a,b]$:

$$[u,v]^*_\Delta = \int_\Delta (u'v' + puv)\,\mathrm{d}x + \frac{1}{2}\int_{\Delta\times\Delta} (u(x)-u(s))(v(x)-v(s))\,\mathrm{d}\xi.$$

Let W_Δ be the set of functions absolutely continuous on Δ, satisfying the inequality

$$[u,u]^*_\Delta < \infty.$$

The same abstract scheme from Section 2 can be applied to the form $[u, v]^*$. So, this corresponds to the operator $\mathcal{L}_\Delta$ (see (1.3)):

$$[u, u]^*_\Delta \to \text{operator } \mathcal{L}_\Delta.$$

We use two different spaces, the actual W_Δ and the subspace $\{u \in W_\Delta \colon u(a) = u(b) = 0\}$. For each of these spaces the scheme from Section 2 can be used. For the former we have the corresponding spectral problem (1.5), for the latter it is (1.4). Thus, from (2.4) we have the equalities

$$\lambda(\Delta) = \inf_{\substack{u \in W_\Delta, u \neq 0 \\ u(a)=u(b)=0}} \frac{[u, u]^*_\Delta}{(Tu, Tu)_\Delta}, \tag{3.4}$$

$$\mu(\Delta) = \inf_{u \in W_\Delta, u \neq 0} \frac{[u, u]^*_\Delta}{(Tu, Tu)_\Delta}. \tag{3.5}$$

We also need similar eigenvalues for the ordinary operator $\mathcal{L}_0$ to be considered on the segment Δ only. Let

$$[u, v]^0_\Delta = \int_\Delta (u'v' + puv)\,\mathrm{d}x$$

and let W^0_Δ be the set of functions absolutely continuous on Δ, satisfying the inequality

$$[u, u]^0_\Delta < \infty.$$

Denote the corresponding minimal eigenvalues of the operator $\mathcal{L}_0$ on Δ by $\lambda_0(\Delta)$ and $\mu_0(\Delta)$. Then

$$\lambda_0(\Delta) = \inf_{\substack{u \in W^0_\Delta, u \neq 0 \\ u(a)=u(b)=0}} \frac{[u, u]^0_\Delta}{(Tu, Tu)_\Delta}, \tag{3.6}$$

$$\mu_0(\Delta) = \inf_{u \in W^0_\Delta, u \neq 0} \frac{[u, u]^0_\Delta}{(Tu, Tu)_\Delta}. \tag{3.7}$$

The equalities (3.4), (3.5), (3.6), (3.7) immediately imply the inequalities

$$\mu(\Delta) \leqslant \lambda(\Delta), \quad \mu_0(\Delta) \leqslant \lambda_0(\Delta), \tag{3.8}$$

and

$$\lambda_0(\Delta) \leqslant \lambda(\Delta), \quad \mu_0(\Delta) \leqslant \mu(\Delta). \tag{3.9}$$

Introduce one more value, analogous to $\mu(\Delta)$. It is

$$\widetilde{\mu}(\Delta)=\inf_{u\in W,u\neq 0}\frac{[u,u]_\Delta}{(Tu,Tu)_\Delta}. \tag{3.10}$$

For any segment Δ we have

$$\mu(\Delta)\leqslant\widetilde{\mu}(\Delta). \tag{3.11}$$

This follows from the inequality

$$[u,u]^*_\Delta=[u,u]_\Delta-\frac{1}{2}\int_{\Delta\times(\mathbb{R}\setminus\Delta)}(u(x)-u(s))^2\,\mathrm{d}\xi\leqslant[u,u]_\Delta.$$

The principle of localization in our case can be expressed by means of a pseudo-eigenvalue $\widetilde{\mu}(\Delta)$ (Corollary 5.1 to Lemma 5.8):

Theorem 3.2. *The spectrum of $\mathcal{L}$ is discrete if and only if $\widetilde{\mu}(\Delta)\to\infty$, when the segment $\Delta\to\infty$, for Δ of any fixed length.*

To conclude this section we present two auxiliary statements.

3.3. Two lemmas.

Lemma 3.1. *Suppose* (1.7) *holds. Then for any Δ*

$$\lambda(\Delta)\leqslant(1+2M)\lambda_0(\Delta). \tag{3.12}$$

Proof. Let $u\in W_\Delta$. We can estimate

$$\begin{aligned}\frac{1}{2}\int_{\Delta\times\Delta}(u(x)-u(s))^2\,\mathrm{d}\xi&\leqslant\int_{\Delta\times\Delta}(u(x)^2+u(s)^2)\,\mathrm{d}\xi=2\int_{\Delta\times\Delta}u(x)^2\,\mathrm{d}\xi\\&=2\int_\Delta u(x)^2\,\mathrm{d}x\int_\Delta\mathrm{d}_s r(x,s)\leqslant2\int_\Delta q(x)u(x)^2\,\mathrm{d}x.\end{aligned}$$

Thus

$$\begin{aligned}[u,u]^*_\Delta&\leqslant[u,u]^0_\Delta+2\int_\Delta q(x)u(x)^2\,\mathrm{d}x\\&\leqslant[u,u]^0_\Delta+2M\int_\Delta p(x)u(x)^2\,\mathrm{d}x\leqslant(1+2M)[u,u]^0_\Delta.\end{aligned}$$

The statement (3.12) follows from (3.4), (3.6). □

Lemma 3.2. *Suppose* (1.7) *holds. Let* Δ *be a segment,* $u \in W$*, and* $u(x) = 0$ *if* $x \notin \Delta$*. Then*

$$[u,u]_\Delta \leqslant \Big(1 + \frac{1}{2}M\Big)[u,u]^*_\Delta. \tag{3.13}$$

Proof.

$$\frac{1}{2}\int_{\Delta\times(\mathbb{R}\setminus\Delta)} (u(x)-u(s))^2\,\mathrm{d}\xi = \frac{1}{2}\int_{\Delta\times(\mathbb{R}\setminus\Delta)} u(x)^2\,\mathrm{d}\xi = \frac{1}{2}\int_\Delta u(x)^2\,\mathrm{d}x \int_{\mathbb{R}\setminus\Delta} \mathrm{d}_s r(x,s)$$
$$\leqslant \frac{1}{2}\int_\Delta q(x)u(x)^2\,\mathrm{d}x.$$

Hence

$$[u,u]_\Delta \leqslant [u,u]^*_\Delta + \frac{1}{2}\int_\Delta q(x)u(x)^2\,\mathrm{d}x \leqslant \Big(1+\frac{1}{2}M\Big)[u,u]^*_\Delta.$$

□

4. Proofs of theorems

4.1. Proof of Theorem 1.1. For discreteness of the spectrum of $\mathcal{L}_0$ it is necessary and sufficient that $\mu_0(\Delta) \to \infty$ when $\Delta \to \infty$ conserving its length [14]. In view of inequalities (3.9) and (3.11) and Corollary 5.1 to Lemma 3.2 operator T is compact. Hence the spectrum of $\mathcal{L}$ is discrete. □

4.2. Proof of Theorem 1.2. Suppose T is compact. Let Δ be a segment, and let u be the eigenfunction of the problem (1.4) that corresponds to the eigenvalue $\lambda(\Delta)$. We can define $u(x) = 0$ out of the segment Δ. By virtue of Lemma 3.2

$$\lambda(\Delta) = \frac{[u,u]^*_\Delta}{(Tu,Tu)_\Delta} \geqslant \frac{2}{(2+M)}\frac{[u,u]_\Delta}{(Tu,Tu)_\Delta} \geqslant \frac{2}{(2+M)}\widetilde{\mu}(\Delta) \to \infty, \quad \text{if } N \to \infty.$$

□

4.3. Proof of Theorem 1.3. From Lemma 3.1 and from (3.4), (3.6) it follows that for any segment Δ

$$\lambda(\Delta) \leqslant (1+2M)\lambda_0(\Delta).$$

If the spectrum of $\mathcal{L}$ is discrete then $\lambda(\Delta) \to \infty$ when $\Delta \to \infty$. Then $\lambda_0(\Delta) \to \infty$. But this is the condition of Ismagilov for discreteness of the spectrum of $\mathcal{L}_0$. □

5.1. Properties of the space W.

Lemma 5.1. *The space W is a Hilbert space.*

Proof. The integral $\int_{\mathbb{R}\times\mathbb{R}}(u(x)-u(s))(v(x)-v(s))\,\mathrm{d}\xi$ is finite (convergent), if $u,v\in W$. Thus $[u,v]$ in (3.2) is defined correctly. Now we have to show that W is complete. Let u_n be a sequence satisfying

$$\|u_n-u_m\|^2=\int_{-\infty}^{\infty}((u_n'-u_m')^2+p(x)(u_n-u_m)^2)\,\mathrm{d}x + \int_{\mathbb{R}\times\mathbb{R}}((u_n(x)-u_m(x))-(u_n(s)-u_m(s)))^2\,\mathrm{d}\xi\to 0, \tag{5.1}$$

when $n,m\to 0$. Then there exist two functions $u\in L_2(\mathbb{R},p)$ and $\varphi\in L_2(\mathbb{R})$ such that $u_n\to u$ in $L_2(\mathbb{R},p)$ and $u_n'\to\varphi$ in $L_2(\mathbb{R})$.

Let $[a,b]$ be an arbitrary segment. It is clear that $u_n\to u$ in $L_2([a,b],p)$ and $u_n'\to\varphi$ in $L_2([a,b])$. Let $u_n'=\varphi+\delta_n$. Thus,

$$u_n(x)=u_n(a)+\int_a^x\varphi(s)\,\mathrm{d}s+\int_a^x\delta_n(s)\,\mathrm{d}s. \tag{5.2}$$

Consequently,

$$\int_a^b p(x)\Big(u_n(a)+\int_a^x\varphi(s)\,\mathrm{d}s+\int_a^x\delta_n(s)\,\mathrm{d}s-u(x)\Big)^2\,\mathrm{d}x\to 0.$$

The third term tends to zero uniformly on $[a,b]$:

$$\Big(\int_a^x\delta_n(s)\,\mathrm{d}s\Big)^2\leqslant\int_a^x\delta_n(s)^2\,\mathrm{d}s\cdot\int_a^x 1\,\mathrm{d}x\leqslant\int_a^b\delta_n(s)^2\,\mathrm{d}s\cdot\int_a^b 1\,\mathrm{d}x\to 0.$$

Thus, this term converges to zero in $L_2([a,b],p)$ and can be excluded:

$$\int_a^b p(x)\Big(u_n(a)+\int_a^x\varphi(s)\,\mathrm{d}s-u(x)\Big)^2\,\mathrm{d}x\to 0.$$

It follows that there exists $\lim u_n(a)=c$, and

$$c+\int_a^x\varphi(s)\,\mathrm{d}s-u(x)=0,\ \ x\in[a,b].$$

Thus, $u(x)$ is absolutely continuous on $[a,b]$ and $u'(x) = \varphi(x)$. Since the segment $[a,b]$ is arbitrary, $u'(x) = \varphi(x)$ on the whole axis.

To prove the convergence $u_n - u \to 0$ in W note that the convergence

$$\int_{-\infty}^{\infty} ((u_n' - u')^2 + p(u_n - u)^2)\,\mathrm{d}x \to 0$$

follows from the definitions of u and $\varphi = u'$. To show that

$$\int_{\mathbb{R}\times\mathbb{R}} ((u_n(x) - u(x)) - (u_n(s) - u(s)))^2\,\mathrm{d}\xi \to 0,$$

denote $g(x,s) = u(x) - u(s)$, $g_n(x,s) = u_n(x) - u_n(s)$. From (5.2) it follows that $u_n \to u$ uniformly on any segment. So, $g_n(x,s) \to u(x) - u(s)$ for all x, s. By virtue of (5.1), $g_n \to \widetilde{g}$ in $L_2(\mathbb{R}\times\mathbb{R}, \xi)$. Thus, $\widetilde{g} = u(x) - u(s)$ for ξ-almost all (x,s). □

Lemma 5.2. *The operator $T\colon W \to L_2$ defined by equality $Tu(x) = u(x)$, $x \in (-\infty, \infty)$, is continuous.*

P r o o f. This follows immediately from comparison of norms. □

Lemma 5.3[2]**.** *Let $h(x)$ be a function square integrable on a segment $[a,b]$. If*

$$\int_a^b h(x)g(x)\,\mathrm{d}x = 0$$

for any function $g(x)$ square integrable on $[a,b]$ such that $\int_a^b g(x)\,\mathrm{d}x = 0$, then $h(x)$ is a constant.

P r o o f. Choose a constant c such that $\int_a^b (h(x) - c)\,\mathrm{d}x = 0$. According to the requirement of the lemma $\int_a^b h(x)(h(x) - c)\,\mathrm{d}x = 0$. Subtracting from this equality the equality $c\int_a^b (h(x) - c)\,\mathrm{d}x = 0$ we obtain

$$\int_a^b (h(x) - c)^2\,\mathrm{d}x = 0.$$

Thus, $h = c$. □

[2] This is a well known assertion, see for example [6], Chapter 1, Lemma 2; it is also a simple fact in functional analysis.

Lemma 5.4. *The image $T(W)$ of the space W is dense in L_2.*

Proof. Note that $W \subset L_2$ as sets. If the closure $\widetilde{W}$ in L_2 is not the L_2, there exists a function $h \in L_2$, $h \neq 0$, that is orthogonal to $\widetilde{W}$:

$$\int_{-\infty}^{\infty} u(x)h(x)\,\mathrm{d}x = 0,\ \forall u \in W.$$

Consider now an arbitrary segment $[a,b]$ and all functions $u \in W$ that are equal to zero out of the segment $[a,b]$. In this case $u(a) = u(b) = 0$, and

$$0 = \int_a^b u(x)h(x)\,\mathrm{d}x = -\int_a^b H(x)u'(x)\,\mathrm{d}x,$$

where $H(x) = \int_a^x h(s)\,\mathrm{d}s$.

Thus, the last integral is equal to zero for any square integrable function $u'(x)$ that satisfies the condition $\int_a^b u'(x)\,\mathrm{d}x = 0$. According to Lemma 5.3, $H(x)$ is a constant. Thus, $H(x) = 0$ and $h(x) = 0$ on $[a,b]$. The segment $[a,b]$ is arbitrary, therefore $h(x) = 0$, for all $x \in \mathbb{R}$. This contradiction shows that $\widetilde{W} = L_2$. □

5.2. Euler equation. According to Lemma 2.1 the equation

$$[u,v] = (f, Tv), \quad \forall v \in W,$$

has the unique solution $u = T^*f$ and the operator T^* is an injection. Thus, the operator T^* has an inverse $\mathcal{L} = (T^*)^{-1}$ defined on the set $D_{\mathcal{L}} = T^*L_2$.

Lemma 5.5. *The operator $\mathcal{L}$ has the representation* (1.2). *The domain $D_{\mathcal{L}}$ consists of functions $u \in W$ with locally on $\mathbb{R}$ absolutely continuous derivative, and $u'' \in L_2(\mathbb{R})$.*

Proof. Let u be the solution of $[u,v] = (f, Tv)$. So, for all $v \in W$,

$$\int_{\mathbb{R}} (u'v' + puv)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} (u(x)-u(s))(v(x)-v(s))\,\mathrm{d}\xi = \int_{\mathbb{R}} fv\,\mathrm{d}x. \tag{5.3}$$

By virtue of Lemma 5.9 for a ξ-measurable function f we have

$$\int_{\mathbb{R}\times\mathbb{R}} f(x,s)\,\mathrm{d}\xi = \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(x,y)\,\mathrm{d}_s r(x,s).$$

Using this formula and considering the symmetry of ξ one can represent the second term in (5.3) in the form

$$\frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} (u(x)-u(s))(v(x)-v(s))\,\mathrm{d}\xi = \int_{\mathbb{R}} v(x)\,\mathrm{d}x \int_{\mathbb{R}} (u(x)-u(s))\,\mathrm{d}_s r(x,s).$$

Let $[a,b]$ be a segment. Consider all functions $v \in W$ that are equal to zero out of (a,b): $v = 0$ if $x \notin [a,b]$. Let $h(x) = -pu - \int_{\mathbb{R}}(u(x) - u(s))\,\mathrm{d}_s r(x,s) + f$, $H = \int_a^x h(s)\,\mathrm{d}s$. Thus,

$$\int_a^b u'v'\,\mathrm{d}x = \int_a^b hv\,\mathrm{d}x = -\int_a^b Hv'\,\mathrm{d}x,$$

or $\int_a^b (u'+H)v'\,\mathrm{d}x = 0$. According to Lemma 5.3 this implies that $u'+H$ is a constant, the derivative u'' exists, and $u''+h=0$. Finally, on $[a,b]$

$$-u'' + pu + \int_{\mathbb{R}}(u(x) - u(s))\,\mathrm{d}_s r(x,s) = f.$$

Since $[a,b]$ is an arbitrary interval, the left hand side is an expression for the operator $\mathcal{L}$. From $u''+h=0$ it follows that $u'' \in L_2(\mathbb{R})$. □

5.3. Compactness of the operator T. By virtue of the criterium of Gelfand, (see Theorem 5.1) the necessary and sufficient condition of compactness is the uniform convergence on $\{Tu\colon [u,u] \leqslant 1\}$ of any sequence $f_n \in L_2$ that converges for any $z \in L_2$, i.e., $(f_n, z) \to 0$.

The following theorem [9], page 318, can be used to show compactness.

Theorem 5.1 (Gelfand)**.** *A set E from a separable Banach space X is relatively compact if and only if for any sequence of linear continuous functionals that converge to zero at each point, i.e.*

$$f_n(x) \to 0, \quad \forall x \in X, \tag{5.4}$$

the convergence (5.4) *is the uniform on E.*

Lemma 5.6. *Suppose $f_n \in L_2$, and $(f_n, z) \to 0$ for any $z \in L_2$. For any segment $\Delta = [a,b]$ the convergence $(f_n, Tu)_\Delta$ is uniform for $\|u\| \leqslant 1$.*

P r o o f. The set $\{u \in W\colon \|u\| \leqslant 1\}$ is the set of functions u satisfying

$$\int_{\mathbb{R}} ({u'}^2 + pu^2)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} (u(x) - u(s))^2\,\mathrm{d}\xi \leqslant 1.$$

Since

$$\int_a^b f_n(x)u(x)\,\mathrm{d}x = u(a)\int_a^b f_n(x)\,\mathrm{d}x + \int_a^b f_n(x)\int_a^x u'(s)\,\mathrm{d}s\,\mathrm{d}x$$

and $u(a)$ is bounded (because of $\int_{\mathbb{R}}((u')^2+u^2)\,\mathrm{d}x \leqslant 1$) on the set $\|u\| \leqslant 1$, it remains to show that

$$\int_a^b f_n(x) \int_a^x u'(s)\,\mathrm{d}s\,\mathrm{d}x \to 0$$

uniformly. Since

$$\begin{aligned}\left(\int_a^b f_n(x) \int_a^x u'(s)\,\mathrm{d}s\,\mathrm{d}x\right)^2 &= \left(\int_a^b u'(s)\,\mathrm{d}s \int_s^b f_n(x)\,\mathrm{d}x\right)^2 \\ &\leqslant \int_a^b u'(s)^2\,\mathrm{d}s \int_a^b \varphi_n(s)^2\,\mathrm{d}s \leqslant \int_a^b \varphi_n(s)^2\,\mathrm{d}s,\end{aligned}$$

where

$$\varphi_n(s) = \int_s^b f_n(x)\,\mathrm{d}x,$$

it is sufficient to show that $\varphi_n \to 0$ in the space L_2. In fact, $\varphi_n \to 0$ uniformly. To show this consider

$$z_s(x) = \begin{cases} 0 & \text{if } x \notin [s,b], \\ 1 & \text{if } x \in [s,b]. \end{cases}$$

Note that

$$\varphi_n(s) = f_n(z_s)$$

(on the right hand side f_n is considered as a functional). It is clear that the set $S = \{z_s\colon\ s \in [a,b]\}$ is relatively compact in L_2. By virtue of the same criterium of Gelfand f_n converges uniformly on S. But this is the uniform convergence of $\varphi_n(s)$. □

By Lemma 5.6 the question about compactness is reduced to the behavior on infinity.

Lemma 5.7. *The operator T is compact if and only if*

$$\lim_{N\to\infty} \sup_{u\in W, u\neq 0} \frac{(Tu,Tu)_{|x|>N}}{[u,u]_{|x|>N}} = 0.$$

P r o o f. Sufficiency. Let f_n be a sequence $f_n \in L_2$, convergent for any $z \in L_2$, i.e., $(f_n, z) \to 0$. Then it is bounded, $(f_n, f_n) \leqslant M$. Let $\varepsilon > 0$. Choose N such that

$$\sup_{u\in W, u\neq 0} \frac{(Tu,Tu)_{|x|>N}}{[u,u]_{|x|>N}} < \frac{\varepsilon}{2M}.$$

Then for $\|u\| \leqslant 1$

$$(f_n, Tu)^2_{|x|>N} \leqslant (f_n, f_n)(Tu,Tu)_{|x|>N} \leqslant M \cdot \frac{\varepsilon}{2M} = \frac{\varepsilon}{2}.$$

On $[-N,N]$ uniform convergence is fulfilled, and for sufficiently large n and all $\|u\| \leqslant 1$

$$(f_n, Tu)^2_{[-N,N]} < \frac{\varepsilon}{2}.$$

N e c e s s i t y. Suppose T is compact but there exist $\varepsilon > 0$ and sequences $N_n \to \infty$ and u_n such that $[u_n, u_n]_{D_n} = 1$, where $D_n = \{x\colon\ |x| > N_n\}$ and

$$(Tu_n, Tu_n)_{D_n} \geqslant \varepsilon.$$

Let $f_n = \chi_{D_n} Tu_n / \|\chi_{D_n} Tu_n\|$, where χ is the characteristic function of D_n. This sequence converges at any $z \in L_2$:

$$(f_n, z)^2 = (f_n, z)^2_{D_n} \leqslant (f_n, f_n)(z, z)_{D_n} = (z, z)_{D_n} \to 0.$$

However,

$$f_n(Tu_n) = \frac{1}{\|\chi_{D_n} Tu_n\|}(Tu_n, Tu_n)_{D_n} \geqslant \sqrt{\varepsilon},$$

which contradicts the criterium of compactness of Gelfand. □

R e m a r k 5.1. From this proof of necessity we can see that instead of $|x| > N$ we can consider any segment Δ. Since $\inf\limits_{u \in W, u \neq 0} [u, u]_\Delta / (Tu, Tu)_\Delta = \widetilde{\mu}(\Delta)$ (see (3.10)), the condition

$$\lim_{\Delta \to \infty} \widetilde{\mu}(\Delta) = \infty \tag{5.5}$$

is necessary for the compactness of T.

Lemma 5.8. *If the operator T is not compact, there exists an $\varepsilon > 0$ such that for any $d > 0$ there exists a sequence of segments Δ_n of length d that tends to infinity and*

$$\sup_{u \in W, u \neq 0} \frac{(Tu, Tu)_{\Delta_n}}{[u, u]_{\Delta_n}} \geqslant \varepsilon. \tag{5.6}$$

P r o o f. According to Lemma 5.7, if T is not compact, there exist an $\varepsilon > 0$, a sequence $N_n \to \infty$ and a sequence u_n such that

$$(Tu_n, Tu_n)_{|x| > N_n} \geqslant \varepsilon [u_n, u_n]_{|x| > N_n}. \tag{5.7}$$

Let us fix n, $N = N_n$ and $u = u_n$. Divide the set $\{|x| > N\}$ in segments of the length d, then for one segment Δ the inequality (5.6) will be satisfied. If not, we could sum the inequalities

$$(Tu, Tu)_\Delta < \varepsilon [u, u]_\Delta$$

and obtain a contradiction with (5.7). □

This together with the remark to Lemma 5.7 yields

Corollary 5.1. *T is compact if and only if $\widetilde{\mu}(\Delta) \to \infty$ when $\Delta \to \infty$ (for Δ of any fixed length).*

5.4. One generalization of the Fubini theorem. Reduction of double integral to repeated integral needs a generalization of the Fubini theorem. We are grateful to I. Shragin who found the relevant source.

Lemma 5.9 ([3]). *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, let μ be a measure on $(X, \mathcal{A})$, and $K\colon X \times \mathcal{B} \to [0, \infty]$ a kernel (i.e. for μ-a.a. $x \in X$, $K(x, \cdot)$ is a measure on $(Y, \mathcal{B})$, for all $B \in \mathcal{B}$, $K(\cdot, B)$ is μ-measurable on X). Then*

(1) *The function ν defined on $\mathcal{A} \times \mathcal{B}$ by the equality*

$$\nu(E) = \int_X K(x, E_x)\mu(\mathrm{d}x), \quad E_x = \{y\colon (x, y) \in E\},$$

is a measure,

(2) *if $f\colon X \times Y \to [-\infty, \infty]$ is ν-integrable on $X \times Y$, then*

$$\int_{X\times Y} f(x, y)\, \mathrm{d}\nu = \int_X \left(\int_Y f(x, y) K(x, \mathrm{d}y) \right) \mu(\mathrm{d}x).$$

R e m a r k 5.2. The function ν is the Lebesgue expansion from the set of all rectangles

$$\nu(A \times B) = \int_A K(x, B)\mu(\mathrm{d}x), \quad A \in \mathcal{A},\ B \in \mathcal{B}.$$

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