INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Bases and Borel selectors for tall families

Jan Grebík

Carlos Uzcátegui

Preprint No. 108-2017

PRAHA 2017

BASES AND BOREL SELECTORS FOR TALL FAMILIES

JAN GREBÍK AND CARLOS UZCÁTEGUI

Abstract. Given a family $\mathcal{C}$ of infinite subsets of $\mathbb{N}$, we study when there is a Borel function $S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for every infinite $x \in 2^{\mathbb{N}}$, $S(x) \in \mathcal{C}$ and $S(x) \subseteq x$. We show that the family of homogeneous sets (with respect to a partition of a front) as given by the Nash-Williams' theorem admits such a Borel selector. However, we also show that the analogous result for Galvin's lemma is not true by proving that there is a F_σ tall ideal on $\mathbb{N}$ without a Borel selector, the proof is not constructive since it is based on descriptive set theoretic considerations. We construct a $\mathbf{\Pi}^1_2$ tall ideal on $\mathbb{N}$ without a tall closed subset.

1. Introduction

A family $\mathcal{C}$ of subsets of $\mathbb{N}$ is tall if for every infinite $x \subseteq \mathbb{N}$ there is an infinite $y \in \mathcal{C}$ such that $y \subseteq x$. We are interested in tall families $\mathcal{C}$ which are in addition definable as subsets of $2^{\mathbb{N}}$. Take for example the set $hom(c)$ of all monochromatic subsets of $\mathbb{N}$ for some coloring $c : [\mathbb{N}]^2 \to 2$. This is, by Ramsey theorem, a tall family and moreover it is a closed subset of $2^{\mathbb{N}}$. We deal with the question of when we can effectively witness that a family is tall, i.e. when there is a Borel function $S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for every infinite $x \in 2^{\mathbb{N}}$, $S(x) \in \mathcal{C}$, $S(x)$ is infinite and $S(x) \subseteq x$. We call such a function S a Borel selector for $\mathcal{C}$. Note that if there is a Borel selector S for $\mathcal{C}$, then $\mathcal{C}$ contains an analytic subfamily which is also tall. This leads to a natural basis problem of whether a given tall family $\mathcal{C}$ contains a simpler tall subfamily $\mathcal{C}' \subseteq \mathcal{C}$. By simpler we mean that $\mathcal{C}'$ is of lower complexity (for example closed) or is of a specific form (for example $hom(c)$ for some coloring c).

An important source of examples of tall families are tall Borel ideals on $\mathbb{N}$. Up to now, all known examples of Borel tall ideals (see, for instance,[5, 6]) have a Borel selector (see section 3.3). One of the main results of this article it to show that there is a F_σ tall ideal without a Borel selector. The proof of this result is based on the following facts. Every F_σ ideal can be coded by a closed collection of sets, i.e. by an element of the hyperspace $K(2^{\mathbb{N}})$. In [4] it is proved that the set of codes of tall F_σ ideals is a $\mathbf{\Pi}^1_2$−complete subset of $K(2^{\mathbb{N}})$. To show that there is an F_σ ideal without a selector we prove that the complexity of the set of codes of F_σ ideals with a Borel selector is $\mathbf{\Sigma}^1_2$. However, it is an open question to find a concrete example of such F_σ ideal. This result is a generalization of the fact that there is a closed subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ whose projection is $\mathbb{N}^{\mathbb{N}}$ but without a Borel uniformization (see Corollary 4.19).

Another important class of tall families are the collection of homogeneous sets with respect to a partition of $[\mathbb{N}]^\omega$, the infinite subsets of $\mathbb{N}$. Given $\mathcal{O} \subseteq [\mathbb{N}]^\omega$, a set $x \subseteq \mathbb{N}$ is called $\mathcal{O}$-homogeneous, if either $[x]^\omega \subseteq \mathcal{O}$ or $[x]^\omega \cap \mathcal{O} = \emptyset$. A well known theorem of Silver

The first-listed author was supported by the GACR project 15-34700L and RVO: 67985840.

[11] says that for every analytic subset $\mathcal{O}$ of $[\mathbb{N}]^\omega$ the collection $hom(\mathcal{O})$ of $\mathcal{O}$-homogeneous sets is tall. When $\mathcal{O}$ is open (resp. clopen), the corresponding Ramsey result is called Galvin's lemma [2] (resp. Nash-Williams' theorem [10]). The existence of Borel selectors for families of the form $hom(\mathcal{O})$ is a consequence of the fact that the corresponding Ramsey theorem holds uniformly. For instance, the fact that the Random ideal $\mathcal{R}$ [5] has a Borel selector is due to the fact there is uniform approach of finding an infinite monochromatic subset of a given set $x \subseteq \mathbb{N}$ (or having a Borel proof of Ramsey theorem) [6]. Analogously, we show that Nash-Williams' theorem also has a uniform version and thus $hom(\mathcal{O})$ has a Borel selector for every clopen set $\mathcal{O}$. However, we show there is an open set $\mathcal{O}$ such that $hom(\mathcal{O})$ does not have a Borel selector and therefore Galvin's lemma does not admit a uniform version.

Ramsey type theorems have been analyzed from a related but different complexity point of view. Solovay ([13]) showed that if $\mathcal{O} \subseteq [\mathbb{N}]^\omega$ is open and $[x]^\omega \subseteq \mathcal{O}$ for every $x \in hom(\mathcal{O})$, then $hom(\mathcal{O})$ contains an element which is hyperarithmetical on the code of $\mathcal{O}$ (see also [1]).

Finally, we show that the basis problem also has a negative answer. We construct a $\mathbf{\Pi}^1_2$ tall ideal $\mathcal{I}$ such that $hom(\mathcal{O}) \not\subseteq \mathcal{I}$ for all open set $\mathcal{O} \subseteq [\mathbb{N}]^\omega$, in particular, $\mathcal{I}$ does not contain any tall closed subset. It is still an open question whether every tall Borel (analytic) ideal contains a closed tall subset.

2. Preliminaries

In this section we fix our notation, give some basic definitions and results that are later used. We consider the natural isomorphism $\mathcal{P}(\mathbb{N}) \approx 2^{\mathbb{N}}$ and view all relations such as $\subseteq, \cap, [\_]^{<\omega}$, etc, as defined on $2^{\mathbb{N}}$ i.e. we use $x \subseteq y$, $x \cap y, [x]^{<\omega}$, etc, for $x, y \in 2^{\mathbb{N}}$. We use the standard descriptive set theoretic notions and notations (as in [7]). The projective classes are denoted $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.

Definition 2.1. *Let $\mathcal{C} \subseteq 2^{\mathbb{N}}$ be a tall family. We say that $\mathcal{C}$ has a Borel selector, if there is a Borel function $S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for every $x \in 2^{\mathbb{N}}$*

- *$S(x) \subseteq x$,*
- *if $|x|$ is infinite then $|S(x)|$ is infinite,*
- *$S(x) \in \mathcal{C}$.*

Note that we define the notion of a Borel selector only for tall families so if we say that $\mathcal{C}$ has a Borel selector it automatically means that $\mathcal{C}$ is tall. We say that a family $\mathcal{C}$ is hereditary if $y \in \mathcal{C}$ whenever there is $x \in \mathcal{C}$ such that $y \subseteq x$. We say that $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an ideal on $\mathbb{N}$ if it is hereditary and it is closed under finite unions. As usual, we define $\mathcal{I}^+$ as $2^{\mathbb{N}} \setminus \mathcal{I}$.

The following characterization of an F_σ ideal on $\mathbb{N}$ was given by Mazur [9]. Recall that a map $\varphi : 2^{\mathbb{N}} \to [0, \infty]$ is a *lower-semicontinuous submeasure (lcsms)* if for all $x, y \in \mathbb{N}$

- $\varphi(\emptyset) = 0$,
- $x \subseteq y$ implies $\varphi(x) \leq \varphi(y)$,
- $\varphi(x \cup y) \leq \varphi(x) + \varphi(y)$,

- $\varphi(x) = \lim_{n\to\infty} \varphi(x \cap n)$.

Each lcsms φ naturally corresponds to the F_σ ideal $Fin(\varphi) := \{x : \varphi(x) < \infty\}$.

Theorem 2.2 (Mazur [9])**.** *An ideal $\mathcal{I}$ is F_σ if and only if there is lcsms φ such that $\mathcal{I} = Fin(\varphi)$.*

From this characterization one easily deduces (see for example [4]) the following result which allows us to consider $K(2^{\mathbb{N}})$, the hyperspace of closed subsets of $2^{\mathbb{N}}$ endowed with its usual metric topology, as a space of codes of F_σ ideals. For $K \in K(2^{\mathbb{N}})$, let $\mathcal{I}_K$ be ideal generated by K, i.e. $x \in \mathcal{I}_K$ if and only if there are $y_0, ..., y_{n-1} \in K$ such that $\bigcup_{i<n} y_i \subseteq x$. Clearly, $\mathcal{I}_K$ is F_σ.

Proposition 2.3. *For every F_σ ideal $\mathcal{I}$ there is $K \in K(2^{\mathbb{N}})$ such that $\mathcal{I} = \mathcal{I}_K$.*

Let $\mathcal{T}$ be the collection of all $K \in K(2^{\mathbb{N}})$ such that $\mathcal{I}_K$ is tall. The following result is crucial for our purposes.

Theorem 2.4. [4] *$\mathcal{T}$ is $\mathbf{\Pi}^1_2$−complete subset of $K(2^{\mathbb{N}})$.*

Next we state the combinatorial theorems (as presented in [14]). Let $s, t \in [\mathbb{N}]^{<\omega}$. We write $s \sqsubseteq t$ when there is $n \in \omega$ such that $s = t \cap \{0, 1, \cdots, n\}$ and we say that s is an initial segment of t.

Theorem 2.5 (Galvin)**.** *Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ and an infinite $x \in 2^{\mathbb{N}}$. Then there is an infinite $y \subseteq x$ such that one of the following holds*

- *for all $z \in [y]^\omega$ there is $s \in \mathcal{F}$ such that $s \sqsubseteq z$,*
- *$[y]^{<\omega} \cap \mathcal{F} = \emptyset$.*

We can think of $\mathcal{F}$ as a coloring of $[\mathbb{N}]^{<\omega}$ and put $hom(\mathcal{F}) \subseteq 2^{\mathbb{N}}$ for the family of all y that satisfy one of the conditions in the conclusion of Galvin's theorem, such sets are called $\mathcal{F}$-homogeneous. It is clear that $hom(\mathcal{F})$ is an hereditary tall collection. Moreover, the family of all sets that satisfy the second condition is closed and the family of sets that satisfy the first condition is $\mathbf{\Pi}^1_1$. We write $\mathbb{P}_2$ for the set of all those $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ such that first condition in the conclusion of Galvin's theorem is never satisfied.

A special type of coloring of $[\mathbb{N}]^{<\omega}$ are as follows. We say that $\mathcal{B} \subseteq [\mathbb{N}]^{<\omega}$ is a *front* on an infinite $x \in 2^{\mathbb{N}}$ if

- every two elements of $\mathcal{B}$ are $\sqsubseteq$-incomparable,
- every infinite $y \subseteq x$ has an initial segment in $\mathcal{B}$.

Theorem 2.6 (Nash-Williams)**.** *Let $\mathcal{B}$ be a front on $\mathbb{N}$ and $\mathcal{F} \subseteq \mathcal{B}$. Then for every infinite $x \in 2^{\mathbb{N}}$ there is an infinite $y \subseteq x$ such that one of the following holds*

- *$[y]^{<\omega} \cap \mathcal{B} \subseteq \mathcal{F}$,*
- *$[y]^{<\omega} \cap \mathcal{F} = \emptyset$.*

Let $\mathcal{F} \subseteq \mathcal{B}$ as above, it is easy to verify that $y \in hom(\mathcal{F})$ iff y satisfies one of the conditions from the Nash-Williams' theorem. Moreover, the family $hom(\mathcal{F})$ is easily seen to be closed, hereditary and tall.

Proposition 2.7. *For every closed, tall and hereditary $K \subseteq 2^{\mathbb{N}}$ there is $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ such that $hom(\mathcal{F}) = K$.*

Proof. Define $\mathcal{F}_K = \{s \in [\mathbb{N}]^{<\omega} : s \notin K\}$. We claim that $hom(\mathcal{F}_K)$ is equal to $\{y \in [\omega]^{\omega} : [y]^{<\infty} \cap \mathcal{F}_K = \emptyset\}$. Let $y \in hom(\mathcal{F}_K)$ and suppose y satisfies the first condition in the conclusion of Galvin's theorem. Since K is tall there is an infinite $z \subseteq y$ such that $z \in K$. As y satisfies the first condition, there is $s \in \mathcal{F}_K$ such that $s \sqsubseteq z$ but since K is hereditary we have $s \in K$ and this contradicts the definition of $\mathcal{F}_K$.

It remains to check that $K = hom(\mathcal{F}_K)$. Clearly $\subseteq$ holds. For the opposite take $x \notin K$. Since K is hereditary and closed there must be some $n \in \mathbb{N}$ such that $x \cap n \notin K$ then we have $x \cap n \in \mathcal{F}_K$. Thus $x \notin hom(\mathcal{F}_K)$. □

Proposition 2.8. *The set $\mathbb{P}_2$ is $\mathbf{\Pi}^1_2$−complete.*

Proof. This is a generalization of previous argument. Consider the continuous map $\psi : K(2^{\mathbb{N}}) \to \mathcal{P}(\mathbb{N}^{<\omega})$ given by

$$s \in \psi(K) \Leftrightarrow \forall x \in K \ s \not\subseteq x.$$

One may check that $\mathcal{T} = \psi^{-1}(\mathbb{P}_2)$ and the desired result follows since $\mathbb{P}_2$ is easily seen to be $\mathbf{\Pi}^1_2$. □

3. Positive results

In this section we prove the uniform version of the Nash-Williams's theorem. To state our theorem in full generality we must first introduce several definitions.

3.1. **Uniformly p^+, q^+ and selective ideals.** Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. We say that $\mathcal{I}$ is q^+ if for all $x \in \mathcal{I}^+$ and every partition $\{s_n\}_n$ of x into finite sets there is $y \subseteq x$ such that $y \in \mathcal{I}^+$ and $|y \cap x_n| \leq 1$ for all $n \in \mathbb{N}$. It is p^+ if for every decreasing sequence $(x_n)_n$ of sets in $\mathcal{I}^+$ there is $x \in \mathcal{I}^+$ such that $x \setminus x_n$ is finite for all n. It is *selective*, if for every decreasing sequence $(x_n)_n$ of sets in $\mathcal{I}^+$ there is $x \in \mathcal{I}^+$ such that $x/n \subseteq x_n$ for all $n \in x$. We are interested in the uniform versions of these notions. We say that a Borel ideal $\mathcal{I}$ is *uniformly selective* if there is a Borel function F such that whenever $(x_n)_n$ is a decreasing sequence of sets in $\mathcal{I}^+$, then $x = F((x_n)_n)$ is in $\mathcal{I}^+$ and $x/n \subseteq x_n$ for all $n \in x$. In an analogous way, we define when an ideal is uniformly p^+ or q^+.

Lemma 3.1. *A Borel ideal $\mathcal{I}$ is uniformly selective iff it is uniformly p^+ and q^+.*

Proof. Follow a standard proof of the fact that an ideal is selective iff it is p^+ and q^+ (see for instance [15, Lemma 7.4]). □

Theorem 3.2. *Let $\mathcal{I}$ be a F_σ ideal. Then*

(i) *$\mathcal{I}$ is uniformly p^+.*
(ii) *if $\mathcal{I}$ is q^+, then it is uniformly q^+.*

In particular, every selective F_σ ideal is uniformly selective.

Proof. Let $\{s_k\}_k$ be an enumeration of $[\mathbb{N}]^{<\omega}$ and let μ be the lower semicontinuous submeasure such that $\mathcal{I} = \{x \in 2^{\mathbb{N}} : \mu(x) < \infty\}$. First we claim that for each $n \in \mathbb{N}$ there is a Borel function $G_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for all $x \notin \mathcal{I}$, $G_n(x)$ is a finite subset of x and $\mu(G_n(x)) \geq n$. Define $G_n(x) = \emptyset$ for $x \in \mathcal{I}$. For $x \in \mathcal{I}^+$ let $G_n(x) = s_k$ where k is the minimal index such that $s_k \subseteq x$ and $\mu(s_k) \geq n$.

(i) Let $(x_n)_n$ be a decreasing sequence of sets in $\mathcal{I}^+$. Define $G((x_n)_n) = \bigcup_n G_n(x_n)$. Then G is Borel and has the required property.

(ii) We define inductively a sequence of Borel functions $(F_n)_n$ where $F_n : 2^{\mathbb{N}} \times ([\mathbb{N}]^{<\omega})^{\mathbb{N}} \to [\mathbb{N}]^{<\omega}$ and for $(x, (t_i)_i) \in 2^{\mathbb{N}} \times ([\mathbb{N}]^{<\omega})^{\mathbb{N}}$ we have

- $F_0(x, (t_i)_i) = \emptyset$,
- if $n > 0$, $x \in \mathcal{I}^+$ and $(t_i)_i$ is a partition of x then let $F_n(x, (t_i)_i) = s_k$ where k is the minimal index such that s_k is a partial selector, $\mu(s_k) \geq n$ and $F_{n-1}(x, (t_i)_i) \subseteq s_k$,
- otherwise put $F_n(x, (t_i)_i) = \emptyset$.

These are clearly Borel conditions and the functions are well defined since $\mathcal{I}$ is q^+. Finally put $F(x, (t_i)_i) = \bigcup_{n \in \mathbb{N}} F_n(x, (t_i)_i)$. □

Corollary 3.3. Fin *is uniformly selective.*

Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$ and $\mathcal{I}(\mathcal{A})$ be the ideal generated by $\mathcal{A}$. By a result of Mathias [8], $\mathcal{I}(\mathcal{A})$ is selective. It is easy to verify that when $\mathcal{A}$ is closed (as a subset of $2^{\mathbb{N}}$), then $\mathcal{I}(\mathcal{A})$ is F_σ. Hence from Theorem 3.2 we get the following

Corollary 3.4. *Let $\mathcal{A}$ be a closed almost disjoint family. Then $\mathcal{I}(\mathcal{A})$ is uniformly selective.*

The previous result naturally suggest the following.

Question 3.5. *Is $\mathcal{I}(\mathcal{A})$ uniformly selective for any almost disjoint Borel family $\mathcal{A}$? More generally, is any Borel selective ideal uniformly selective?*

3.2. **Uniform Ramsey type theorems.** Recall that the lexicographic order $<_{lex}$ on $[\mathbb{N}]^{<\omega}$ is defined by $s <_{lex} t$ if $min(s \triangle t) \in s$. Let $x \in 2^{\mathbb{N}}$ be infinite and $\mathcal{B} \subseteq [x]^{<\omega}$ be a front on x then the restriction of $<_{lex}$ on $\mathcal{B}$ is a well-order and its order type is called the rank of $\mathcal{B}$ (denoted $rank(\mathcal{B})$).

For $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ we define $\overline{\mathcal{F}} = \{s \in [\mathbb{N}] :^{<\omega} s \sqsubseteq t \text{ for some } t \in \mathcal{F}\}$.

Lemma 3.6. *Let $\mathcal{B}$ be a front and $\mathcal{F} \subseteq \overline{\mathcal{B}}$. Let $\widehat{\mathcal{F}} = \{s \in [\mathbb{N}]^{<\omega} : \exists t \in \mathcal{F}, \exists t' \in \mathcal{B}, t \sqsubseteq s \sqsubseteq t'\}$. Then $x \in hom(\mathcal{F})$ if and only if $[x]^{<\omega} \cap \mathcal{F} = \emptyset$ or $[x]^{<\omega} \cap \overline{\mathcal{B}} \subseteq \overline{\mathcal{F}} \cup \widehat{\mathcal{F}}$.*

Proof. Let $x \in hom(\mathcal{F})$. Suppose the first item in the conclusion of Theorem 2.5 holds. Let $s \subset x$ with $s \in \overline{\mathcal{B}}$ and put $y = s \cup \{n \in x : n > \max s\}$. Thus there is $t \in \mathcal{F}$ such that $t \sqsubset y$. Hence $s \sqsubseteq t$ or $t \sqsubseteq s$. In either case, $s \in \overline{\mathcal{F}} \cup \widehat{\mathcal{F}}$. Conversely, suppose that $[x]^{<\omega} \cap \overline{\mathcal{B}} \subseteq \overline{\mathcal{F}} \cup \widehat{\mathcal{F}}$ and let $y \in [x]^{<\omega}$. Since $\mathcal{B}$ is a front, there is $t \in \mathcal{B}$ such that $t \sqsubset y$. Then $t \in \overline{\mathcal{F}} \cup \widehat{\mathcal{F}}$. Since $t \in \mathcal{B}$, there is $s \sqsubseteq t$ with $s \in \mathcal{F}$. Hence $x \in hom(\mathcal{F})$. □

Theorem 3.7. *Let $\mathcal{B}$ be a front on some set $z \in [\mathbb{N}]^\omega$ and $\mathcal{I}$ be a uniformly selective Borel ideal on ω. There is a Borel map $S : 2^{\overline{\mathcal{B}}} \times (\mathcal{I}^+ \restriction z) \to \mathcal{I}^+$ such that $S(\mathcal{F}, x)$ is a $\mathcal{F}$-homogeneous subset of x for all $x \in \mathcal{I}^+$ and $x \subseteq z$.*

Proof. We may assume that $\mathcal{B}$ is a front on $\mathbb{N}$ and proceed by induction on $\alpha = rank(\mathcal{B})$. If $rank(\mathcal{B}) = \omega$, then $\mathcal{B} = [B]^1$. Let $S(\mathcal{F}, x) = (\bigcup \mathcal{F}) \cap y$, if $(\bigcup \mathcal{F}) \cap x \in \mathcal{I}^+$. Otherwise, $S(\mathcal{F}, x) = x \setminus \bigcup \mathcal{F}$. Since $\mathcal{I}^+$ is Borel, then S is a Borel function.

Now suppose that the claim holds for all fronts on some set $z \in [\mathbb{N}]^\omega$ of rank less then α. For each $n \in \mathbb{N}$ and $\mathcal{F} \subseteq \mathcal{B}$, let

$$\mathcal{F}_{\{n+1\}} = \{t \in [\mathbb{N}]^{<\omega} : \ n < \min(t) \ \& \ \{n\} \cup t \in \mathcal{F}\}.$$

Observe that $\mathcal{B}_{\{n+1\}}$ is a front on $x/(n+1) = \{m \in x : \ n < m\}$ with rank less than α and the function

$$\Gamma : 2^{\overline{\mathcal{B}}} \times \mathcal{I}^+ \to \prod_{n \in \mathbb{N}} (2^{\overline{\mathcal{B}_{\{n\}}}} \times \mathcal{I}^+ \restriction (\mathbb{N} \setminus n))$$

where $\Gamma(\mathcal{F}, x) = ((\mathcal{F}_{\{n\}}, x \setminus n))_{n \in \mathbb{N}}$ is Borel. By the inductive hypothesis there is Borel function

$$S : \prod_{n \in \mathbb{N}} (2^{\overline{\mathcal{B}_{\{n\}}}} \times \mathcal{I}^+ \restriction (\mathbb{N} \setminus n)) \to \prod_{n \in \mathbb{N}} (I^+ \restriction (\mathbb{N} \setminus n))$$

that satisfies the conclusion of the theorem for each coordinate. Denote the composition of Γ, S and projection to n-th coordinate as S_n.

We define a sequence of Borel functions $\{H_n\}_{n<\omega}$. For $(\mathcal{F}, x) \in 2^{\overline{\mathcal{B}}} \times \mathcal{I}^+$ define inductively

- $H_0(\mathcal{F}, x) = x$,
- $H_{n+1}(\mathcal{F}, x) = S_{n+1}(\mathcal{F}, x)$ if $n \in x$ otherwise $H_{n+1}(\mathcal{F}, x) = H_n(\mathcal{F}, x)$.

Since $\mathcal{I}$ is uniformly selective, we can extract, in a Borel way, from the sequence $\{H_n(\mathcal{F}, x)\}_{n<\omega}$ a set $y \in \mathcal{I}^+$ such that

$$y/(n+1) \subseteq H_{n+1}(\mathcal{F}, x) \ \text{ for all } n \in y.$$

Lemma 3.6 naturally provides the notion of i-homogeneous for $\mathcal{F}$ for $i = 0, 1$. Let

$$y_i = \{n \in y : \ H_{n+1}(\mathcal{F}, x) \text{ is } i\text{-homogeneous for } \mathcal{F}_{\{n+1\}}\}.$$

Then y_i is i-homogeneous for $\mathcal{F}$. In fact, for $i = 0$, let t be a finite subset of y_0 and let $n = \min(t)$. Then $t/(n+1) \subseteq H_{n+1}(\mathcal{F}, x)$ as $n \in y$. Therefore $t/(n+1) \notin \mathcal{F}_{\{n+1\}}$, as $H_{n+1}(\mathcal{F}, x)$ is 0-homogeneous. Thus $t = \{n\} \cup t/(n+1) \notin \mathcal{F}$. Using Lemma 3.6, a similar argument works for $i = 1$.

By Lemma 3.6, being i-homogeneous for $\mathcal{F}$ is a Borel property, therefore the function $y \mapsto (y_0, y_1)$ is Borel. Since $y \in \mathcal{I}^+$, then at least one of the sets y_0 or y_1 belongs to $\mathcal{I}^+$. Let $S(\mathcal{F}, x) = y_0$ if $y_0 \in \mathcal{I}^+$ and y_1, otherwise. As $\mathcal{I}^+$ is Borel, we can pick in a Borel way the alternative that holds. Thus S is Borel. □

Since **Fin** is uniformly selective (corollary 3.3), we get the uniform version of Nash-Williams' theorem.

Corollary 3.8. *Let $\mathcal{B}$ be a front on $\mathbb{N}$. There is a Borel map $S : 2^{\mathcal{B}} \times [\mathbb{N}]^{<\omega} \to [\mathbb{N}]^{<\omega}$ such that $S(\mathcal{F}, x)$ is a $\mathcal{F}$-homogeneous subset of x, for all $x \in [\mathbb{N}]^{<\omega}$ and all $\mathcal{F} \subseteq \mathcal{B}$.*

Using the front $[\mathbb{N}]^n$, we get that the classical Ramsey's theorem holds uniformly (the case $n = 2$ appeared in [6]).

Corollary 3.9. *For each $n \in \mathbb{N}$, there is a Borel function $S : 2^{[\mathbb{N}]^n} \times [\mathbb{N}]^{<\omega} \to [\mathbb{N}]^{<\omega}$ such that $S(\mathcal{F}, x)$ is an infinite subset of x homogeneous for $\mathcal{F} \subseteq [\mathbb{N}]^n$.*

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two tall hereditary families with Borel selector. It is easy to verify that $\mathcal{C}_1 \cap \mathcal{C}_2$ has a Borel selector and thus it is natural to ask the following.

Question 3.10. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ two fronts on $\mathbb{N}$ and $\mathcal{F}_i \subseteq \mathcal{B}_i$, $i \in 2$. Is there a front $\mathcal{B}_3$ and $\mathcal{F}_3 \subseteq \mathcal{B}_3$ such that $hom(\mathcal{F}_3) \subseteq hom(\mathcal{F}_1) \cap hom(\mathcal{F}_2)$?*

3.3. **Some examples.** We present some examples showing that the search for a Borel selector for a tall family $\mathcal{C}$ can be reduced, in some instances, to find an appropriated coloring c such that $hom(c) \subseteq \mathcal{C}$ and then use Corollary 3.9.

Let us start recalling that an ideal $\mathcal{I}$ is Katětov below an ideal $\mathcal{J}$ ($\mathcal{I} \leq_K \mathcal{J}$) if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $f^{-1}[x] \in \mathcal{J}$ for every $x \in \mathcal{I}$. Let $\mathcal{R}$ be the ideal on $\mathbb{N}$ generated by the homogeneous sets of the random graph ([6]). It follows from the universal property of the random graph that $\mathcal{R} \leq_K \mathcal{I}$ iff there is a $\mathcal{F} \subseteq [\mathbb{N}]^2$ such that $hom(\mathcal{F}) \subseteq \mathcal{I}$. In particular, if $\mathcal{R} \leq_K \mathcal{I}$, then $\mathcal{I}$ has a Borel selector. All ideals studied in [5, 6] are Katetov above $\mathcal{R}$, and therefore they admit a Borel selector. Even Solecki's ideal $\mathcal{S}$ ([12]) has a Borel selector [4] (even thought, it is open whether $\mathcal{R} \leq_K \mathcal{S}$). It is proved in [4] that having a Borel selector is closed upwards in the Katětov order and if $\mathcal{I}$ is a tall Borel ideal with a Borel selector then there is a tall Borel ideal $\mathcal{J}$ such that $\mathcal{I} \not\leq_K \mathcal{J}$.

Example 3.11. Let $WO(\mathbb{Q})$ be the collection of all well-ordered subsets of $\mathbb{Q}$ respect the usual order. Let $WO(\mathbb{Q})^*$ the collection of well ordered subsets of $(\mathbb{Q}, <^*)$ where $<^*$ is the reversed order of the usual order of $\mathbb{Q}$. Let $\mathcal{C} = WO(\mathbb{Q}) \cup WO(\mathbb{Q})^*$. Notice that $\mathcal{C}$ is a complete co-analytic set. To see that $\mathcal{C}$ has a Borel selector, fix an enumeration $(r_n)_n$ of $\mathbb{Q}$. Let $c : [\mathbb{Q}]^2 \to 2$ be the Sierpinski's coloring which is given by $c\{r_n, r_m\} = 0$ iff $n < m$ and $r_n < r_m$. Then $hom(c) \subseteq \mathcal{C}$.

Example 3.12. Let $(x_n)_n$ be a sequence on a compact metric space X. Let

$$\mathcal{C}(x_n)_n = \{y \subseteq \mathbb{N} :\ (x_n)_{n\in y} \text{ is convergent}\}.$$

Then $\mathcal{C}(x_n)_n$ is clearly tall. We show that there is a coloring c such that $hom(c) \subseteq \mathcal{C}(x_n)_n$. In fact, let $f : 2^{\mathbb{N}} \to X$ be a continuous surjection. Pick $y_n \in 2^{\mathbb{N}}$ such that $f(y_n) = x_n$ for each $n \in \mathbb{N}$. Let $\preceq$ be the usual lexicografic order on $2^{\mathbb{N}}$. Consider the Sierpinsky coloring $c\{n, m\}_< = 0$ iff $y_n \prec y_m$. Then $hom(c) \subseteq \mathcal{C}(x_n)_n$.

Example 3.13. Let (X, τ) be a regular space without isolated points over a countable set X. There is a coloring $c : [X]^2 \to 2$ such that $hom(c) \subseteq nwd(X, \tau)$. The Sierpinski coloring c on $[\mathbb{Q}]^2$ satisfies that $hom(c) \subseteq nwd(\mathbb{Q})$. Let $(V_n)_n$ be a countable collection of τ-open sets that separates points. Let ρ be the topology generated by the V_n's. Then (X, ρ) is homeomorphic to $\mathbb{Q}$. Therefore the Sierpinski coloring on $\mathbb{Q}$ can be defined on $[X]^2$ such that every c-homogeneous set is a ρ-discrete subset of X. Since $\rho \subseteq \tau$, then $hom(c) \subseteq nwd(X, \tau)$.

Example 3.14. Let $e : [\mathbb{N}]^\omega \to \mathbb{N}^\mathbb{N}$ be the increasing enumeration function, i.e. $e(x)(n)$ is the nth element of x in its natural order. Notice that e is continuous. Let $\gamma : [\mathbb{N}]^\omega \times [\mathbb{N}]^\omega \to [\mathbb{N}]^\omega$ be given by

$$\gamma(x, y) = \{e(x)(n) : \ n \in y\}.$$

Then $\gamma(x, y) \subseteq x$ and γ is continuous. For each $y \in [\mathbb{N}]^\omega$, let

$$\mathcal{C}_y = \{\gamma(x, y) : \ x \in [\mathbb{N}]^\omega\}.$$

Then $\mathcal{C}_y$ is a tall family and obviously $S(x) = \gamma(x, y)$ is a Borel selector for $\mathcal{C}_y$.

We will show that $\mathcal{C}_y$ contains $hom(c)$ for some coloring c. Let $(y_n)_n$ be the increasing enumeration of y. We assume that $y_0 \geq 1$. If $(z_n)_n$ is the increasing enumeration of an infinite set z, then

$$z \in \mathcal{C}_y \quad \Leftrightarrow \quad (\forall n)(y_{n+1} - y_n \leq z_{n+1} - z_n) \ \& \ y_0 \leq z_0.$$

Consider the following coloring:

$$c\{k, l\} = 0 \quad \text{iff} \quad l - k \geq y_k \ \& \ \ k \geq y_0.$$

It is easy to verify that any c-homogeneous infinite set is necessarily 0-homogeneous and also that $hom(c) \subseteq \mathcal{C}_y$.

An important open question stated in [5] is whether $\mathcal{R} \leq_K \mathcal{S}$. An analogous question is the following.

Question 3.15. *Are there a front $\mathcal{B}$ and $\mathcal{F} \subseteq \mathcal{B}$ such that $hom(\mathcal{F}) \subseteq \mathcal{S}$?*

4. Negative results

In this section we show that there is a tall F_σ ideal without a Borel selector and deduce from this fact that there is no uniform version of Galvin's theorem. We also show that there is a $\mathbf{\Pi}^1_2$ tall ideal $\mathcal{I}$ such that $hom(\mathcal{F}) \not\subseteq \mathcal{I}$ for every $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$.

4.1. **A F_σ ideal without a selector and no uniform version of Galvin's theorem.** Recall that the hyperspace $K(2^\mathbb{N})$ serves as a space of codes for F_σ ideals (see Proposition 2.3). In [4] it is proved that the set of codes of tall F_σ ideals is $\mathbf{\Pi}^1_2-$complete. To show that there is an F_σ ideal without a selector we prove that the complexity of the set of codes of F_σ ideals with a Borel selector is $\mathbf{\Sigma}^1_2$.

We start by modifying a bit the notion of tallness and Borel selector. For $K \in K(2^\mathbb{N})$, let

$$\downarrow K = \{x : \exists y \in K \ x \subseteq y\}.$$

Definition 4.1. *We say that $K \in K(2^\mathbb{N})$ is pseudo-tall if for every infinite $x \in 2^\mathbb{N}$ there is infinite $y \in \downarrow K$ such that $y \subseteq x$.*

One can verify that as a function $\downarrow : K(2^\mathbb{N}) \to K(2^\mathbb{N})$ is continuous and K is pseudo-tall if and only if $\mathcal{I}_K$ is tall.

Proposition 4.2. [4] *Given $K \in K(2^\mathbb{N})$, there is a Borel function $\phi : \mathcal{I}_K \to K^{<\omega}$ such that $x \subseteq \bigcup \phi(x)$.*

Proposition 4.3. *Let $K \in K(2^{\mathbb{N}})$ be pseudo-tall. Then $\mathcal{I}_K$ has a Borel selector S if and only if there is a Borel selector S' such that $rng(S') \subseteq\downarrow K$.*

Proof. Using Proposition 4.2, it is enough to realize that if x is infinite then at least one set in $\phi(x)$ must have infinite intersection with x and since $\phi(x)$ is finite we can pick such a set in a Borel way. □

This leads to a modified definition of a selector.

Definition 4.4. *Let $K \in K(2^{\mathbb{N}})$ be a pseudo-tall. We say that K has a Borel* pseudo-selector *if there is a Borel function $S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that*

- *$S(x) \in\downarrow K$,*
- *if $|x| = \omega$ then $|S(x)| = \omega$,*
- *$S(x) \subseteq x$.*

By the previous proposition, $K \in K(2^{\mathbb{N}})$ has a pseudo-selector if and only if $\mathcal{I}_K$ has a selector and therefore it suffices to consider only pseudo-selectors of closed subsets of $2^{\mathbb{N}}$, in other words the questions of existence of a Borel selector for F_σ ideals and hereditary tall closed subsets of $2^{\mathbb{N}}$ are equivalent. Let us summarize this in the following proposition.

Proposition 4.5. *Let $K \in K(2^{\mathbb{N}})$ be tall. The following are equivalent:*

- *there is a Borel selector for K,*
- *there is a Borel pseudo-selector for K,*
- *the F_σ ideal $\mathcal{I}_K$ has a Borel selector,*
- *the smallest ideal $\mathcal{I}$ that contains K and* **Fin** *has a Borel selector.*

Proof. It can be easily verified that the ideal $\mathcal{I}$ in the fourth condition is also F_σ. The only implication that is not clear from previou arguments is how to get a Borel selector from a Borel pseudo-selector.

Let $S : 2^{\mathbb{N}} \to \mathbb{N}$ be a Borel pseudo-selector for K. Define

$$\{(x, y) : S(x) \subseteq y \subseteq x,\ y \in K\} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}.$$

This is a Borel set with each vertical section compact and therefore it has a Borel uniformization by a classical uniformization theorem (see, for instance, [7, Theorem 35.46]). The uniformizing function is a Borel selector for K.

□

4.1.1. *Coding of Borel functions.* Now we are going to present how to code Borel functions. For that end, first we need to code Borel sets. This coding is somewhat standard (see for instance [3, pag. 19]), but we need to present it with some detail. We define a set of labeled well-founded trees which will be the codes of Borel sets.

Definition 4.6. *Let $\mathcal{LT}$ be the set of all trees on $\mathbb{N}$ where each node is labeled by an element of $\{0, 1\}$.*

So, formally, every element of $\mathcal{LT}$ is a tuple (T, f) where $T \subseteq \mathbb{N}^{<\omega}$ is a tree and $f : T \to 2$. However, we will always write only $T \in \mathcal{LT}$ and $(s, i) \in T$ meaning that $f(s) = i$.

One can easily check that there $\mathcal{LT}$ is a closed subset of the Polish space of all trees on $\mathbb{N} \times 2$, thus $\mathcal{LT}$ is a Polish space. Moreover, the set of all well-founded labeled trees $WF\mathcal{LT}$ is $\mathbf{\Pi}^1_1$.

We are interested in a closed subspace of $\mathcal{LT}$ which will contain all codes for Borel subsets of $2^{\mathbb{N}}$.

Definition 4.7. *Let $\mathcal{LT}_c \subseteq \mathcal{LT}$ be the set of all labeled trees satisfying the following condition.*

- *if $(s,1) \in T$ then $(s^\frown(0),0) \in T$ and it is the only immediate successor of $(s,1)$.*

One can easily verify that $\mathcal{LT}_c$ is a closed subspace of $\mathcal{LT}$ and the set of well-founded trees $WF\mathcal{LT}_c \subseteq \mathcal{LT}_c$ is $\mathbf{\Pi}^1_1$.

Now we will define, for each $T \in WF\mathcal{LT}_c$, the Borel set A_T coded by T. And conversely, for each Borel set $A \subseteq 2^{\mathbb{N}}$ there will be a $T \in WF\mathcal{LT}_c$ such that $A = A_T$. The definition of A_T is by recursion on the rank of T.

Let $\{t_n : n \in \mathbb{N}\}$ be an enumeration of all basic open sets of $2^{\mathbb{N}}$, i.e. each t_n is a finite binary sequence. Recursively define what each $(s,i) \in T$ codes:

- if $(s,0)$ is a leaf then it codes the basic open set $t_{s(|s|-1)}$ (in the case of $s = \emptyset$, we put $t_{\emptyset(|\emptyset|-1)} = t_0$),
- if $(s,0)$ is not a leaf, then it codes the union of the sets coded by $(s^\frown n, i)$ where $(s^\frown n, i) \in T$,
- $(s,1)$ codes the complement of what $(s^\frown(0),0)$ codes.

Finally, A_T is the set coded by $(\emptyset, i)$.

Lemma 4.8. *For every Borel set $A \subseteq 2^{\mathbb{N}}$ there is $T \in WF\mathcal{LT}_c$ such that $A = A_T$. And conversely, A_T is Borel for each $T \in WF\mathcal{LT}_c$.*

Proof. Given $T \in WF\mathcal{LT}_c$, one easily shows for induction on the rank of T that A_T is Borel. Conversely, given a Borel set $A \subseteq 2^{\mathbb{N}}$, by induction on the Borel complexity of A it is easy to construct a $T \in WF\mathcal{LT}_c$ such that $A = A_T$ □

Let $\mathcal{C}_i \subseteq 2^{\mathbb{N}} \times \mathcal{LT}_c$, $i \in 2$, be given by

$$(x,T) \in \mathcal{C}_1 \text{ if and only if } T \in WF\mathcal{LT}_c \text{ and } x \in A_T$$

and

$$(x,T) \in \mathcal{C}_0 \text{ if and only if } T \in WF\mathcal{LT}_c \text{ and } x \notin A_T.$$

The following is a crucial result.

Lemma 4.9. *The relation $\mathcal{C}_i$ is $\mathbf{\Pi}^1_1$ for $i \in 2$.*

For the proof we need some auxiliary results. We define the following subset $G \subseteq 2^{\mathbb{N}} \times \mathcal{LT}_c \times \mathcal{LT}$.

Definition 4.10. *A triple (x,T,S) is in $G \subseteq 2^{\mathbb{N}} \times \mathcal{LT}_c \times \mathcal{LT}$ if and only if*

- *$(s,i) \in T$ for some $i \in 2$ if and only if $(s,j) \in S$ for some $j \in 2$,*
- *if $(s,0) \in T$ is leaf then $(s,1) \in S$ if and only if $t_{s(|s|-1)} \sqsubseteq x$,*

- *if* $(s,1) \in T$ *then* $(s,1) \in S$ *if and only if* $(s^\frown(0),0) \in S$,
- *if* $(s,0) \in T$ *not a leaf then* $(s,1) \in S$ *if and only if there is* $n \in \mathbb{N}$ *such that* $(s^\frown n,1) \in S$.

Note that if $(x,T,S) \in G$ then S has the same tree structure as T, it only has different labeling. Also note that if T is well-founded then the labeling of S is uniquely determined by the values on its leafs. This can be proved by induction on the rank of S. Since the label of the leafs of S are uniquely determined by (x,T), we can conclude that for each $T \in WF\mathcal{L}T_c$ and every $x \in 2^{\mathbb{N}}$ there is exactly one S such that $(x,T,S) \in G$.

Claim 4.11. *The set* G *is Borel.*

Proof. We verify that each condition is Borel. The first and the third conditions are independent of the first coordinate and are closed.

For the second condition. Let $P_s := \{T \in \mathcal{L}T_c : \ s \text{ is a leaf of } T\}$ and $Q_s := \{T \in \mathcal{L}T : (s,1) \in T\}$ for each $s \in \mathbb{N}^{<\omega}$. Then P_s and Q_s are easily seen to be closed. Define

$$R_s := (2^{\mathbb{N}} \times (\mathcal{L}T_c \setminus P_s) \times \mathcal{L}T) \cup (t_{s(|s|-1)} \times P_s \times Q_s) \cup ((2^{\mathbb{N}} \setminus t_{s(|s|-1)}) \times P_s \times (\mathcal{L}T \setminus Q_s)).$$

Then $\bigcap_{s \in \mathbb{N}^{<\omega}} R_s$ is the collection of all (x,T,S) satisfying the second condition.

The fourth condition is also independent of the first coordinate and one can verify that

$$Q'_s := \{S \in \mathcal{L}T : (s,1) \in T \iff \exists n \in \mathbb{N}(s^\frown(n),1) \in S\}$$

is Borel. Combination of P_s, Q'_s and their complements gives us the desired result. □

For each $(s,i) \in T$, let $T_{(s,i)} := \{(t,j) : (s^\frown t, j) \in T\}$. Consider the following continuous bijection $\Gamma : \mathcal{L}T_c \to \mathcal{L}T_c$ where

- if $(\emptyset,0) \in T$ then $\Gamma(T) = R$ where $(\emptyset,1) \in R$ and $T_{(\emptyset,0)} = R_{((0),0)}$,
- if $(\emptyset,1) \in T$ then $\Gamma(T) = R$ where $(\emptyset,0) \in R$ and $T_{((0),0))} = R_{(\emptyset,0)}$.

In other words, $\Gamma \restriction WF\mathcal{L}T_c$ is the bijection switching the codes for a set and its complement.

Claim 4.12. *Let* $T \in WF\mathcal{L}T_c$ *and* $x \in 2^{\mathbb{N}}$ *then* $|\{S : (x,T,S) \in G\}| = 1$ *and for the unique* $(x,T,S) \in G$ *we have that* $(\emptyset,1) \in S$ *if and only if* x *is in the set coded by* T. *Moreover, let* $(x,T,S),(x,\Gamma(T),S') \in G$, *then* $(\emptyset,1)$ *is in* S *or* S' *but not in both of them.*

Proof. This follows from the discussion after the Definition 4.10 and the definition of Γ. □

Proof of Lemma4.9. Let $G_i := \{(x,T,S) \in G : (\emptyset,i) \in S\}$ for $i \in 2$. One can easily see that $G = G_0 \cup G_1$ and both sets are Borel. Let $proj(G_i) := \{(x,T) : \exists S \in \mathcal{L}T \ (x,T,S) \in G_i\}$. Then from Claim 4.12 we have

$$\mathcal{C}_1 = (2^{\mathbb{N}} \times WF\mathcal{L}T_c) \cap proj(G_1)$$

and

$$\mathcal{C}_0 = (2^{\mathbb{N}} \times WF\mathcal{L}T_c) \cap proj(G_0).$$

Finally, we show that the set $(2^{\mathbb{N}} \times WF\mathcal{L}T_c) \cap proj(G_i)$ is $\mathbf{\Pi}^1_1$ for $i < 2$. This follows from the classical result that if $A \subseteq X \times Y$ is Borel, then $\{x \in X : \exists! y \in Y (x,y) \in A\}$ is $\mathbf{\Pi}^1_1$. But we can also give a direct proof as follows.

The sets $H_i := (2^{\mathbb{N}} \times \mathcal{L}T_c) \setminus proj(G_i)$ are clearly $\mathbf{\Pi}^1_1$ and so are $M_i := WF\mathcal{L}T_c \cap H_i$ for $i < 2$. But then using the Claim 4.12 we see that $(2^{\mathbb{N}} \times WF\mathcal{L}T_c) \cap proj(G_i) = M_{1-i}$. □

Next we define a coding of Borel functions from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$. Let

$$C_n := \{x \in 2^{\mathbb{N}} : x(n) = 1\}.$$

Let $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a Borel function and let $A_n := f^{-1}(C_n)$. Then f is described by the sequence $\{A_n\}_{n\in\omega}$ because $f(x)(n) = 1$ if and only if $x \in A_n$. Thus the following is the natural definition of codes for Borel functions.

Definition 4.13. *Let $\mathcal{F}T = (\mathcal{L}T_c)^\omega$ and $WF\mathcal{F}T = (WF\mathcal{L}T_c)^\omega$.*

The product topology on $\mathcal{F}T$ is Polish and $WF\mathcal{F}T \subseteq \mathcal{F}T$ is $\mathbf{\Pi}^1_1$. We denote the elements of $\mathcal{F}T$ also by T and the n-th element of T as $T(n)$.

Lemma 4.14. *The set $WF\mathcal{F}T$ codes Borel functions from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$ i.e. every sequence $T \in WF\mathcal{F}T$ is a code for a function f_T and for every Borel function f there is a sequence $T \in WF\mathcal{F}T$ such that $f_T = f$.*

Proof. As it was mentioned above, every Borel function f is coded by a sequence of Borel sets $(A_n)_n$. Let $T = (T(n))_n$ be such that $T(n) \in WF\mathcal{L}T_c$ codes A_n for each $n \in \mathbb{N}$. □

4.1.2. *Coding of selectors and F_σ ideals.* Now we will show that the codes for F_σ ideals with Borel selector is $\mathbf{\Sigma}^1_2$ and then conclude with the main results of this section.

Consider the following map $\Omega : 2^{\mathbb{N}} \times WF\mathcal{F}T \to 2^{\mathbb{N}}$ by $\Omega(x,T)(n) = 1$ if and only if x is in the set coded by $T(n)$. From the definitions of $\mathcal{C}_i$, Ω and Lemma 4.9 the following is straightforward.

Lemma 4.15. *Let $\mathcal{R} \subseteq 2^{\mathbb{N}} \times \mathcal{F}T \times 2^{\mathbb{N}}$ be given by $(x,T,y) \in \mathcal{R}$ if and only if*

$$\forall n \in \mathbb{N}\,[\,(\,(x,T(n)) \in \mathcal{C}_1 \to y(n) = 1)\ \wedge\ (\,(x,T(n)) \in \mathcal{C}_0 \to y(n) = 0)\,].$$

Then $\mathcal{R}$ is $\mathbf{\Sigma}^1_1$ and for all $(x,T,y) \in 2^{\mathbb{N}} \times WF\mathcal{F}T \times 2^{\mathbb{N}}$ we have

$$\Omega(x,T) = y \iff (x,T,y) \in \mathcal{R}.$$

□

Consider the following set $\mathcal{M} \subseteq 2^{\mathbb{N}} \times \mathcal{F}T \times K(2^{\mathbb{N}})$ defined by $(x,T,K) \in \mathcal{M}$ if and only if

- $T \in WF\mathcal{F}T$,
- $\Omega(x,T) \in\downarrow K$,
- $\Omega(x,T) \subseteq x$,
- if $|x| = \omega$, then $|\Omega(x,T)| = |x|$.

Lemma 4.16. *$\mathcal{M}$ is a $\mathbf{\Pi}^1_1$ subset of $2^{\mathbb{N}} \times \mathcal{F}T \times K(2^{\mathbb{N}})$.*

Proof. It follows from Lemma 4.15. For instance, the second condition can be expressed as follows:

$$T \in WF\mathcal{F}T\ \wedge\ \Omega(x,T) \in\downarrow K \iff T \in WF\mathcal{F}T \wedge \forall y \in 2^{\mathbb{N}}((x,T,y) \in \mathcal{R} \to y \in\downarrow K).$$

□

Theorem 4.17. *The set of all $K \in K(2^{\mathbb{N}})$ that have a Borel pseudo-selector is $\mathbf{\Sigma}^1_2$.*

Proof. This set may be described as

$$\{K \in K(2^\omega) : \exists T \in \mathcal{F}T\, \forall x \in 2^\omega (x, T, K) \in \mathcal{M}\}$$

which is $\mathbf{\Sigma}^1_2$. □

Theorem 4.18. *There is a F_σ tall ideal without a Borel selector.*

Proof. The codes of F_σ ideals with a Borel selector are clearly a subset of all tall F_σ ideals and the former set is $\mathbf{\Sigma}^1_2$ but the later is $\mathbf{\Pi}^1_2-$complete (see Theorem 2.4). □

Corollary 4.19. [7] *There is a closed subset of $A \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $\mathbb{N}^{\mathbb{N}} = proj(A) = \{x \in \mathbb{N}^{\mathbb{N}} : \exists y \in \mathbb{N}^{\mathbb{N}} \text{ s.t. } (x, y) \in A\}$ and it does not have a Borel uniformization.*

Proof. The space $X := 2^{\mathbb{N}} \setminus \{x : \exists n \text{ s.t. } \forall m > n\ x(m) = 0\}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. The restriction of the relation $S = \{(x, y) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : x \supseteq y\}$ to X is closed in X. By our theorem there is a tall $K \in K(2^{\mathbb{N}})$ without Borel selector. Then $K \cap X$ is closed in X and the closed set $A := S \restriction (X \times X) \cap (X \times (K \cap X))$ has no Borel uniformization. □

Since Theorem 4.18 has an indirect proof we have the following.

Question 4.20. *Find a concrete example of a F_σ tall ideal without a Borel selector.*

4.1.3. *Galvin's theorem.* Now we use some previous results to simply observe that there is no uniform version of Galvin's theorem.

Theorem 4.21. *There is $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ such that there is no Borel function $S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ satisfies $S(x) \in hom(\mathcal{F})$, $S(x) \subseteq x$ and $|S(x)| = \omega$ for every infinite $x \in 2^{\mathbb{N}}$.*

Proof. Combine Theorem4.18 and Proposition2.7. □

4.2. **A $\mathbf{\Pi}^1_2$ tall ideal without a closed tall subset.** We construct a $\mathbf{\Pi}^1_2$ tall ideal which does not contain $hom(\mathcal{F})$ for every $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$. Recall that $hom(\mathcal{F})$ is $\mathbf{\Pi}^1_1$ for every $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ and therefore we have the following.

Observation 4.22. *Let $R \subseteq 2^{[\mathbb{N}]^{<\omega}} \times [\mathbb{N}]^\omega \times [\mathbb{N}]^\omega$ be defined by*

$$R(\mathcal{F}, x, y) \Leftrightarrow y \subseteq x \;\&\; y \in hom(\mathcal{F}).$$

Then R is $\mathbf{\Pi}^1_1$.

Lemma 4.23. [6, Lemma 4.6] *There is a continuous function $\psi : [\mathbb{N}]^\omega \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for every infinite $x \in [\mathbb{N}]^\omega$, the collection $\{\psi(x, y) : y \in 2^{\mathbb{N}}\}$ is an almost disjoint family of infinite subsets of x. Moreover, for all infinite x there is an infinite $z \subseteq x$ such that $z \cap \psi(x, y) = \emptyset$ for all $y \in 2^{\mathbb{N}}$.*

Theorem 4.24. *There is a $\mathbf{\Pi}^1_2$ tall ideal $\mathcal{I}$ such that for all $x \in \mathcal{I}^+$ and all $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ there is $y \subseteq x$ with $y \in hom(\mathcal{F}) \cap \mathcal{I}^+$. In particular, $\mathcal{I}$ does not contain any closed hereditary tall set.*

Proof. The construction is similar to that presented in [6, Theorem 4.7]. We will sketch the argument below. Let $\varphi : 2^{\mathbb{N}} \to 2^{[\mathbb{N}]^{<\omega}}$ be a continuous surjection. By the classical uniformization theorem [7], let $R^* \subseteq R$ be a $\mathbf{\Pi}^1_1$ uniformization for the relation R given by 4.22. Let ψ be given by Lemma 4.23. Let

$$\mathcal{C}_1 = \{y \in [\mathbb{N}]^\omega :\ \exists x \in 2^{\mathbb{N}}, R^*(\varphi(x), \psi(\mathbb{N}, x), y)\},$$

$$\mathcal{C}_{n+1} = \{y \in [\mathbb{N}]^\omega : \exists x \in 2^{\mathbb{N}}, \exists z \in \mathcal{C}_n, R^*(\varphi(x), \psi(z, x), y)\}.$$

Then each $\mathcal{C}_n$ is $\mathbf{\Sigma^1_2}$. Finally, let

$$x \in \mathcal{H} \Leftrightarrow\ (\exists n \in \mathbb{N})\ (\exists y \in \mathcal{C}_n)\ y \subseteq^* x.$$

The proof of Theorem 4.7 in [6] shows that $\mathcal{I} = \mathcal{P}(\mathbb{N}) \setminus \mathcal{H}$ is a tall ideal. We will show that it satisfies the other requirements. It is clearly $\mathbf{\Pi}^1_2$. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ and $y \notin \mathcal{I}$. Then there is $x \in 2^{\mathbb{N}}$ such that $\mathcal{F} = \varphi(x)$. There is also $n \in \mathbb{N}$ and $z \in \mathcal{C}_n$ so that $z \subseteq^* y$. Let w be such that $R^*(\varphi(x), \psi(z, x), w)$. Then $w \subseteq z$ and is $\mathcal{F}$-homogeneous. By definition, $w \in \mathcal{H}$. Then $w \cap y$ is infinite and $\mathcal{F}$-homogeneous.

The last claim follows from Lemma 2.7. □

A corollary of the proof of the previous theorem provides a more general construction of co-analytic tall ideals as in [6].

Theorem 4.25. *Let $\mathcal{B}$ be a front over $\mathbb{N}$. There is a co-analytic tall ideal $\mathcal{I}$ such that hom$(\mathcal{F}) \not\subseteq \mathcal{I}$ for all $\mathcal{F} \subseteq \mathcal{B}$.*

Proof. From the proof of Theorem 4.24 and using Corollary 3.8 instead of the co-analytic uniformizing set R^*, we define the sets $\mathcal{C}_n$, which now are analytic. Thus the ideal constructed is co-analytic. □

In [6] was asked whether every analytic tall ideal contains a F_σ tall ideal. A weaker version of this question is the following.

Question 4.26. *For which tall families $\mathcal{C}$ is there $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ such that hom$(\mathcal{F}) \subseteq \mathcal{C}$ (here hom$(\mathcal{F})$ is not necessarily closed)?*

Acknowledgment: The first author would like to thank the members of Prague Set theory seminar for many useful suggestions and comments on preliminary version of this paper. The second author would like to thank Stevo Todorčević for the discussions we had years ago about some of the results presented in this article.

References

[1] J. Avigad. An effective proof that open sets are Ramsey. *Arch. Math. Logic*, 37(4):235–240, 1998.

[2] F. Galvin and K. Prikry. Borel sets and Ramsey's theorem. *J. Symbolic Logic*, 38:193–198, 1973.

[3] S. Gao. Invariant Descriptive Set Theory. *Pure and Applied Mathematics*, 293. Taylor & Francis Group, 2009. ISBN-13: 978-1-58488-793-5.

[4] J. Grebík and M. Hrušák. No minimal tall Borel ideal in the Katětov order. *arxiv.org/pdf/1708.05322.pdf*, 2017.

[5] M. Hrušák. Katětov order on borel ideals. *Archive for Mathematical Logic*, 56(7-8):831–847, 2017.

[6] M. Hrušák, D. Meza-Alcántara, E. Thümmel, and C. Uzcátegui. Ramsey type properties of ideals. *Annals of Pure and Applied Logic*, 168(11):2022–2049, 2017.
[7] A. S. Kechris. *Classical Descriptive Set Theory*. Springer-Verlag, 1994.
[8] A.R.D. Mathias. Happy families. *Ann. Math. Log.*, 12:59–11, 1977.
[9] K. Mazur, F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$, *Fundamenta Mathematicae*, **138**(2): 103–111, 1991.
[10] C. St. J. A. Nash-Williams. On better-quasi-ordering transfinite sequences. *Mathematical Proceedings of the Cambridge Philosophical Society*, 64(2):273–290, 1968.
[11] J. Silver. Every analytic set is Ramsey. *J. Symbolic Logic*, 35:60–64, 1970.
[12] S. Solecki, Filters and sequences, Fundamenta Mathematicae, **163** (2000), no. 3, 215–228.
[13] R. Solovay. Hyperarithmetically encodable sets. *Trans. Amer. Math. Soc.*, 239:99–122, 1978.
[14] S. Todorčević. Higher dimensional Ramsey theory. In *Ramsey methods in analysis*, Advanced courses in mathematics, CRM Barcelona. Birkhäuser, 2005.
[15] S. Todorčević. *Introduction to Ramsey spaces*. Annals of Mathematical Studies 174. Princeton University Press, 2010.

Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 609/25, 110 00 Praha 1-Nové Město, Czech Republic
E-mail address: grebik@math.cas.cz

Escuela de Matemáticas, Universidad Industrial de Santander, Cra. 27 Calle 9 UIS Edificio 45, Bucaramanga, Colombia
E-mail address: cuzcatea@saber.uis.edu.co