A REMARK TO THE PAPER “ON CONDENSING DISCRETE DYNAMICAL SYSTEMS”

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Abstract. In the paper a new proof of Lemma 11 in the above-mentioned paper is given. Its original proof was based on Theorem 3 which has been shown to be incorrect.

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MSC 2000: 37B99, 47H07, 47H10, 37C99

Introduction

Theorem 3 in [4, p. 292] is not correct as the following example of a non locally connected continuum in $\mathbb{R}^2$ shows. This example was suggested by N. Dancer in [1]. (For similar results, see [3, p. 162], [2, Example 5.1].)

$$X = \{(0,y)\colon\ 0 \leqslant y \leqslant 2\} \cup \{(x,y)\colon\ y = 1 + \sin\tfrac{1}{x},\ 0 < x \leqslant \tfrac{2}{\pi}\} \cup \{(x,2)\colon\ \tfrac{2}{\pi} < x \leqslant 2\}.$$

In view of this, Theorem 4, Remark 4, Lemma 9 and Theorem 5, Lemma 11 in [4] are true in a weaker formulation. They only guarantee the existence of a continuum of sub- and superequilibria and a continuum of equilibria, respectively. They will be rewritten here. Also a new proof of Lemma 11 from the above-mentioned paper will be given. This will guarantee that, with these changes, all results of [4] remain valid.

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Theorem 4. *Let assumption* (H3) *be fulfilled, let* $[z_1, z_2] \subset [a, b]$ *be a positively invariant interval for the operator* T *and let* $z_1, z_2 \in C_2$. *Then the set* F *of all subequilibria and all superequilibria lying in* C_2 *forms a continuous branch connecting the points* z_1, z_2 *and contains a continuum possessing* z_1, z_2.

Remark 4. By Theorem 2, each equilibrium belongs to C_2. Further, if z is a subequilibrium (superequilibrium) and there is a sequence $z_k \to z$ such that z_k are superequilibria (subequilibria), then z is an equilibrium. We also have that the set of all equilibria lying in a continuum C is closed, and thus the set of all sub- and superequilibria in C is open (with respect to that continuum).

Theorem 5. *If assumption* (H3) *is satisfied and* $[z_1, z_2] \subset [a, b]$ *is a singular interval for the mapping* T, *then the set* F_p *of all equilibria lying in* $[z_1, z_2]$ *forms a continuous branch connecting the points* z_1, z_2 *and contains a continuum possessing* z_1, z_2.

Lemma 9. *Let assumption* (H3) *be fulfilled, let* $[z_1, z_2] \subset [a, b]$ *be a positively invariant interval for* T *and let* z_1, z_2 *be two equilibria. Then the following alternative holds: Either*
(a) *there exists a further equilibrium in* $[z_1, z_2]$, *or*
(b) *there exists a continuum* C *in* $[z_1, z_2]$ *containing* z_1, z_2 *such that all points of* C *except* z_1, z_2 *are strict subequilibria, or*
(c) *there exists a continuum* C *in* $[z_1, z_2]$ *containing* z_1, z_2 *such that all points of* C *except* z_1, z_2 *are strict superequilibria.*

Lemma 11. *Let assumption* (H3) *be satisfied, let* z_1, z_2 *be two equilibria such that* $a \leqslant z_1 < z_2 \leqslant b$ *and let* T *be order-preserving in* $[z_1, z_2]$. *Further, let all equilibria in* $[z_1, z_2]$ *be stable. Then there is a continuum of equilibria in* $[z_1, z_2]$ *containing* z_1, z_2.

The proof of this lemma will be based on Theorem 4 and on the following

Lemma. *Let assumption* (H3) *be fulfilled, let* $a \leqslant z_1 < z_2 \leqslant b$ *be two points such that* z_1 *(*z_2*) is a subequilibrium (superequilibrium) and* T *is order-preserving in* $[z_1, z_2]$. *Further, let all equilibria in* $[z_1, z_2]$ *be stable. Denote* F *(*F_p*) the set of all sub- and superequilibria (the set of all equilibria) lying in* $[z_1, z_2]$. *Then:*
(a) *For each* $x \in F$ *there exists* $\lim_{k\to\infty} T^k(x) \in [z_1, z_2]$.
(b) *The mapping* $U\colon F \to F_p$ *defined by*

$$U(x) = \lim_{k\to\infty} T^k(x), \quad x \in F, \tag{1}$$

is continuous.

P r o o f. The statement (a) follows from Lemma 10 and hence the mapping U defined by (1) is well-defined. Let $x \in F$ be an arbitrary point and $\varepsilon > 0$ an arbitrary number. Then by the stability of $y = U(x)$ there exists a $\delta > 0$ such that

(2) $\|y - T^k(u)\| < \varepsilon$ for each $u \in [z_1, z_2]$, $\|u - y\| < \delta$ and each natural k.

Since $\lim\limits_{k\to\infty} T^k(x) = y$, there exists a natural k_0 with the property

$$\|T^{k_0}(x) - y\| < \frac{\delta}{2}. \tag{3}$$

As T^{k_0} is continuous at x, there exists a $\delta_1 > 0$ such that $z \in F$, $\|x - z\| < \delta_1$ implies

$$\|T^{k_0}(x) - T^{k_0}(z)\| < \frac{\delta}{2}. \tag{4}$$

Then for each $z \in F$, $\|x - z\| < \delta_1$, (4) and (3) give that

$$\|T^{k_0}(z) - y\| \leqslant \|T^{k_0}(z) - T^{k_0}(x)\| + \|T^{k_0}(x) - y\| < \delta. \tag{5}$$

Put $u = T^{k_0}(z)$ in (2). In view of (5), (2) implies that

$$\|y - T^{k_0+k}(z)\| < \varepsilon \text{ for each natural } k. \tag{6}$$

Thus we get that $\|x - z\| < \delta_1$, $z \in F$, implies the inequality $\|U(x) - U(z)\| \leqslant \varepsilon$ which means the continuity of U at x. □

P r o o f of Lemma 11. By Theorem 4 above, there is a continuum C containing z_1, z_2 in the set F of all subequilibria and all superequilibria lying in C_2. Lemma assures the existence of a continuous map U which maps C onto a continuum of equilibria in $[z_1, z_2]$ containing z_1, z_2. □

References

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