ROUTE SYSTEMS ON GRAPHS

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Abstract. The well known types of routes in graphs and directed graphs, such as walks, trails, paths, and induced paths, are characterized using axioms on vertex sequences. Thus non-graphic characterizations of the various types of routes are obtained.

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1. Introduction

The concept of route system was introduced by Nebeský [1], [2] to study the set of all geodesics (shortest paths) in a connected graph. Using the idea of route systems, Nebeský was able to give a nice, non-metric characterization of geodesics, see [3].

In this note, motivated by Nebeský's original idea of a route system, we define a quite general notion of route system on a finite non empty set V as a collection of sequences of elements in V satisfying some simple axioms. To specialize various types of route systems, we use 'extension axioms' and 'exclusion axioms'. Given a route system, one can define the underlying graph. An important question is what the routes in the route system signify in the underlying graph. If the route system satisfies a symmetry axiom, then the underlying graph is undirected, otherwise the underlying graph will be directed. We present axioms for the route systems (directed as well as undirected) which give, respectively, the walks, the trails, the paths, the minimal paths, and the triangular minimal paths, in the underlying (di)graph.

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Let V be a nonempty, finite set. By $L(V)$ we denote the set of all finite sequences of elements in V. For α, β in $L(V)$, the sequence $\alpha\beta$ is the concatenation of the sequences α and β. The empty sequence is denoted by ω. Thus, for any sequence α in $L(V)$, we have $\omega\alpha = \alpha\omega = \alpha$. If $\alpha = v_1 v_2 \ldots v_k$, then $|\alpha| = k$ is the length of the sequence, and $\bar{\alpha} = v_k v_{k-1} \ldots v_2 v_1$ is the sequence obtained from α by reversing the order of the elements in α. We call this sequence the *inverse* of α. In particular, $|\omega| = 0$, and $\bar{\omega} = \omega$. In the sequel, we will denote elements of V by roman letters, and sequences in $L(V)$ by Greek letters. Thus, the sequence $u\alpha v$ has u as first element and v as last element, whereas α is a, possibly empty, sequence in $L(V)$.

Let α be a nonempty sequence in $L(V)$. We denote the first element of α by $\Phi(\alpha)$. Let α, β, and γ be, possibly empty, sequences in $L(V)$. Then we write $\beta \in \alpha\beta\gamma$. Note that β is a subsequence consisting of consecutive elements of $\alpha\beta\gamma$. If β is non empty, we write $\alpha < \alpha\beta$. Note that the $<$ symbol is only used for sequences having the same first element. Of course, if we allow β to be empty, then we write $\alpha \leqslant \alpha\beta$.

A *route system* on the set V is a family of nonempty sequences $\mathcal{R} \subseteq L(V)$ satisfying the following axioms:

$(r0)$ $u \in \mathcal{R}$ for any $u \in V$,

$(r1)$ $u\alpha v \in \mathcal{R} \Rightarrow u\alpha, \alpha v \in \mathcal{R}$.

Let $\mathcal{R}$ be a route system. The sequences in $\mathcal{R}$ are called *routes*. The *length* of a route α is $|\alpha| - 1$. The axiom $(r0)$ serves only to exclude certain degenerate cases. Let α be a route in $\mathcal{R}$, and let $\beta \in \alpha$. Then we call β a *subroute* of α. Axiom $(r1)$ states that every subroute of a route is itself a route. Hence we call $(r1)$ the *subroute axiom*.

The *underlying digraph* $D_{\mathcal{R}} = (V, A)$ of $\mathcal{R}$ has vertex set V and (u, v) is an arc in $D_{\mathcal{R}}$ if and only if uv is a route in $\mathcal{R}$. We will write $uv = (u, v)$, for short. Note that, if we would drop $(r0)$, then we should take as vertex set of D the set $W = \{v \in V;\ v \in \mathcal{R}\}$.

The next two axioms are *symmetry axioms*:

$(s0)$ $uv \in \mathcal{R} \Rightarrow vu \in \mathcal{R}$,

$(s1)$ $\alpha \in \mathcal{R} \Rightarrow \bar{\alpha} \in \mathcal{R}$.

Axiom $(s0)$ is the *weak symmetry* axiom, and axiom $(s1)$ is the *strong symmetry* axiom, or the *symmetry axiom*, for short. The *underlying graph* $G_{\mathcal{R}} = (V, E)$ of a weakly symmetric route system has vertex set V and uv is an edge in $G_{\mathcal{R}}$ if and only if uv lies in $\mathcal{R}$.

In the sequel we will prove a number of results on route systems. The symmetric, or undirected analogues usually follow straightforward from the general, or directed case by applying $(s1)$. It turns out that, in each case, we will only require weak symmetry, and that strong symmetry follows from weak symmetry and the other axioms. To obtain the undirected case, sometimes just a simple adaptation of the proof is necessary. We omit proofs for the undirected case.

We start with our most general case, the route system of all walks, and specialize as we proceed. Basically we require two types of axioms for the special cases, namely an *extension axiom* to generate new routes from existing ones and an *exclusion axiom* to exclude the non-routes from the route system. We list the axioms as follows: the extension axiom will have an even number $2i$, while the associated exclusion axiom will have the odd number $2i+1$. Extension axioms $(r2)$ and $(r2)^*$ have no matching exclusion axiom.

There are several possibilities for the extension and exclusion axioms for the various types of routes that we consider. We propose two types. The first type always involves two routes of the same length in the extension axiom. The second type always extends an existing route respectively with an arc or edge only. To distinguish between the two types, we denote the axioms of the second type by $(rk)^*$.

First we list all axioms, where α is understood to be a, possibly empty, sequence:

$(r2)$ $\quad u\alpha x,\ \alpha xv \in \mathcal{R} \Rightarrow u\alpha xv \in \mathcal{R}$,

$(r4)$ $\quad u\alpha x,\ \alpha xv \in \mathcal{R},\ u\Phi(\alpha x) \neq xv, \Rightarrow u\alpha xv \in \mathcal{R}$,

$(r5)$ $\quad u\alpha xv \in \mathcal{R} \Rightarrow u\Phi(\alpha x) \neq xv$;

$(r6)$ $\quad u\alpha x, \alpha xv \in \mathcal{R}, u \neq v \Rightarrow u\alpha xv \in \mathcal{R}$,

$(r7)$ $\quad u\alpha v \in \mathcal{R} \Rightarrow u \neq v$,

$(r8)$ $\quad u\alpha x,\ \alpha xv \in \mathcal{R},\ u \neq v,\ uv \notin \mathcal{R} \Rightarrow u\alpha xv \in \mathcal{R}$,

$(r9)$ $\quad u\alpha xv \in \mathcal{R} \Rightarrow uv \notin \mathcal{R}$,

$(r10)$ $\quad uv, vw \in \mathcal{R}, u \neq w \Rightarrow uvw \in \mathcal{R}$,

$\quad\quad u\alpha xy,\ \alpha xyv \in \mathcal{R},\ u \neq v,\ uv \notin \mathcal{R} \Rightarrow u\alpha xyv \in \mathcal{R}$,

$(r11)$ $\quad u\alpha xyv \in \mathcal{R} \Rightarrow uv \notin \mathcal{R}$,

$(r2)^*$ $\quad \alpha v, vw \in \mathcal{R} \Rightarrow \alpha vw \in \mathcal{R}$,

$(r4)^*$ $\quad \alpha v, vw \in \mathcal{R}, vw \notin \alpha v \Rightarrow \alpha vw \in \mathcal{R}$,

$(r4)^{**}$ $\quad \alpha v, vw \in \mathcal{R}, vw \notin \alpha v,\ wv \notin \alpha v \Rightarrow \alpha vw \in \mathcal{R}$,

$(r5)^*$ $\quad \alpha xv \in \mathcal{R} \Rightarrow xv \notin \alpha x$,

$(r5)^{**}$ $\quad \alpha xv \in \mathcal{R} \Rightarrow xv \notin \alpha x$ and $vx \notin \alpha x$,

$(r6)^*$ $\quad \alpha v,\ vw \in \mathcal{R},\ w \notin \alpha v \Rightarrow \alpha vw \in \mathcal{R}$,

$(r7)^*$ $= (r7)$,

$(r8)^*$ $\quad \alpha v,\ vw \in \mathcal{R},\ w \notin \alpha v$, and $\alpha' w \notin \mathcal{R}$ (for $\alpha' \leqslant \alpha$) $\Rightarrow \alpha vw \in \mathcal{R}$,

$(r9)^*$ $= (r9)$,

$(r10)^*$ $\quad uv, vw \in \mathcal{R}, u \neq w \Rightarrow uvw \in \mathcal{R}$,

$\quad\quad \alpha v,\ vw \in \mathcal{R},\ w \notin \alpha v$, and $\alpha' w \notin \mathcal{R}$ (for $\alpha' < \alpha$) $\Rightarrow \alpha vw \in \mathcal{R}$,

$(r11)^*$ $= (r11)$.

To avoid confusion, we list some terminology on graphs and digraphs. All our graphs will be without multiple edges or arcs, but we allow loops. Let $D = (V, A)$ be a digraph. A *directed walk* in D is a sequence of vertices $v_1 v_2 \ldots v_k$ such that $v_i v_{i+1}$ is an arc for $1 \leqslant i < k$. Similarly, a *walk* in a graph $G = (V, E)$ is a sequence of vertices $v_1v_2 \ldots v_k$ such that v_iv_{i+1} is an edge for $1 \leqslant i < k$. The length of the walk is $k-1$. A (*directed*) *trail* is a (directed) walk without repetition of arcs or edges, respectively. Note that in our terminology a single loop is a trail in the directed as well as in the undirected case. A (*directed*) *path* is a (directed) trail without repetition of vertices. A (*directed*) *minimal path* is a (directed) path in which v_iv_j is not an arc nor an edge, respectively, for $1 \leqslant i < j-1 \leqslant k$. A (*directed*) *triangular minimal path* is a (directed) path, in which v_iv_j is not an arc nor an edge, respectively, for $1 \leqslant i < j-2 < k$. A (*directed*) *geodesic* is a (directed) path of minimal length between its endpoints. A (*directed*) *triangular geodesic* is a (directed) triangular minimal path such that, if we take a minimal (directed) subpath P contained in it, then it is a (directed) geodesic, and each other vertex of the path is adjacent to only two consecutive vertices of P.

Let $D = (V, A)$ be a digraph, and let $G = (V, E)$ be a graph. Clearly, the family of all directed walks in D is a route system satisfying $(r2)$ as well as $(r2)^*$. The family of all directed trails in D is a route system satisfying $(r4)$ and $(r5)$, as well as $(r4)^*$ and $(r5)^*$. The family of all directed paths in D is a route system satisfying $(r6)$ and $(r7)$, as well as $(r6)^*$ and $(r7)^*$. The family of all directed minimal paths in D is a route system satisfying $(r7)$, $(r8)$ and $(r9)$, as well as $(r7)^*$, $(r8)^*$ and $(r9)^*$. The family of directed triangular minimal paths in D is a route system satisfying $(r7)$, $(r10)$ and $(r11)$, as well as $(r7)^*$, $(r10)^*$ and $(r11)^*$. Clearly, the walks, trails, paths, minimal paths, and triangular minimal paths of G all satisfy axioms $(s0)$ and $(s1)$, and the respective extension and exclusion axiom.

A nice and non-trivial characterization of the route system consisting of all geodesics of a graph was given by Nebeský in [3]. Thus he was able to produce a non-metric characterization of the set of all geodesics of a graph. The characterization of the directed route system of all directed geodesics in a digraph is still open, as is the characterization of the (directed) route system of all (directed) triangular geodesics of a (di)graph.

All the lemmata below are proved by induction on the length of the routes. The basic idea for characterizing the route system as the family of directed walks in a directed graph is given in Lemma 1. The undirected analogue is given in Lemma 2.

Lemma 1. *Let $\mathcal{R}$ and $\mathcal{S}$ be route systems on V with $D_{\mathcal{R}} = D_{\mathcal{S}}$. If $\mathcal{R}$ and $\mathcal{S}$ both satisfy $(r2)$, then $\mathcal{R} = \mathcal{S}$.*

P r o o f. By induction on the lengths of the routes in $\mathcal{R}$, we prove that $\mathcal{R} \subseteq \mathcal{S}$. Let $\alpha = v_1v_2 \ldots v_k$ be a route in $\mathcal{R}$. If $k = 1$, then α lies in $\mathcal{S}$, by $(r0)$. If $k = 2$, then α lies in $\mathcal{S}$, by the fact that the underlying digraphs of $\mathcal{R}$ and $\mathcal{S}$ have the same arcs. So assume that $k \geqslant 3$. By $(r1)$, it follows that $v_1v_2 \ldots v_{k-1}$ and $v_2 \ldots v_{k-1}v_k$ are in

$\mathcal{R}$. So, by induction, they are in $\mathcal{S}$, whence, by $(r2)$, it follows that α lies in $\mathcal{S}$, and we are done. Similarly, it follows that $\mathcal{S} \subseteq \mathcal{R}$, by which the lemma is proved. $\square$

The next lemma follows immediately from Lemma 1, and we omit the proof.

Lemma 2. *Let $\mathcal{R}$ and $\mathcal{S}$ be weakly symmetric route systems on V with $G_{\mathcal{R}} = G_{\mathcal{S}}$. If $\mathcal{R}$ and $\mathcal{S}$ both satisfy $(r3)$, then $\mathcal{R} = \mathcal{S}$.*

As observed above, the directed walks in a digraph, as well as the walks in a graph, form a route system satisfying $(r2)$. Hence the next theorem follows immediately from Lemmata 1 and 2.

Theorem 3. *Let V be a finite nonempty set, and let $\mathcal{R}$ be a route system on V. Then $\mathcal{R}$ is the family of all directed walks of $D_{\mathcal{R}}$ if and only if $\mathcal{R}$ satisfies $(r2)$. If $\mathcal{R}$ satisfies $(s0)$, then $\mathcal{R}$ is the family of all walks of $G_{\mathcal{R}}$ if and only if $\mathcal{R}$ satisfies $(r2)$.*

Lemma 4. *Let $\mathcal{R}$ and $\mathcal{S}$ be route systems on V with $D_{\mathcal{R}} = D_{\mathcal{S}}$. If $\mathcal{R}$ and $\mathcal{S}$ both satisfy $(r2i)$ and $(r2i+1)$, then $\mathcal{R} = \mathcal{S}$ for $i = 2, 3, 4, 5$.*

Proof. By induction on the lengths of the routes in $\mathcal{R}$, we prove that $\mathcal{R} \subseteq \mathcal{S}$. Let $\alpha = v_1 v_2 \ldots v_k$ be a route in $\mathcal{R}$. If $k = 1$, then α lies in $\mathcal{S}$, by $(r0)$. If $k = 2$, then α lies in $\mathcal{S}$, by the fact that the underlying digraphs of $\mathcal{R}$ and $\mathcal{S}$ have the same arcs. So assume that $k \geqslant 3$. In the cases $i = 4, 5$ we assume that $v_1 v_k$ is not in $D_{\mathcal{R}} = D_{\mathcal{S}}$. Since $\mathcal{R}$ satisfies $(r2i + 1)$, we know that $v_1 v_2 \neq v_{k-1} v_k$ in the case $i = 2$, or that $v_1 \neq v_k$ in the cases $i = 3, 4, 5$. By $(r1)$, it follows that $v_1 v_2 \ldots v_{k-1}$ and $v_2 \ldots v_{k-1} v_k$ are in $\mathcal{R}$. So, by induction, they are in $\mathcal{S}$. Since $\mathcal{S}$ satisfies $(r2i)$, it follows that α is in $\mathcal{S}$. Similarly, it follows that $\mathcal{S} \subseteq \mathcal{R}$, by which the lemma is proved. $\square$

The weakly symmetric analogue of Lemma 4 is given in the next lemma. The proof is straightforward.

Lemma 5. *Let $\mathcal{R}$ and $\mathcal{S}$ be weakly symmetric route systems on V with $G_{\mathcal{R}} = G_{\mathcal{S}}$. If $\mathcal{R}$ and $\mathcal{S}$ both satisfy $(r2i)$ and $(r2i + 1)$ then $\mathcal{R} = \mathcal{S}$ for $i = 2, 3, 4, 5$.*

Lemmata 4 and 5 provide us with the following batch of theorems, in which the route systems associated with the various types of walks in a (di)graph are characterized.

Theorem 6. *Let V be a finite nonempty set, and let $\mathcal{R}$ be a route system on V. Then $\mathcal{R}$ is the family of all directed trails of $D_{\mathcal{R}}$ if and only if $\mathcal{R}$ satisfies $(r4)$ and $(r5)$. If $\mathcal{R}$ satisfies $(s0)$, then $\mathcal{R}$ is the family of all trails of $G_{\mathcal{R}}$ if and only if $\mathcal{R}$ satisfies $(r4)$ and $(r5)$.*

Theorem 7. *Let V be a finite nonempty set, and let $\mathcal{R}$ be a route system on V. Then $\mathcal{R}$ is the family of all directed paths of $D_{\mathcal{R}}$ if and only if $\mathcal{R}$ satisfies $(r6)$ and $(r7)$. If $\mathcal{R}$ satisfies $(s0)$, then $\mathcal{R}$ is the family of all paths of $G_{\mathcal{R}}$ if and only if $\mathcal{R}$ satisfies $(r6)$ and $(r7)$.*

Theorem 8. *Let V be a finite nonempty set, and let $\mathcal{R}$ be a route system on V. Then $\mathcal{R}$ is the family of all directed minimal paths of $D_{\mathcal{R}}$ if and only if $\mathcal{R}$ satisfies $(r7)$, $(r8)$ and $(r9)$. If $\mathcal{R}$ satisfies $(s0)$, then $\mathcal{R}$ is the family of all minimal paths of $G_{\mathcal{R}}$ if and only if $\mathcal{R}$ satisfies $(r7)$, $(r8)$ and $(r9)$.*

Theorem 9. *Let V be a finite nonempty set, and let $\mathcal{R}$ be a route system on V. Then $\mathcal{R}$ is the family of all directed triangular minimal paths of $D_{\mathcal{R}}$ if and only if $\mathcal{R}$ satisfies $(r7)$, $(r10)$ and $(r11)$. If $\mathcal{R}$ satisfies $(s0)$, then $\mathcal{R}$ is the family of all triangular minimal paths of $G_{\mathcal{R}}$ if and only if $\mathcal{R}$ satisfies $(r7)$, $(r10)$ and $(r11)$.*

All the above results remain true if we replace (rk) by $(rk)^*$ for $k = 2, 4, \ldots, 11$, except in the undirected case of Theorem 6. In that case we replace $(r4), (r5)$ by $(r4)^{**}, (r5)^{**}$. Of course, the proofs of the Lemmata have to be slightly adapted in each case.

In this note we have shown how route systems may be utilized in association with routing in graphs. This is only a start of the study of route systems. The problems of the characterization of the route system of the directed geodesics in a digraph and the (directed) triangular geodesics in (di)graphs are still open. Many other properties may be studied. To name a few examples, we call a route system $\mathcal{R}$ on a set V weakly connected if, for any u and v in V, there exists a route $u\alpha v$ or a route $v\alpha u$. Now the study of various types of connectivity comes into view. Another type of problems arises when we study route systems which consist of only certain walks or paths in a graph. For example, do there exist Menger type theorems for such route systems?

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