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Abstract

A reaction-diffusion system exhibiting Turing's diffusion driven instability is considered. The equation for an activator is supplemented by unilateral terms of the type $s_-(\mathbf{x})u^-$, $s_+(\mathbf{x})u^+$ describing sources and sinks active only if the concentration decreases below and increases above, respectively, the value of the basic spatially constant solution which is shifted to zero. We show that the domain of diffusion parameters in which spatially non-homogeneous stationary solutions can bifurcate from that constant solution is smaller than in the classical case without unilateral terms. It is a dual information to previous results stating that analogous terms in the equation for an inhibitor imply the existence of bifurcation points even in diffusion parameters for which bifurcation is excluded without unilateral sources. The case of mixed (Dirichlet-Neumann) boundary conditions as well as that of pure Neumann conditions is described.

Keywords: reaction-diffusion systems, unilateral terms, Turing's patterns, positively homogeneous operators, maximal eigenvalue
MSC: 35K57, 35B32, 35J57, 35J50, 92C15

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1. Introduction

Let's consider a reaction-diffusion system

$$\frac{\partial u}{\partial t} = d_1 \Delta u + f(u,v) + \tilde{f}_-(\mathbf{x}, u^-) - \tilde{f}_+(\mathbf{x}, u^+),$$
$$\frac{\partial v}{\partial t} = d_2 \Delta v + g(u,v) \quad \text{in } \Omega \times [0, +\infty) \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, d_1 and d_2 are positive parameters (diffusion coefficients), $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are real differentiable functions, $\tilde{f}_-, \tilde{f}_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ are functions satisfying Carathéodory conditions and such that there exist

$$s_-(\mathbf{x}) := \frac{\partial \tilde{f}_-}{\partial \xi}(\mathbf{x}, \xi)|_{\xi=0} \geq 0, \quad s_+(\mathbf{x}) := \frac{\partial \tilde{f}_+}{\partial \xi}(\mathbf{x}, \xi)|_{\xi=0} \geq 0 \quad \text{for a.a. } \mathbf{x} \in \Omega, s_\pm \in L_\infty(\Omega). \tag{2}$$

As usual, $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ denotes the positive and negative, respectively, part of u. We will always assume that

$$f(0,0) = g(0,0) = \tilde{f}_-(\mathbf{x}, 0) = \tilde{f}_+(\mathbf{x}, 0) = 0 \quad \text{for a.a. } \mathbf{x} \in \Omega. \tag{3}$$

Our system will be supplemented by boundary conditions

$$u = v = 0 \quad \text{on } \Gamma_D,$$
$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma_N, \tag{4}$$

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where n is the unit outward-pointing normal vector of the boundary $\partial\Omega$ and Γ_N, Γ_D are open disjoint subsets of $\partial\Omega$, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$.

Apparently the problem $(1), (4)$ has always the trivial solution $[0, 0]$. Our system should describe a reaction of two chemicals, e.g. morphogens, having a basic positive spatially constant steady state $[\overline{u}, \overline{v}]$, that means we should assume in fact $f(\overline{u}, \overline{v}) = g(\overline{v}, \overline{v}) = \tilde{f}_-(\mathbf{x}, \overline{u}) = \tilde{f}_+(\mathbf{x}, \overline{u}) = 0$ instead of (3), but as usually, we can shift the positive steady state to zero and we obtain our system satisfying (3). Let us emphasize that then the functions u, v do not describe concentrations of the reactants, but their differences from the basic constant stationary state $[\overline{u}, \overline{v}]$.

We will consider assumptions under which the problem $(1), (4)$ with $\tilde{f}_- \equiv \tilde{f}_+ \equiv 0$ exhibits diffusion driven instability discovered in the famous Turing's paper [11]. That means if $\tilde{f}_- \equiv \tilde{f}_+ \equiv 0$ then the trivial solution $[0, 0]$ is stable as a solution of the corresponding problem without diffusion (ODE's obtained for $d_1 = d_2 = 0$), but as a solution of the whole system it is unstable for $[d_1, d_2]$ from a certain subdomain D_U of the positive quadrant $\mathbb{R}_+^2$ (domain of instability), and stable only for $[d_1, d_2] \in D_S = \mathbb{R}_+^2 \setminus \overline{D_U}$ (domain of stability). Spatially non-homogeneous steady states bifurcate from the basic constant equilibrium in some points of $\overline{D_U}$, but such a bifurcation is excluded in D_S. Let us note that spatially non-homogeneous steady states can describe spatial patterns in some models in biology.

Our goal is to prove that if we add unilateral terms $\tilde{f}_-(\mathbf{x}, u^-)$, $\tilde{f}_+(\mathbf{x}, u^+)$, then the domain of diffusion coefficients where spatially non-homogeneous steady states can bifurcate is smaller than $\overline{D_U}$. In fact we will prove more, see below. An example of unilateral terms can be

$$\tilde{f}_-(\mathbf{x}, u^-) = s_-(\mathbf{x})\frac{u^-}{1 + \varepsilon u^-}, \quad \tilde{f}_+(\mathbf{x}, u^+) = s_+(\mathbf{x})\frac{u^+}{1 + \varepsilon u^+}.$$

The stationary system corresponding to (1) can be written in the form

$$\begin{aligned} d_1\Delta u + b_{1,1}u + b_{1,2}v + n_1(u, v) + \tilde{f}_-(\mathbf{x}, u^-) - \tilde{f}_+(\mathbf{x}, u^+) &= 0, \\ d_2\Delta v + b_{2,1}u + b_{2,2}v + n_2(u, v) &= 0, \end{aligned} \tag{5}$$

where $\mathbf{B} := (b_{i,j})_{i,j=1,2}$ is the Jacobi matrix of the mappings f, g at $[0, 0]$ and the functions n_1, n_2 are higher order terms, i.e.

$$n_{1,2}(u, v) = o(|u| + |v|) \text{ as } |u| + |v| \to 0. \tag{6}$$

(The nonlinear part in the first equation could be written also in the form $s_-(\mathbf{x})u^- - s_+(\mathbf{x})u^+ + \tilde{n}_1(\mathbf{x}, u, v)$, that means a homogenization + higher order terms dependent on $\mathbf{x}$).

We will always assume that the following conditions necessary for Turing's diffusion driven instability mentioned above are fulfilled:

$$b_{1,1} > 0, b_{2,2} < 0, b_{1,2}b_{2,1} < 0, tr(\mathbf{B}) < 0, det(\mathbf{B}) > 0. \tag{7}$$

The first three conditions in (7) correspond to an activator-inhibitor system (for $b_{1,2} < 0, b_{2,1} > 0$), or to a substrate depletion system (for $b_{1,2} > 0, b_{2,1} < 0$), see e.g. [9]. The last two conditions ensure the stability of $[0, 0]$ as a solution of the system without any diffusion.

We will work mainly with the homogenized system

$$\begin{aligned} d_1\Delta u + b_{1,1}u + b_{1,2}v + s_-(\mathbf{x})u^- - s_+(\mathbf{x})u^+ &= 0, \\ d_2\Delta v + b_{2,1}u + b_{2,2}v &= 0. \end{aligned} \tag{8}$$

We will show more than what is mentioned above, namely that critical points, i.e. couples $[d_1, d_2]$ for which the homogenized problem $(8), (4)$ has a non-trivial solution, can exist only in a smaller domain than in the classical case $\tilde{s}_- = \tilde{s}_+ \equiv 0$. Since any bifurcation point is simultaneously a critical point, the main goal mentioned above will follow. A similar result was proved in [5] for the case of unilateral sources on the boundary described by quasi-variational inequalities, but we consider the description of unilateral sources and sinks by the terms $\tilde{f}_-(\mathbf{x}, u^-)$, $\tilde{f}_+(\mathbf{x}, u^+)$ more

natural. We will briefly discuss also problems with unilateral terms of the type $s_-(x)u^-$, $s_+(x)u^+$ on the boundary.

Main ideas are similar to those from [5]. Considering a weak formulation, we will write our problem as a system of operator equations in Sobolev space and we will consider an arbitrary fixed d_2. Expressing the variable v from the second equation and substituting it to the first equation, we reduce the originally non-symmetric problem to a single equation with a positively homogeneous operator having a potential. A variational characterization of its largest eigenvalue enables us to compare the largest eigenvalue corresponding to the problem with and without unilateral terms, which is simultaneously the largest d_1 for which $[d_1, d_2]$ is a critical point of the original system with and without unilateral terms.

Let us note that if unilateral sources of the second variable v (inhibitor) are supplemented in the second equation then bifurcation of spatial patterns occurs even in the domain D_S, where it is excluded for the classical case without unilateral sources. See e.g. [7] and references therein for the case of sources described by variational inequalities, [4] for unilateral sources described by multivalued maps and [3], [6] for the case of unilateral terms similar to the current paper. These results motivated numerical experiments [12] showing that for a concrete model also spatial patterns arise from small initial perturbations for diffusion parameters from D_S, where it is not the case without unilateral sources. The sense of these results is positive because one of the problems of Turing's theory is that the set of diffusion parameters for which diffusion-driven instability occurs is too small, so unilateral sources for v improve this situation. The result of the current paper is opposite, unilateral sources for u makes larger the set of diffusion parameters for which bifurcation of spatial patterns is exluded, i.e. for which no small spatial patterns can exist. We believe that, at least in some cases, it is a signal that the same is true for the set of parameters for which spatial patterns evolve from small perturbations of the basic spatially constant steady state. It agrees with numerical experiments which will be published in a forthcoming paper. This seems to be a negative result, but perhaps there are situations when it would be valuable to understand how to prevent evolution of spatial patterns. For instance, patterns play a role in models of tumors, see e.g. [1] and references therein. In spite of that the paper [1] has completely different goals, it can be perhaps motivating from the point of view mentioned, in particular its Section 5.

We present the basic general assumptions and definitions in Section 2. Main results of this paper are formulated and discussed in Section 3. In Section 4 we formulate our problem as a system of operator equations in Sobolev space and we describe properties of the corresponding operators. Section 5 concerns a reduction of our system to a single equation with a positively homogeneous operator and a variational characterization of its largest eigenvalue. A comparison of largest eigenvalues and consequently also critical points with and without unilateral terms by using this variational characterization is given. The proofs of the main results are done in Section 6.

2. Basic assumptions and definitions

We will always suppose that there exists $c \in \mathbb{R}$ such that

$$|n_j(\chi, \xi)| \leq c(1 + |\chi|^{q-1} + |\xi|^{q-1}) \quad \text{for all } \chi, \xi \in \mathbb{R}, j = 1, 2, \tag{9}$$

$$|\tilde{f}_{\mp}(\mathbf{x}, \xi)| \leq c(1 + |\xi|^{q-1}) \quad \text{for all } \xi \in \mathbb{R} \text{ and a.a. } \mathbf{x} \in \Omega, \tag{10}$$

with some $q > 2$ if $N = 2$ or $2 < q < \frac{2N}{N-2}$ if $N > 2$. In the dimension $N = 1$ no growth assumptions are necessary.

Besides systems (8) and (5) we will discuss systems

$$\begin{aligned} d_1 \Delta u + b_{1,1} u + b_{1,2} v &= 0, \\ d_2 \Delta v + b_{2,1} u + b_{2,2} v &= 0 \end{aligned} \tag{11}$$

and

$$\begin{aligned} d_1 \Delta u + b_{1,1} u + b_{1,2} v + n_1(u, v) &= 0, \\ d_2 \Delta v + b_{2,1} u + b_{2,2} v + n_2(u, v) &= 0. \end{aligned} \tag{12}$$

By solutions we will always mean weak solutions in the space

$$H_D^1(\Omega) := \{\phi \in W^{1,2}(\Omega) : \phi = 0 \text{ on } \Gamma_D \text{ in the sense of traces}\}. \tag{13}$$

If $\Gamma_D = \emptyset$, then the space H_D^1 is actually the whole Sobolev space $W^{1,2}$ equipped with the standard inner product

$$(u, \varphi)_{H_D^1} = (u, \varphi)_{W^{1,2}} = \int_\Omega (\nabla u \nabla \varphi + u\varphi) \; d\Omega \tag{14}$$

and the Sobolev norm $\|u\|_{W^{1,2}} = \left(\int_\Omega (\nabla u)^2 + u^2 \; d\Omega\right)^{\frac{1}{2}}$. If $\Gamma_D \neq \emptyset$, then we will use the inner product

$$(u, \varphi)_{H_D^1} = \int_\Omega \nabla u \nabla \varphi \; d\Omega \tag{15}$$

and the norm $\|u\|_{H_D^1} = \left(\int_\Omega (\nabla u)^2 \; d\Omega\right)^{\frac{1}{2}}$ equivalent to the classical Sobolev norm.

Definition 2.1 (Critical point).
A parameter $d = [d_1, d_2] \in \mathbb{R}_+^2$ will be called a critical point of $(11), (4)$ or $(8), (4)$ if there exists a non-trivial (weak) solution of $(11), (4)$ or $(8), (4)$, respectively.

Definition 2.2 (Bifurcation point).
A parameter $d^0 = [d_1^0, d_2^0] \in \mathbb{R}_+^2$ will be called a bifurcation point of $(12), (4)$ or $(5), (4)$ if in any neighbourhood of $[d^0, 0, 0] \in \mathbb{R}_+^2 \times H_D^1 \times H_D^1$ there exists $[d, W] = [d, u, v]$, $\|W\| \neq 0$ satisfying $(12), (4)$ or $(5), (4)$, respectively.

Remark 2.1. *Let's consider the problem*

$$\begin{aligned} -\Delta u &= \kappa u, \\ u &= 0 \; on \; \Gamma_D, \\ \frac{\partial u}{\partial n} &= 0 \; on \; \Gamma_N. \end{aligned} \tag{16}$$

The eigenvalues of (16) form a non-negative non-decreasing sequence κ_j with $j = 1, 2, \ldots$ (for $\Gamma_D \neq \emptyset$) or $j = 0, 1, 2, \ldots$ (for $\Gamma_D = \emptyset$). The first eigenvalue is always simple. In the case $\Gamma_D \neq \emptyset$, the eigenfunction e_1 corresponding to the first eigenvalue κ_1 does not change the sign on the domain Ω. In the case $\Gamma_D = \emptyset$, the eigenfunction e_0 corresponding to the first eigenvalue $\kappa_0 = 0$ is constant. Other eigenfunctions change the sign in both cases. We can choose an orthonormal basis e_j in H_D^1, $j = 1, 2, \ldots$ (for $\Gamma_D \neq \emptyset$) or $j = 0, 1, 2, \ldots$ (for $\Gamma_D = \emptyset$) composed of the eigenfunctions of (16).

Let's remind that the conditions (7) are always considered. The sets

$$C_j := \left\{ [d_1, d_2] \in \mathbb{R}_+^2 : \; d_1 = \frac{1}{\kappa_j}\left(\frac{b_{1,2}b_{2,1}}{d_2\kappa_j - b_{2,2}} + b_{1,1}\right) \right\}, \; j = 1, 2, \ldots \tag{17}$$

are hyperbolas (or more specifically their parts) in the positive quadrant $\mathbb{R}_+^2$. Let's note that we present hyperbolas in the different form than usually, namely with respect to d_1. It is of course equivalent to the standard form derived from the relation

$$(\kappa_j d_1 - b_{1,1})(\kappa_j d_2 - b_{2,2}) - b_{1,2}b_{2,1} = 0$$

(see e.g. [9]). If $\Gamma_D = \emptyset$, for $j = 0$ the last equality is never satisfied, because $det(B)$ is positive by (7). The envelope

$$C_E := \left\{ d = [d_1, d_2] \in \mathbb{R}_+^2 : \; d_1 = \max_{\tilde{d}_1 \in \mathbb{R}_+} \left\{ \tilde{d}_1 : [\tilde{d}_1, d_2] \in \bigcup_{j=1}^\infty C_j \right\} \right\} \tag{18}$$

divides the positive quadrant $\mathbb{R}_+^2$ onto two sets D_U and D_S (see Figure 1).
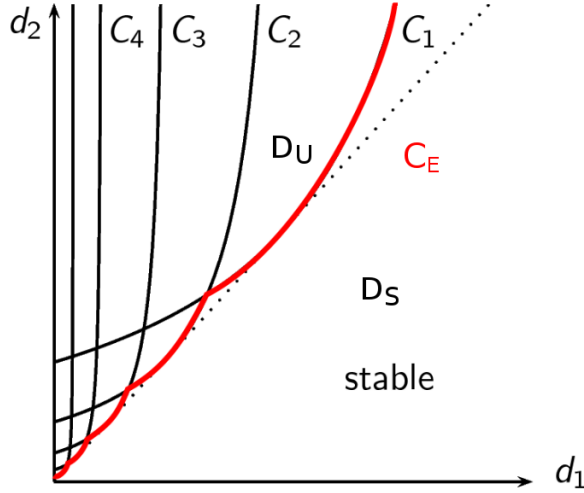
Figure 1: Illustration of the hyperbolas C_j and the envelope C_E. The case when all eigenvalues κ_j are simple.

Remark 2.2. *If all eigenvalues of* (16) *are simple, i.e. $\kappa_j < \kappa_{j+1}$ for all $j \in \mathbb{N}$, then $C_j \neq C_{j+1}$ for all $j > 0$. If an eigenvalue κ_j has a multiplicity k, then $\kappa_{j-1} < \kappa_j = \ldots = \kappa_{j+k-1} < \kappa_{j+k}$ and $C_{j-1} \neq C_j = \ldots = C_{j+k-1} \neq C_{j+k}$. The sets*

$$D_U := \{d = [d_1, d_2] \in \mathbb{R}_+^2 : d \text{ is on the left of } C_E\},$$
$$D_S := \{d = [d_1, d_2] \in \mathbb{R}_+^2 : d \text{ is on the right of } C_E\}$$

are called the domain of instability and the domain of stability. It is known that if $[d_1, d_2] \in D_S$, then all eigenvalues λ of the problem deciding about stability of the trivial solution of the evolution system corresponding to (12), (4) *have negative real parts and if $[d_1, d_2] \in D_U$, then there is an eigenvalue λ with positive real part (for a particular case see [8],[10] and for a general case [2]). In particular, the trivial solution of* (12), (4) *is linearly stable for $[d_1, d_2] \in D_S$ and unstable for $[d_1, d_2] \in D_U$.*

Remark 2.3. *The following properties of the curves C_j are known, see e.g. [10],[8] for a particular case, or [2] for the general case.*

- *A point $d = [d_1, d_2]$ is a critical point of* (11), (4) *if and only if there exists j such that $d \in C_j$. In particular, the domain of stability D_S does not contain any critical point of* (11), (4) *or bifurcation point of* (12), (4)*. Under some additional assumptions, e.g. if the eigenvalue κ_j is simple or of odd multiplicity, the points on C_j are simultaneously bifurcation points (see e.g. [10]).*

- *If $d \in C_n$ for $n = j, \ldots, j + k - 1$ (either k is the multiplicity of the eigenvalue κ_j or d is in the intersection of two hyperbolas C_j, C_m and k is the sum of multiplicities of κ_j, κ_m, see Remark 2.2), then $\mathrm{span}\left(\left[\frac{d_2 \kappa_j - b_{2,2}}{b_{2,1}} e_j, e_j \right]_{n=j}^{j+k-1} \right)$ is the set of the solutions of* (11), (4)*.*

3. Main results

Let's recall that the assumptions (9), (10) are automatically supposed. Besides the notions introduced in Section 2 we will use the following symbols.

Notation 3.1.
Let $r, R, \varepsilon \in \mathbb{R}_+$ and $r < R$. We define
$C_r^R := \{d = [d_1, d_2] \in C_E : d_2 \in [r, R]\}$,
$C_r^R(\varepsilon) := \{d = [d_1, d_2] \in C_E \cup D_U : d_2 \in [r, R] \wedge dist(d, C_E) < \varepsilon\}$.

The following theorem is the main result of this paper.

Theorem 3.1. i) *The domain of stability D_S contains neither critical points of* $(8), (4)$ *nor bifurcation points of* $(5), (4)$.

ii) *Let* $0 < r < R$. *Let* $C_j, \ldots, C_{j+k-1}$ *be all hyperbolas which have a non-empty intersection with* C_r^R. *Let any linear combination e of the eigenfunctions of* (16) *corresponding to* $\kappa_j, \ldots, \kappa_{j+k-1}$ *satisfy*

$$s_- e^- - s_+ e^+ \not\equiv 0. \tag{19}$$

Then there exists $\varepsilon > 0$ such that there are neither critical points of $(8), (4)$ *nor bifurcation points of* $(5), (4)$ *in* $C_r^R(\varepsilon)$.

We emphasize that if the condition (19) is not satisfied for some linear combination e mentioned, then there are critical points of $(8), (4)$ directly on C_r^R due to Remark 2.3. Let's note that if all hyperbolas $C_j, \ldots, C_{j+k-1}$ do not coincide, i.e. it is not $\kappa_j = \kappa_{j+1} = \ldots = \kappa_{j+k-1}$, then the eigenfunctions $e_j, \ldots, e_{j+k-1}$ do not correspond to the same eigenvalue and their linear combination need not be an eigenfunction. We discuss possible situations in the following two examples:

- First let's assume that C_r^R has a non-empty intersection with exactly two non-coinciding hyperbolas C_k and C_{k+1}. If both $e = e_k$ and $e = e_{k+1}$ satisfy (19), then there are no critical points of $(8), (4)$ on $C_r^R \setminus (C_k \cap C_{k+1})$. However, it can happen that there is a linear combination e of e_k, e_{k+1} such that $s_- e^- - s_+ e^+ \equiv 0$, and in this case the intersection point $C_k \cap C_{k+1}$ is a critical point of $(8), (4)$ (see also Remark 2.3).

- In an other scenario we take C_r^R which consists of a part of two coinciding hyperbolas $C_k = C_{k+1}$, i.e. $\kappa_k = \kappa_{k+1}$. In this case the assumption of Theorem 3.1 ii) means that every eigenfunction corresponding to $\kappa_k = \kappa_{k+1}$ must satisfy (19). Otherwise the critical points of $(8), (4)$ are on the whole C_k, in particular on C_r^R (see Remark 2.3).
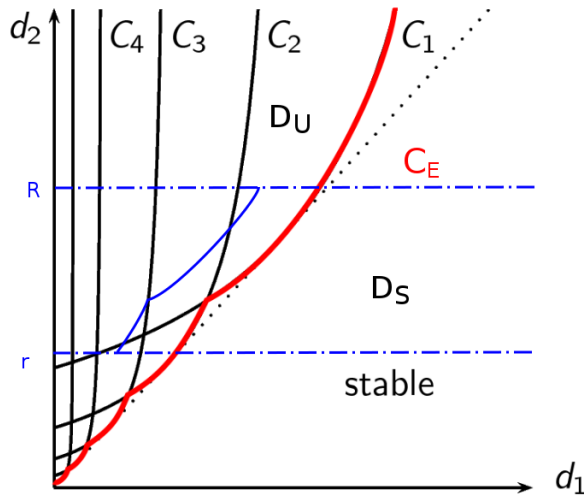
The result is illustrated on Figure 2.

Figure 2: Illustration of the result of Theorem 3.1. The critical points are no longer in the region between C_E (red curve) and blue curve. Assuming the case when all eigenvalues κ_j are simple, i.e. $C_j \neq C_k$ for all $k \neq j$, and any linear combination of eigenfunctions e_1, e_2 corresponding to κ_1, κ_2 satisfy (19).

Corollary 3.1. i) *For any compact part M of D_S there exists $\delta > 0$ such that for any $[d_1, d_2] \in M$ there are no non-trivial solutions of* $(5), (4)$ *with* $0 < \|u\|_{H_D^1} + \|v\|_{H_D^1} < \delta$.

ii) *Under the assumption from Theorem 3.1 ii), for any compact part M of $D_S \cup C_r^R(\varepsilon)$ there exists $\delta > 0$ such that for any $[d_1, d_2] \in M$ there are no non-trivial solutions of (5),(4) with $0 < \|u\|_{H_D^1} + \|v\|_{H_D^1} < \delta$.*

Proof.
Indeed, it is easy to see that if this were not true, then a bifurcation point of (5),(4) would exist in M, which would contradict Theorem 3.1. $\qquad \square$

There are two important particular cases for $\Gamma_D \neq \emptyset$ and $\Gamma_D = \emptyset$:

Theorem 3.2. *Let $\Gamma_D \neq \emptyset$. Let one of the functions s_+, s_- be identically zero and the other positive a.e. on Ω. Let d_2^I be the second coordinate of the intersection point of C_1 and C_2.*

  i) *Any $d \in C_1$, in particular any $d \in C_r^R$ with $d_2^I \leq r < R$, is a critical point of (8),(4).*
  ii) *If $0 < r < R < d_2^I$, then there exists $\varepsilon > 0$ such that there are neither critical points of (8),(4) nor bifurcation points of (5),(4) in $C_r^R(\varepsilon)$.*

Theorem 3.3. *Let $\Gamma_D = \emptyset$. Let one of the functions s_+, s_- be identically zero and the other positive a.e. on Ω. Then for any $0 < r < R$ there exists $\varepsilon > 0$ such that there are neither critical points of (8),(4) nor bifurcation points of (5),(4) in $C_r^R(\varepsilon)$.*

Remark 3.1. *The size of ε in Theorems 3.1-3.3 depends on r and R. Actually $\varepsilon \to 0$ as $R \to d_2^I$ or $r \to 0$ in Theorem 3.2 and $\varepsilon \to 0$ as $r \to 0$ in Theorem 3.3. The following theorem states that if the source and sink are in some sense small enough, then there exists at least one critical point $[d_1, d_2] \in D_U \cup C_E$ with a given d_2. A question if sometimes (for a strong source or sink) no critical point with a given d_2 exists remains an open problem. Cf. Remark 5.5 in Section 5.*

Theorem 3.4. *Let $d_2 > 0$ be arbitrary fixed. Let j_0 be such that $\left[\frac{1}{\kappa_{j_0}} \left(\frac{b_{1,2}b_{2,1}}{d_2\kappa_{j_0} - b_{2,2}} + b_{1,1} \right), d_2 \right] \in C_E$ (see (17),(18)). If $\max\{\|s_-\|_\infty, \|s_+\|_\infty\} < b_{1,1} + \frac{b_{1,2}b_{2,1}}{d_2\kappa_{j_0} - b_{2,2}}$, then there exists at least one d_1 such that $[d_1, d_2] \in D_U \cup C_E$ is a critical point of the problem (8),(4).*

The last theorem of this section is a modification of Theorem 3.1 for the case of unilateral terms in boundary conditions, namely for systems (11) and (12) with boundary conditions

$$u = v = 0 \quad \text{on} \quad \Gamma_D,$$
$$\frac{\partial u}{\partial n} = s_-(\mathbf{x})u^- - s_+(\mathbf{x})u^+ \quad \text{on} \quad \Gamma_N, \tag{20}$$
$$\frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Gamma_N.$$

Let us note that we consider only positively homogeneous boundary conditions because introducing more general boundary terms as $\tilde{f}_\mp$ in the case of sources and sinks in the interior of the domain would mean additional technical complications.

Theorem 3.5.   i) *The domain of stability D_S contains neither critical points of (11),(20) nor bifurcation points of (12),(20).*
  ii) *Let $0 < r < R$. Let $C_j, \ldots, C_{j+k-1}$ be all hyperbolas which have a non-empty intersection with C_r^R. Let any linear combination e of the eigenfunctions of (16) corresponding to $\kappa_j, \ldots, \kappa_{j+k-1}$ satisfy*

$$s_- e^- - s_+ e^+ \not\equiv 0 \quad on \ \Gamma_N. \tag{21}$$

  Then there exists $\varepsilon > 0$ such that there are neither critical points of (11),(20) nor bifurcation points of (12),(20) in $C_r^R(\varepsilon)$.

Analogous consequence as in Corollary 3.1 can be formulated for Theorems 3.2, 3.3 and 3.5.

4. Abstract formulation

We define the operator $A : H_D^1 \mapsto H_D^1$ as

$$(Au, \varphi) = \int_\Omega u\varphi \, d\Omega \quad \text{for all } u, \varphi \in H_D^1(\Omega). \tag{22}$$

Remark 4.1. *The operator A defined by (22) is linear, bounded, symmetric and compact due to compact embedding $W^{1,2} \hookrightarrow\hookrightarrow L^2$. Simple calculation gives that the eigenvalues of the operator A are $\mu_j = \frac{1}{\kappa_j}, j = 1, 2, \dots$ for $\Gamma_D \neq \emptyset$ and $\mu_j = \frac{1}{\kappa_j + 1}, j = 0, 1, 2, \dots$ for $\Gamma_D = \emptyset$, and the corresponding eigenvectors of A coincide with the eigenfunctions e_j of (16). In particular, the maximal eigenvalue of A is always one and therefore $(Au, u) \leq \|u\|_{H_D^1}^2$, where the equality holds only for all multiples u of e_1 or e_0 if $\Gamma_D \neq \emptyset$ or $\Gamma_D = \emptyset$, respectively, see also Remark 2.1. Hence, $((I - A)u, u) > 0$ for all $u \notin span\{e_1\}$ in the case $\Gamma_D \neq \emptyset$ and for all $u \notin span\{e_0\}$ in the case $\Gamma_D = \emptyset$.*

We define two non-linear operators $N_1, N_2 : H_D^1 \times H_D^1 \mapsto H_D^1$ as

$$(N_i(u, v), \varphi) = \int_\Omega n_i(u, v)\varphi \, d\Omega \quad \text{for all } u, v, \varphi \in H_D^1, \ i = 1, 2. \tag{23}$$

These two operators are well-defined and continuous due to the theorem about Nemytskii operators and the assumptions (9).

Remark 4.2. *It is known that under the assumptions (6),(9) we have*

$$\lim_{\|u\|_{H_D^1} + \|v\|_{H_D^1} \to 0} \frac{N_i(u, v)}{\|u\|_{H_D^1} + \|v\|_{H_D^1}} = 0, \quad i = 1, 2. \tag{24}$$

For details see e.g. Appendix A.1 of [5].

Furthermore we define operators $\beta^-, \beta^+ : H_D^1 \mapsto H_D^1$ by

$$(\beta^\mp(u), \varphi) = \mp \int_\Omega s_\mp u^\mp \varphi \, d\Omega \quad \text{for all } u, \varphi \in H_D^1 \tag{25}$$

and $\beta : H_D^1 \mapsto H_D^1$ as

$$\beta := \beta^+ + \beta^-. \tag{26}$$

Due to the theorem about Nemytskii operators and (10) we can also define operators $\tilde{F}_-, \tilde{F}_+ : H_D^1 \mapsto H_D^1$ by

$$(\tilde{F}_\mp(u), \varphi) = \mp \int_\Omega \tilde{f}_\mp(\mathbf{x}, u^\mp)\varphi \, d\Omega \quad \text{for all } u, \varphi \in H_D^1 \tag{27}$$

and $\tilde{F} : H_D^1 \mapsto H_D^1$ as

$$\tilde{F} := \tilde{F}_+ + \tilde{F}_-. \tag{28}$$

Lemma 4.1. *The operator β is positively homogeneous (i.e. $\beta(tu) = t\beta(u)$ for all $t > 0, u \in H_D^1$) and*

$$\begin{align}
&i) \quad \exists c \in \mathbb{R} : \|\beta(u)\|_{H_D^1} \leq c\|s_-\|_\infty \|u^-\|_{H_D^1} + c\|s_+\|_\infty \|u^+\|_{H_D^1} \quad \forall u \in H_D^1, \tag{29}\\
&ii) \quad u_n \rightharpoonup u \implies \beta(u_n) \to \beta(u), \tag{30}\\
&iii) \quad (\beta(u), u) \geq 0 \quad \forall u \in H_D^1, \tag{31}\\
&iv) \quad u_n \to 0, \frac{u_n}{\|u_n\|_{H_D^1}} \rightharpoonup w \implies \frac{\tilde{F}(u_n)}{\|u_n\|_{H_D^1}} \to \beta(w). \tag{32}
\end{align}$$

Proof.
The positive homogeneity is apparent.

i) Using the continuous embedding $H_D^1 \hookrightarrow L^2$ and Hölder's inequality we get

$$\|\beta(u)\| = \sup_{\|\varphi\|_{H_D^1} \leq 1} |(\beta(u), \varphi)| = \sup_{\|\varphi\|_{H_D^1} \leq 1} \left| \int_\Omega s_+ u^+ \varphi \, d\Omega - \int_\Omega s_- u^- \varphi \, d\Omega \right| \leq$$

$$\leq \|s_+\|_\infty \sup_{\|\varphi\|_{H_D^1} \leq 1} \left\{ \|u^+\|_{L^2} \cdot \|\varphi\|_{L^2} \right\} + \|s_-\|_\infty \sup_{\|\varphi\|_{H_D^1} \leq 1} \left\{ \|u^-\|_{L^2} \cdot \|\varphi\|_{L^2} \right\} \leq$$

$$\leq c\|s_+\|_\infty \sup_{\|\varphi\|_{H_D^1} \leq 1} \left\{ \|u^+\|_{H_D^1} \cdot \|\varphi\|_{H_D^1} \right\} + c\|s_-\|_\infty \sup_{\|\varphi\|_{H_D^1} \leq 1} \left\{ \|u^-\|_{H_D^1} \cdot \|\varphi\|_{H_D^1} \right\} \leq$$

$$\leq c\|s_+\|_\infty \|u^+\|_{H_D^1} + c\|s_-\|_\infty \|u^-\|_{H_D^1}.$$

ii) Let's have a sequence $(u_n) \subset H_D^1$ such that $u_n \rightharpoonup u \in H_D^1$. Then by the compact embedding $W^{1,2} \hookrightarrow\hookrightarrow L^2$, we get $u_n \to u$ in L^2. It is easy to see that $|u_n^- - u^-| \leq |u_n - u|$ holds almost everywhere on Ω. Hence,

$$\|\beta^-(u_n) - \beta^-(u)\|_{H_D^1} = \sup_{\|\varphi\|_{H_D^1} \leq 1} |(\beta^-(u_n) - \beta^-(u), \varphi)| \leq \sup_{\|\varphi\|_{H_D^1} \leq 1} \int_\Omega |u_n^- - u^-| \cdot |\varphi| \, d\Omega \leq$$

$$\leq C\|u_n - u\|_{L^2} \to 0.$$

The same can be shown for β^+ and the assertion follows.

iii) Let $u \in H_D^1$ be arbitrary and Ω_+, Ω_- subsets of the domain Ω such that $\Omega = \Omega_+ \cup \Omega_-$, $u \geq 0$ a.e. on Ω_+ and $u < 0$ a.e. on Ω_-. Hence

$$(\beta(u), u) = \int_\Omega s_+ u^+ u \, d\Omega - \int_\Omega s_- u^- u \, d\Omega = \int_{\Omega_+} s_+ u^2 d\Omega_+ + \int_{\Omega_-} s_- u^2 d\Omega_-$$

and our assertion follows.

iv) Now we will define a new auxiliary operator $F : H_D^1 \mapsto H_D^1$ by

$$(F(u), \varphi) = -\int_\Omega (\tilde{f}_-(\mathbf{x}, u) - s_- u)\varphi \, d\Omega \quad \text{for all } u, \varphi \in H_D^1.$$

We have

$$\lim_{\xi \to 0} \frac{\tilde{f}_-(\mathbf{x}, \xi) - s_- \xi}{\xi} = 0 \quad \text{for a.a. } \mathbf{x} \in \Omega$$

by assumption (2). The growth conditions (10) and Proposition 3.2 of [4] give

$$\lim_{u \to 0} \frac{F(u)}{\|u\|_{H_D^1}} = 0. \tag{33}$$

If $u_n \to 0$, then $u_n^- \to 0$ (see [13]) and using (33) we get

$$\lim_{n \to +\infty} \frac{\|\tilde{F}_-(u_n) - \beta^-(u_n)\|_{H_D^1}}{\|u_n\|_{H_D^1}} = \lim_{n \to +\infty} \frac{\|F(u_n^-)\|_{H_D^1}}{\|u_n\|_{H_D^1}} \leq \lim_{n \to +\infty} \frac{\|F(u_n^-)\|_{H_D^1}}{\|u_n^-\|_{H_D^1}} = 0.$$

If $u_n \to 0$, $\frac{u_n}{\|u_n\|_{H_D^1}} \rightharpoonup w$ then

$$\frac{\tilde{F}_-(u_n)}{\|u_n\|_{H_D^1}} \to \beta^-(w)$$

due to positive homogeneity of β^- and (30).
The same can be shown for $\tilde{F}_+$ and β^+ and the assertion is proved.

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$\square$

In order to give an operator formulation of the problem (11) or (12) with unilateral sources and sinks on the boundary (20), we define operators $\beta_N^{\pm} : H_D^1 \mapsto H_D^1$ as

$$(\beta_N^{\mp}(u), \varphi) = \mp \int_{\Gamma_N} s_{\mp} u^{\mp} \varphi \, d\Gamma_N \quad \text{for all } u, \varphi \in H_D^1 \tag{34}$$

and $\beta_N : H_D^1 \mapsto H_D^1$ as

$$\beta_N = \beta_N^+ + \beta_N^-. \tag{35}$$

Remark 4.3. *The operator β_N possess the same properties as the operator β (see Lemma 4.1).*

Let's emphasize that for cases $\Gamma_D = \emptyset$ and $\Gamma_D \neq \emptyset$ we have two different inner products and therefore operators defined above are in these two cases also different. In the case $\Gamma_D \neq \emptyset$ we consider the function space H_D^1 equipped with the inner product $(u, \varphi) = \int_\Omega \nabla u \nabla \varphi \, d\Omega$. A weak solution of the problem (8),(4) or (5),(4) is then a pair of functions $u, v \in H_D^1$ satisfying

$$\begin{aligned} d_1 u - b_{1,1} A u - b_{1,2} A v + \beta(u) &= 0, \\ d_2 v - b_{2,1} A u - b_{2,2} A v &= 0 \end{aligned} \tag{36}$$

or

$$\begin{aligned} d_1 u - b_{1,1} A u - b_{1,2} A v - N_1(u, v) + \tilde{F}(u) &= 0, \\ d_2 v - b_{2,1} A u - b_{2,2} A v - N_2(u, v) &= 0, \end{aligned} \tag{37}$$

respectively.

If $\Gamma_D = \emptyset$, the function space H_D^1 is identical with $W^{1,2}$ and is equipped with the inner product $(u, \varphi) = \int_\Omega (\nabla u \nabla \varphi + u \varphi) \, d\Omega$. A weak solution of (8),(4) or (5),(4) is then a pair of functions $u, v \in W^{1,2}$ satisfying

$$\begin{aligned} d_1 (I - A) u - b_{1,1} A u - b_{1,2} A v + \beta(u) &= 0, \\ d_2 (I - A) v - b_{2,1} A u - b_{2,2} A v &= 0 \end{aligned} \tag{38}$$

or

$$\begin{aligned} d_1 (I - A) u - b_{1,1} A u - b_{1,2} A v - N_1(u, v) + \tilde{F}(u) &= 0, \\ d_2 (I - A) v - b_{2,1} A u - b_{2,2} A v - N_2(u, v) &= 0, \end{aligned} \tag{39}$$

respectively.

For the problem (11), (20) or (12), (20) we will get analogous systems, we just replace operators β and $\tilde{F}$ with β_N.

5. Critical points for fixed d_2

In this Section we will assume that $d_2 > 0$ is fixed and we will use the notation from Sections 2 and 4. As usually, by an eigenvalue of a positively homogeneous operator P we mean a number λ such that the equation $P(u) = \lambda u$ has a non-trivial solution. More generally, by an eigenvalue of a problem with a positively homogeneous operator we mean a parameter for which the problem under consideration has a non-trivial solution.

5.1. Reduction to one operator equation for the case $\Gamma_D \neq \emptyset$

Let's suppose $\Gamma_D \neq \emptyset$. Since the operator A is positive by Remark 4.1 and $b_{2,2} < 0$ by the assumption (7), the number $\frac{d_2}{b_{2,2}}$ is not its eigenvalue. Therefore the operator $d_2 I - b_{2,2} A$ is invertible and surjective. Hence, we can express v from the second equation in (36), substitute it into the first one and get

$$d_1 u - b_{1,1} A u - b_{1,2} A (d_2 I - b_{2,2} A)^{-1} b_{2,1} A u + \beta(u) = 0.$$

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Introducing the operator $S_{d_2} : H_D^1 \mapsto H_D^1$ as

$$S_{d_2} := b_{1,1}A + b_{1,2}A(d_2I - b_{2,2}A)^{-1}b_{2,1}A, \tag{40}$$

we can write the system (36) as

$$d_1u - S_{d_2}u + \beta(u) = 0, \tag{41a}$$
$$v = (d_2I - b_{2,2}A)^{-1}b_{2,1}Au. \tag{41b}$$

In particular, the system of the operator equations

$$d_1u - S_{d_2}u = 0, \tag{42a}$$
$$v = (d_2I - b_{2,2}A)^{-1}b_{2,1}Au \tag{42b}$$

is equivalent with the system

$$
\begin{aligned}
d_1u - b_{1,1}Au - b_{1,2}Av &= 0, \\
d_2v - b_{2,1}Au - b_{2,2}Av &= 0.
\end{aligned}
\tag{43}
$$

Remark 5.1. *The operator $S_{d_2} : H_D^1 \mapsto H_D^1$ defined by (40) is linear, bounded, symmetric and compact. It follows from simple calculations and Remark 4.1 that the eigenvalues of the operator S_{d_2} are*

$$d_1^j = \frac{1}{\kappa_j}\left(\frac{b_{1,2}b_{2,1}}{d_2\kappa_j - b_{2,2}} + b_{1,1}\right), \; j = 1, 2, \ldots \tag{44}$$

and since $\kappa_j \to \infty$ as $j \to \infty$, we get $d_1^j \to 0$ as $j \to \infty$. The eigenvectors of S_{d_2} corresponding to d_1^j coincide with those of the operator A corresponding to μ_j, i.e. with the eigenfunctions of (16) corresponding to κ_j.

5.2. Reduction to one operator equation for the case $\Gamma_D = \emptyset$

Let's consider the case $\Gamma_D = \emptyset$. It follows from Remark 4.1 that the number d_2 is not an eigenvalue of the operator $d_2A + b_{2,2}A$. Indeed, we have $d_2 \neq \frac{d_2+b_{2,2}}{\kappa_j+1}$, because $d_2\kappa_j \neq b_{2,2}$ ($b_{2,2}$ is negative by (7)). Hence, the operator $d_2I - d_2A - b_{2,2}A$ (in (38)) is surjective and invertible. Similarly as in Section 5.1 we can transform the system (38) to the system

$$d_1(I - A)u - S_{d_2}u + \beta(u) = 0, \tag{45a}$$
$$v = (d_2I - d_2A - b_{2,2}A)^{-1}b_{2,1}Au, \tag{45b}$$

with the new operator

$$S_{d_2} := b_{1,1}A + b_{1,2}A(d_2I - d_2A - b_{2,2}A)^{-1}b_{2,1}A. \tag{46}$$

In particular, the system of the operator equations

$$d_1(I - A)u - S_{d_2}u = 0, \tag{47a}$$
$$v = (d_2I - d_2A - b_{2,2}A)^{-1}b_{2,1}Au \tag{47b}$$

is equivalent with the system

$$
\begin{aligned}
d_1(I - A)u - b_{1,1}Au - b_{1,2}Av &= 0, \\
d_2(I - A)v - b_{2,1}Au - b_{2,2}Av &= 0.
\end{aligned}
\tag{48}
$$

Remark 5.2. *The operator S_{d_2} defined by (46) is linear, continuous, symmetric and compact. Simple calculations and Remark 4.1 imply that the eigenvalues of the operator S_{d_2} are*

$$\lambda^j = \frac{1}{\kappa_j + 1}\left(\frac{b_{1,2}b_{2,1}}{d_2\kappa_j - b_{2,2}} + b_{1,1}\right), \; j = 0, 1, 2, \ldots \tag{49}$$

and the eigenvectors of S_{d_2} corresponding to λ^j coincide with those of A corresponding to μ_j, i.e. with the eigenfunctions of (16) corresponding to κ_j. However, the eigenvalues d_1^j of the problem (47a) are the same as those of the operator S_{d_2} defined by (40) in the case $\Gamma_D \neq \emptyset$, i.e. they are given by (44). (There is no eigenvalue with $j = 0$.)

5.3. Maximal eigenvalues and critical points

Notation 5.1. We will denote by d_1^{MAX} the maximal eigenvalue of the operator S_{d_2} or of the problem (47a) in the case $\Gamma_D \neq \emptyset$ or $\Gamma_D = \emptyset$, respectively. We will also denote by $d_1^{MAX,\beta}$ the maximal eigenvalue of the operator $S_{d_2} - \beta$ or of the problem (45a) in the case $\Gamma_D \neq \emptyset$ or $\Gamma_D = \emptyset$, respectively, if it exists.

Observation 5.1. *We can see from the form (44) of the eigenvalues d_1^j (see Remarks 5.1, 5.2) and from (17), that a point $[d_1, d_2]$ lies on a hyperbola C_j for some $j \in \mathbb{N}$ if and only if d_1 is an eigenvalue of S_{d_2} in the case $\Gamma_D \neq \emptyset$ or an eigenvalue of (47a) in the case $\Gamma_D = \emptyset$. For the maximal eigenvalue d_1^{MAX} of S_{d_2} in the case $\Gamma_D \neq \emptyset$ and of (47a) in the case $\Gamma_D = \emptyset$ we have $[d_1^{MAX}, d_2] \in C_E$. It follows from (44) and Remark 5.2 that the operator S_{d_2} in the case $\Gamma_D \neq \emptyset$ and the problem (47a) in the case $\Gamma_D = \emptyset$ have infinitely many positive eigenvalues and maximally finite number of negative eigenvalues. See also Figure 1.*

Lemma 5.1. *If $\Gamma_D \neq \emptyset$, then a point $[d_1, d_2] \in \mathbb{R}_+^2$ is a critical point of the system (11), (4) or (8), (4) if and only if d_1 is an eigenvalue of the operator S_{d_2} or $S_{d_2} - \beta$, respectively.*

If $\Gamma_D = \emptyset$, then a point $[d_1, d_2] \in \mathbb{R}_+^2$ is a critical point of the system (11), (4) or (8), (4) if and only if d_1 is an eigenvalue of the problem (47a) or (45a), respectively.

Proof. Let $\Gamma_D \neq \emptyset$. A point $[d_1, d_2] \in \mathbb{R}_+^2$ is a critical point of the system (11), (4) or (8), (4) if and only if there exists a non-trivial solution $[u, v]$ of (43) or (36), respectively. This is true if and only if there exists a non-trivial solution $u \in H_D^1$ of (42a) or (41a), i.e. d_1 is an eigenvalue of the operator S_{d_2} or $S_{d_2} - \beta$, respectively (see Section 5.1). The proof for the case $\Gamma_D = \emptyset$ is analogous, we only use (48), (38), (47a), (45a) and the result of Section 5.2. $\qquad\square$

We will use a variational characterization of the largest eigenvalue of an eigenvalue problem with a positively homogeneous operator to a study of critical points of the problem (8), (4). The following abstract theorem is a slight modification of the result proved for the particular case $L \equiv 0$ in [6] and for the general case in a forthcoming paper of J. Navrátil. Let's remind that $Ker(I - L)$ is the kernel of the operator $I - L$.

Theorem 5.1. *Let H be a Hilbert space, $P : H \mapsto H$ a positively homogeneous, continuous operator such that*

$$u_n \rightharpoonup u \implies P(u_n) \to P(u)$$

and $L : H \mapsto H$ a linear, continuous, symmetric and compact operator. In the case $L \not\equiv 0$ we suppose that the maximal eigenvalue of L is in the interval $(0, 1]$. Let there exist $u_0 \in H, u_0 \notin Ker(I - L)$ such that

$$\lambda_0 := \max_{\substack{u \in H \\ u \notin Ker(I-L)}} \frac{(P(u), u)}{((I - L)u, u)} = \frac{(P(u_0), u_0)}{((I - L)u_0, u_0)} > 0 \tag{50}$$

and

$$\lim_{t \to 0} \frac{1}{t}(P(u_0 + th) - P(u_0), u_0) = (P(u_0), h) \quad \forall h \in H. \tag{51}$$

Then λ_0 is the maximal eigenvalue of the problem

$$\lambda(I - L)u - P(u) = 0 \tag{52}$$

and u_0 is a corresponding eigenvector. If $u_1 \notin Ker(I - L)$ is an arbitrary eigenvector of (52) corresponding to λ_0 then it satisfies (50) with u_0 replaced by u_1.

Let us note that the problem (52) has an eigenvector in $Ker(I-L)$ only if there is $u \in Ker(I-L)$ such that $P(u) = 0$. In this case any λ is an eigenvalue.

Proof.

We will assume that $L \not\equiv 0$, the case $L \equiv 0$ is simpler. Let us denote by μ_L^{MAX} the maximal eigenvalue of L. Since $\mu_L^{MAX} \in (0,1]$, we have $\max\limits_{\substack{u \neq 0 \\ \|u\|_{H_D^1}=1}} (Lu, u) = \mu_L^{MAX} \leq 1$. If $\mu_L^{MAX} < 1$, then $(Lu, u) < 1$ and therefore $((I - L)u, u) > 0$ for all u. If $\mu_L^{MAX} = 1$, then $\max\limits_{\substack{u \neq 0 \\ \|u\|_{H_D^1}=1}} (Lu, u) = 1$, but the maximum is attained only in the elements of $Ker(I - A)$. Hence, $((I - L)u, u) > 0$ for all $u \notin Ker(I - L)$ and the expression in (50) makes sense.

Let $u_0 \notin Ker(I - L)$ be arbitrary such that (50) and (51) are fulfilled, and let $h \in H_D^1$ be arbitrary fixed. Then for $t \in \mathbb{R}$ small such that $(u_0 + th) \notin Ker(I - L)$ we have

$$\frac{(P(u_0 + th), u_0 + th)}{((I - L)(u_0 + th), u_0 + th)} \leq \frac{(P(u_0), u_0)}{((I - L)u_0, u_0)} =: \lambda_0.$$

We can rewrite this inequality as

$$(P(u_0 + th), u_0) + t(P(u_0 + th), h) \leq \frac{(P(u_0), u_0)}{((I - L)u_0, u_0)} \left[((I - L)u_0, u_0) + 2t((I - L)u_0, h) + t^2((I - L)h, h) \right]$$

and eventually as

$$(P(u_0 + th), u_0) - (P(u_0), u_0) + t(P(u_0 + th), h) \leq \lambda_0 \left[2t((I - L)u_0, h) + t^2((I - L)h, h) \right].$$

We divide it by $2t$ and get

$$\frac{1}{2t} \left[(P(u_0 + th), u_0) - (P(u_0), u_0) \right] + \frac{1}{2}(P(u_0 + th), h) \leq \lambda_0 \left[((I - L)u_0, h) + \frac{t}{2}((I - L)h, h) \right], \quad t > 0,$$

$$\frac{1}{2t} \left[(P(u_0 + th), u_0) - (P(u_0), u_0) \right] + \frac{1}{2}(P(u_0 + th), h) \geq \lambda_0 \left[((I - L)u_0, h) + \frac{t}{2}((I - L)h, h) \right], \quad t < 0.$$

Let $t \to 0$. We use the condition (51) and continuity of P to get

$$(P(u_0), h) \leq \lambda_0((I - L)u_0, h),$$
$$(P(u_0), h) \geq \lambda_0((I - L)u_0, h).$$

Since h was arbitrary, we have

$$(P(u_0), h) = \lambda_0((I - L)u_0, h) \quad \text{for all } h \in H_D^1,$$

that means

$$P(u_0) = \lambda_0(I - L)u_0.$$

Hence, the number λ_0 is an eigenvalue of the problem (52) and u_0 is a corresponding eigenvector.

Let λ_1 be another eigenvalue of the problem (52) and let $u_1 \notin Ker(I - L)$ be a corresponding eigenvector. Then we have

$$P(u_1) = \lambda_1(I - L)u_1$$

and if we multiply it by u_1 and divide by $((I - L)u_1, u_1)$, we get

$$\lambda_1 = \frac{(P(u_1), u_1)}{((I - L)u_1, u_1)} \leq \frac{(P(u_0), u_0)}{((I - L)u_0, u_0)} = \lambda_0.$$

Hence, λ_0 is the maximal eigenvalue. If $\lambda_1 = \lambda_0$, then we have equality in the last estimate, that means u_1 is a maximizer of the expression (50). That means an arbitrary eigenvector corresponding to λ_0 not lying in $Ker(I - L)$ satisfies (50) with u_0 replaced by u_1. $\qquad\square$

If the condition (51) is fulfilled for any u_0, then it actually means that P has a potential $\Phi = \frac{1}{2}(Pu, u)$.

Remark 5.3. *In the particular case $L \equiv 0$, $\lambda_0 := \max\limits_{\substack{u \in H \\ u \neq o}} \frac{(P(u),u)}{\|u\|_H^2}$ is the maximal eigenvalue of P.*

Theorem 5.2. *Let $\Gamma_D \neq \emptyset$ and let S_{d_2} be the operator from (40). If there exists a function $\varphi \in H_D^1$ such that*

$$(S_{d_2}\varphi, \varphi) - (\beta(\varphi), \varphi) > 0, \tag{53}$$

then the maximal eigenvalue of the operator $S_{d_2} - \beta$ is

$$d_1^{MAX,\beta} := \max\limits_{\substack{u \in H_D^1 \\ u \neq o}} \frac{(S_{d_2}u, u) - (\beta(u), u)}{\|u\|_{H_D^1}^2} = \max\limits_{\substack{u \in H_D^1 \\ \|u\|_{H_D^1} = 1}} (S_{d_2}u, u) - (\beta(u), u) > 0. \tag{54}$$

Maximizers of the expression in (54) are exactly all eigenvectors of $S_{d_2} - \beta$ corresponding to $d_1^{MAX,\beta}$.

Proof.
Let's prove that the maximum in (54) exists. Let

$$M := \sup\limits_{\substack{u \in H_D^1 \\ \|u\|_{H_D^1} = 1}} (S_{d_2}u, u) - (\beta(u), u).$$

The existence of φ satisfying (53) implies $M > 0$. We can choose a sequence $(u_n) \subset H_D^1$ with $\|u_n\|_{H_D^1} = 1$ such that

$$\lim_{n \to \infty} (S_{d_2}u_n, u_n) - (\beta(u_n), u_n) = M. \tag{55}$$

We can assume $u_n \rightharpoonup u_0 \in H_D^1$. Since S_{d_2} is linear and compact and β satisfies (30), we get

$$(S_{d_2}u_n, u_n) - (\beta(u_n), u_n) \to (S_{d_2}u_0, u_0) - (\beta(u_0), u_0) = M. \tag{56}$$

Now we will show that $\|u_0\|_{H_D^1} = 1$. We know that $\|u_0\|_{H_D^1} \leq 1$. If $0 < \|u_0\|_{H_D^1} < 1$, then $\left(S_{d_2}\frac{u_0}{\|u_0\|_{H_D^1}}, \frac{u_0}{\|u_0\|_{H_D^1}}\right) - \left(\beta\left(\frac{u_0}{\|u_0\|_{H_D^1}}\right), \frac{u_0}{\|u_0\|_{H_D^1}}\right) = \frac{M}{\|u_0\|_{H_D^1}^2} > M$ due to positive homogeneity of β (see Lemma 4.1), which contradicts the fact that M is supremum. If $u_0 = 0$, then $M = 0$, which is not the case. Therefore the last maximum in (54) exists and it is attained at u_0 with $\|u_0\|_{H_D^1} = 1$. The equality between two maxima in (54) follows from the positive homogeneity of β.

It is known that $P = \beta$ and therefore also $P = S_{d_2} - \beta$ satisfies (51) for any u_0 (see Lemma A.3 in Appendix). The operator $P = S_{d_2} - \beta$ satisfies also the other assumptions of Theorem 5.1 (see Remark 5.1 and Lemma 4.1). Hence, the assertions of Theorem 5.2 follow from Theorem 5.1, where we choose $L = 0$, that means we have $Ker(I - L) = \{0\}$. □

Remark 5.4. *Let's consider the case $\Gamma_D = \emptyset$. The definition of the inner product and of the operator A (Section 4) give $((I - A)u, \varphi) = \int_\Omega (\nabla u, \nabla \varphi)\, d\Omega$ for all u, φ. It follows that $(I - A)u = 0$ is equivalent to $((I - A)u, u) = 0$, and this holds if and only if u is a constant function. In other words, $Ker(I - A) = span\{e_0\}$, e_0 being the eigenfunction of (16) corresponding to κ_0. Due to Remark 5.2, any non-trivial $u_0 \in Ker(I - A)$ is simultaneously an eigenvector of S_{d_2} from (46) corresponding to λ^0. Hence, by using (7) we get*

$$(S_{d_2}u_0, u_0) = (\lambda^0 u_0, u_0) = \left(b_{1,1} + \frac{b_{1,2}b_{2,1}}{-b_{2,2}}\right)\|u_0\|_{W^{1,2}}^2 = \frac{-det(\mathbf{B})}{-b_{2,2}}\|u_0\|_{W^{1,2}}^2 < 0. \tag{57}$$

Theorem 5.3. *Let $\Gamma_D = \emptyset$ and let S_{d_2} be the operator from (46). If there exists a function $\varphi \in W^{1,2}$ satisfying (53), then the maximal eigenvalue of the problem (45a) is*

$$d_1^{MAX,\beta} := \max\limits_{\substack{u \in W^{1,2} \\ u \notin Ker(I-A)}} \frac{(S_{d_2}u, u) - (\beta(u), u)}{((I - A)u, u)} > 0. \tag{58}$$

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Maximizers of the expression in (58) *are exactly all eigenvectors of the problem* (45a) *corresponding to* $d_1^{MAX,\beta}$.

Proof.
Let's denote

$$M := \sup_{\substack{u \in W^{1,2} \\ u \notin Ker(I-A)}} \frac{(S_{d_2}u, u) - (\beta(u), u)}{((I-A)u, u)}.$$

Since $((I-A)u, u) = \int_\Omega (\nabla u)^2 \ d\Omega \geq 0$ for every u and we assume that there exists a function φ satisfying (53), we have $M > 0$.

We can choose a sequence $u_n \notin Ker(I-A)$ with $\|u_n\|_{W^{1,2}} = 1$ such that

$$\lim_{n \to \infty} \frac{(S_{d_2}u_n, u_n) - (\beta(u_n), u_n)}{((I-A)u_n, u_n)} = M.$$

We can assume that $u_n \rightharpoonup u_0$. If $u_0 = 0$, then we have

$$((I-A)u_n, u_n) = 1 - (Au_n, u_n) \to 1 - (Au_0, u_0) = 1$$

due to the compactness of A, and

$$(S_{d_2}u_n, u_n) - (\beta(u_n), u_n) \to (S_{d_2}u_0, u_0) - (\beta(u_0), u_0) = 0$$

by the compactness of S_{d_2} and (30). This means that $M = 0$, which contradicts the positivity of M.

Further, let's show that $u_0 \notin Ker(I-A) \setminus \{0\}$, i.e. u_0 is not a constant function. Let u_0 be a non-zero constant function. Then $(S_{d_2}u_0, u_0) < 0$ by Remark 5.4. Since we have $-(\beta(u), u) \leq 0$ for every u by (31), we get $(S_{d_2}u_0, u_0) - (\beta(u_0), u_0) < 0$ and consequently

$$\lim_{n \to \infty} \frac{(S_{d_2}u_n, u_n) - (\beta(u_n), u_n)}{((I-A)u_n, u_n)} \leq 0.$$

That contradicts the fact that u_n is a maximizing sequence and the supremum M is positive. Hence, we have $u_0 \notin Ker(I-L)$.

We need to show that $\|u_0\|_{W^{1,2}} = 1$. We already know that $0 < \|u_0\|_{W^{1,2}} \leq 1$. Now let $0 < \|u_0\|_{W^{1,2}} < 1$. We have $1 - (Au_0, u_0) > 0$ (see Remark 4.1) and

$$\frac{(S_{d_2}u_n, u_n) - (\beta(u_n), u_n)}{((I-A)u_n, u_n)} \to \frac{(S_{d_2}u_0, u_0) - (\beta(u_0), u_0)}{1 - (Au_0, u_0)} = M$$

by the compactness of S_{d_2}, A and the condition (30). Simultaneously $\|u_0\|_{W^{1,2}}^2 - (Au_0, u_0) > 0$ because of $u_0 \notin Ker(I-A)$ (see Remarks 4.1 and 5.4). It follows that

$$\frac{(S_{d_2}u_0, u_0) - (\beta(u_0), u_0)}{\|u_0\|_{W^{1,2}}^2 - (Au_0, u_0)} > \frac{(S_{d_2}u_0, u_0) - (\beta(u_0), u_0)}{1 - (Au_0, u_0)} = M > 0,$$

which contradicts that fact that M is a supremum. Hence, we have $\|u_0\|_{W^{1,2}} = 1$.

We use compactness of S_{d_2}, A, the property (30) of β and the fact that $\|u_n\|_{W^{1,2}} = 1 = \|u_0\|_{W^{1,2}}$ to get

$$\frac{(S_{d_2}u_n, u_n) - (\beta(u_n), u_n)}{((I-A)u_n, u_n)} \to \frac{(S_{d_2}u_0, u_0) - (\beta(u_0), u_0)}{((I-A)u_0, u_0)}. \tag{59}$$

Hence, the maximum exists and it is attained at the function $u_0 \notin Ker(I-A)$ with $\|u_0\|_{W^{1,2}} = 1$.

It is know that $P = S_{d_2} - \beta$ satisfies (51) for any u_0 (see Lemma A.3 in Appendix). The operators $P = S_{d_2} - \beta$ and $L = A$ also satisfy the other assumptions of Theorem 5.1 (see Remark 4.1, Remark 5.2 and Lemma 4.1). Hence, $d_1^{MAX,\beta}$ is the maximal eigenvalue and u_0 is a corresponding eigenvector of the problem (45a) by Theorem 5.1.

Let's show that if u_1 is an arbitrary eigenvector of (45a) corresponding to $d_1^{MAX,\beta}$ then $u_1 \notin Ker(I - A)$. If $u_1 \in Ker(I - A) \setminus \{0\}$, then we have $-(S_{d_2}u_1, u_1) + (\beta(u_1), u_1) > 0$ (see (57) and Lemma 4.1) and $d_1^{MAX,\beta}((I - A)u_1, u_1) = 0$, which contradicts the equation (45a) with $u = u_1$ multiplied by u_1. Hence, $u_1 \notin Ker(I - A)$, and the last assertion of Theorem 5.3 follows also from Theorem 5.1.

$\square$

Remark 5.5. *The assumption* (53) *is clearly satisfied if there exists φ such that $(\beta(\varphi), \varphi) = 0$ and $(S_{d_2}\varphi, \varphi) > 0$, which is easier to verify. If there is no φ satisfying* (53) *then the supremum in proofs of Theorems 5.2 and 5.3 is not positive. It follows that there is no positive eigenvalue of the operator $S_{d_2} - \beta$ in the case $\Gamma_D \neq \emptyset$ or of the problem (45a) in the case $\Gamma_D = \emptyset$. Indeed:*

- *in the case $\Gamma_D \neq \emptyset$, if $d_1 > 0$ were an eigenvalue, then we would have $(S_{d_2}u, u) - (\beta(u), u) = d_1(u, u) > 0$ for the corresponding eigenvector u, which would contradict the non-positivity of the supremum.*

- *in the case $\Gamma_D = \emptyset$, an eigenvector u cannot be constant (see the end of the proof of Theorem 5.3), and if u were non-constant, then we would have $(S_{d_2}u, u) - (\beta(u), u) = d_1((I-A)u, u) > 0$ by the last assertion of Remark 4.1, which would be a contradiction again.*

It follows that in the situation of Theorem 3.4 there exists φ satisfying (53) *because that theorem guarantees the existence of a critical point in $D_U \cup C_E$ and consequently also existence of a positive eigenvalue of the operator $S_{d_2} - \beta$ in the case $\Gamma_D \neq \emptyset$ or of the problem (45a) in the case $\Gamma_D = \emptyset$ (see Sections 5.1, 5.2).*

The following theorem is formulated for both cases $\Gamma_D \neq \emptyset$ and $\Gamma_D = \emptyset$.

Theorem 5.4. *If $[d_1, d_2]$ is a critical point of (8), (4), then always $d_1 \leq d_1^{MAX}$. If $[d_1^{MAX}, d_2] \in C_i$ exactly for $i = j, \ldots, j + k - 1$, all linear combinations e of $e_j, \ldots, e_{j+k-1}$ satisfy (19) and $[d_1, d_2]$ is a critical point of (8), (4), then $d_1 < d_1^{MAX}$. Moreover, if the assumption (53) is satisfied, then $d_1 \leq d_1^{MAX,\beta} < d_1^{MAX}$.*

The assumption concerning a position of $[d_1^{MAX}, d_2]$ is fulfilled either if $[d_1^{MAX}, d_2]$ lies in fact only on one hyperbola $C_j = \ldots C_{j+k-1}$ (the eigenvalue κ_j has the multiplicity k) or in the intersection of two different hyperbolas $C_j = \ldots C_{j+l-1} \neq C_{j+l} = \ldots C_{j+k-1}$ (κ_j has the multiplicity l, κ_{j+l} has the multiplicity $k - l$). See also Remark 2.2. Cf. also comments after Theorem 3.1, where the assumptions are related to a set C_r^R, while in Theorem 5.4 they concern only one point $[d_1^{MAX}, d_2]$ with a given fixed d_2.

Proof.
First let's consider the case $\Gamma_D \neq \emptyset$.
Let's show that if (53) were fulfilled with no φ then no critical point of (8), (4) with d_2 under consideration would exist. If $[d_1, d_2]$ were a critical point with $d_1 > 0$, then d_1 would be an eigenvalue of $S_{d_2} - \beta$ (see Lemma 5.1). Hence, we would have u with $\|u\|_{H_D^1} = 1$ satisfying (41a). It would follow that $(S_{d_2}u, u) - (\beta(u), u) = d_1\|u\|_{H_D^1} > 0$ and the condition (53) would be satisfied with $\varphi = u$, which is a contradiction.
Hence, in the following we can assume that (53) is fulfilled with some φ. Due to Theorem 5.2 and Lemma 4.1 we get

$$d_1^{MAX,\beta} = \max_{\substack{u \in H_D^1 \\ u \neq o}} \frac{(S_{d_2}u, u) - (\beta(u), u)}{\|u\|_{H_D^1}^2} \leq \max_{\substack{u \in H_D^1 \\ u \neq o}} \frac{(S_{d_2}u, u)}{\|u\|_{H_D^1}^2} = d_1^{MAX}.$$

As above, if $[d_1, d_2]$ is a critical point (8), (4), then d_1 is an eigenvalue of $S_{d_2} - \beta$ (see Lemma 5.1). Hence, the first assertion of Theorem 5.4 is true.
There exists $u_0 \in H_D^1$ such that

$$d_1^{MAX,\beta} = \frac{(S_{d_2}u_0, u_0) - (\beta(u_0), u_0)}{\|u_0\|_{H_D^1}^2}.$$

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Due to Lemma 4.1 we have

$$d_1^{MAX,\beta} \leq \frac{(S_{d_2} u_0, u_0)}{\|u_0\|_{H_D^1}^2} \leq d_1^{MAX}. \tag{60}$$

Let $(\beta(u_0), u_0) = 0$. Let us show that then the last inequality is strict. Indeed, if we had equality in (60), then u_0 would be an eigenvector of S_{d_2} corresponding to d_1^{MAX}, that means a linear combination of $e_j, \ldots, e_{j+k-1}$ (see Remark 5.1). Hence, (19) with e replaced by u would be fulfilled by our assumptions and we would get $(\beta(u_0), u_0) > 0$. This contradiction implies that the inequality in (60) must be strict and we get $d_1^{MAX,\beta} < d_1^{MAX}$.
If $(\beta(u_0), u_0) > 0$, then the first inequality in (60) is strict and consequently $d_1^{MAX,\beta} < d_1^{MAX}$ again.

If $[d_1, d_2]$ is a critical point of the problem (8), (4), then d_1 is an eigenvalue of $S_{d_2} - \beta$ by Lemma 5.1 and therefore

$$d_1 \leq d_1^{MAX,\beta} < d_1^{MAX}.$$

The proof for $\Gamma_D = \emptyset$ is analogous, but we must use Remark 5.2 and Theorem 5.3, in particular formula (58) instead of (54). $\qquad\square$

6. Proofs of main results

We will use notation from the previous sections.

Proof of Theorem 3.1

i) Since $d_2 > 0$ was arbitrary in Section 5 and $[d_1^{MAX}, d_2] \in C_E$ (see Observation 5.1), it follows from Theorem 5.4 that there are no critical points of (8), (4) in D_S (see also Figure 1). Consequently there are also no bifurcation points of (5), (4) in D_S (see Lemma A.2 in Appendix).

ii) Let's consider the case $\Gamma_D \neq \emptyset$. Let's suppose the opposite, i.e. the assumptions of the second part of Theorem 3.1 are satisfied and there are critical points of (8), (4) in $C_r^R(\varepsilon)$ for every $\varepsilon > 0$. We can choose a sequence $d^n = [d_1^n, d_2^n] \in D_U$ and $W_n = [u_n, v_n]$ such that $d^n \to d^0 \in C_r^R$, $\|W_n\| = \|u\|_{H_D^1} + \|v\|_{H_D^1} \neq 0$ and d^n, W_n satisfy (36). We can assume that $\frac{W_n}{\|W_n\|} \rightharpoonup W = [w, z]$. Let's divide (36) by $\|W_n\|$ to get

$$\begin{aligned} d_1^n \frac{u_n}{\|W_n\|} - b_{1,1} A \frac{u_n}{\|W_n\|} - b_{1,2} A \frac{v_n}{\|W_n\|} + \beta \left(\frac{u_n}{\|W_n\|} \right) &= 0, \\ d_2^n \frac{v_n}{\|W_n\|} - b_{2,1} A \frac{u_n}{\|W_n\|} - b_{2,2} A \frac{v_n}{\|W_n\|} &= 0. \end{aligned} \tag{61}$$

By the compactness of A and (30), we get $A\frac{u_n}{\|W_n\|} \to Aw$ and $\beta\left(\frac{u_n}{\|W_n\|}\right) \to \beta(w)$, analogously for v_n and z. Hence, it follows easily from (61) that $\frac{u_n}{\|W_n\|} \to w$, $\frac{v_n}{\|W_n\|} \to z$ and

$$\begin{aligned} d_1^0 w - b_{1,1} Aw - b_{1,2} Az - \beta(w) &= 0, \\ d_2^0 z - b_{2,1} Aw - b_{2,2} Az &= 0. \end{aligned}$$

Therefore the point $d^0 = [d_1^0, d_2^0] \in C_r^R$ is a critical point of the system (8), (4), which contradicts Theorem 5.4 for $d_2 = d_2^0$. Hence, there exists $\varepsilon > 0$ such that there are no critical points of (8), (4) and consequently no bifurcation points of (5), (4) in $C_r^R(\varepsilon)$ (see Lemma A.2 in Appendix).

The proof for $\Gamma_D = \emptyset$ is analogous, we only use the system (38) instead of the system (36).

$\qquad\square$

Proof of Theorem 3.2

i) Under the assumptions about $s_\pm$, either $e = e_1$ or $e = -e_1$ satisfies $s_- e^- - s_+ e^+ \equiv 0$. Since any point $d \in C_1$ is a critical point of the problem $(11),(4)$ with a non-trivial solution $\left[\frac{d_2\kappa_1 - b_{2,2}}{b_{2,1}} e_1, e_1\right]$ due to Remark 2.3, it is also a critical point of the problem $(8),(4)$.

ii) Due to the definition of d_2^I, for $d_2 < d_2^I$ we have $[d_1^{MAX}, d_2] \in C_j$ for a finite number of indices $j > 1$ (see Section 2 and also Observation 5.1). Any linear combination e of the eigenfunctions $e_j, j > 1$ changes the sign (see Lemma A.1 in Appendix). Hence, under the assumptions of Theorem 3.2 we have $s_- e^- - s_+ e^+ \not\equiv 0$. Therefore any critical point $[d_1, d_2]$ with $d_2 < d_2^I$ of the problem $(8),(4)$ satisfies $d_1 < d_1^{MAX}$ by Theorem 5.4. Now, it is possible to repeat the part ii) of the proof of Theorem 3.1.

$\square$

Proof of Theorem 3.3

For any $d_2 > 0$ we have $[d_1^{MAX}, d_2] \in C_j$ for a finite number of indices $j > 0$ (see Section 2 and also Observation 5.1). Any linear combination e of the eigenfunctions $e_j, j > 0$ changes the sign (see Lemma A.1 for details), therefore the relation $s_- e^- - s_+ e^+ \not\equiv 0$ is always satisfied. Hence, any critical point $[d_1, d_2]$ of the problem $(8),(4)$ satisfies $d_1 < d_1^{MAX}$ by Theorem 5.4. Now, it is possible to repeat the part ii) of the proof of Theorem 3.1.

$\square$

Proof of Theorem 3.4

The assumption on j_0 directly implies that the j_0-th eigenvalue of the operator S_{d_2} in the case $\Gamma_D \neq \emptyset$ or of the problem $(47a)$ in the case $\Gamma_D = \emptyset$ is positive (see Remarks 5.1, 5.2 and $(17),(18)$). Hence, we have $(S_{d_2} e_{j_0}, e_{j_0}) > 0$, where e_{j_0} is the corresponding eigenvector. Let's denote $\tau := \max\{\|s_-\|_\infty, \|s_+\|_\infty\}$. We get

$$(S_{d_2} e_{j_0}, e_{j_0}) - (\beta(e_{j_0}), e_{j_0}) = (S_{d_2} e_{j_0}, e_{j_0}) - \int_\Omega s_+ e_{j_0} e_{j_0}^+ \, d\Omega - \int_\Omega -s_- e_{j_0} e_{j_0}^- \, d\Omega \geq$$

$$\geq (S_{d_2} e_{j_0}, e_{j_0}) - \|s_+\|_\infty \int_\Omega (e_{j_0}^+)^2 \, d\Omega - \|s_-\|_\infty \int_\Omega (e_{j_0}^-)^2 \, d\Omega$$

$$\geq (S_{d_2} e_{j_0}, e_{j_0}) - \tau \left(\int_\Omega (e_{j_0}^+)^2 \, d\Omega + \int_\Omega (e_{j_0}^-)^2 \, d\Omega\right) \geq$$

$$\geq (S_{d_2} e_{j_0}, e_{j_0}) - \tau (A e_{j_0}, e_{j_0}).$$

Since e_{j_0} is non-trivial, we have $(A e_{j_0}, e_{j_0}) > 0$. Hence, if $\tau < \frac{(S_{d_2} e_{j_0}, e_{j_0})}{(A e_{j_0}, e_{j_0})}$, then $(S_{d_2} e_{j_0}, e_{j_0}) - (\beta(e_{j_0}), e_{j_0}) > 0$.

If $\Gamma_D \neq \emptyset$ then we get

$$\frac{(S_{d_2} e_{j_0}, e_{j_0})}{(A e_{j_0}, e_{j_0})} = \frac{\frac{1}{\kappa_{j_0}}\left(\frac{b_{1,2}b_{2,1}}{d_2\kappa_{j_0} - b_{2,2}} + b_{1,1}\right)\|e_{j_0}\|_{H_D^1}^2}{\frac{1}{\kappa_{j_0}}\|e_{j_0}\|_{H_D^1}^2} = \frac{b_{1,2}b_{2,1}}{d_2\kappa_{j_0} - b_{2,2}} + b_{1,1} \tag{62}$$

(see Remarks 5.1 and 4.1) and if $\Gamma_D = \emptyset$ we get

$$\frac{(S_{d_2} e_{j_0}, e_{j_0})}{(A e_{j_0}, e_{j_0})} = \frac{\frac{1}{\kappa_{j_0}+1}\left(\frac{b_{1,2}b_{2,1}}{d_2\kappa_{j_0} - b_{2,2}} + b_{1,1}\right)\|e_{j_0}\|_{H_D^1}^2}{\frac{1}{\kappa_{j_0}+1}\|e_{j_0}\|_{H_D^1}^2} = \frac{b_{1,2}b_{2,1}}{d_2\kappa_{j_0} - b_{2,2}} + b_{1,1}. \tag{63}$$

(see Remarks 5.2 and 4.1). Hence, if $\tau < \frac{b_{1,2}b_{2,1}}{d_2\kappa_{j_0} - b_{2,2}} + b_{1,1}$, then the assumption (53) of Theorems 5.2, 5.3 is satisfied with $\varphi = e_{j_0}$ and therefore $d_1^{MAX,\beta} > 0$ exists. A point $[d_1^{MAX,\beta}, d_2]$ is a critical point of $(8),(4)$ by Lemma 5.1 and it lies in $D_U \cup C_E$ by Theorem 5.4 and because $[d_1^{MAX}, d_2] \in C_E$ (see Observation 5.1).

$\square$

Proof of Theorem 3.5

If unilateral terms in boundary conditions are considered, we replace the operators $\tilde{F}$ and β in (36),(37),(38),(39) by the operator β_N, which has the same properties as β. Then it is necessary to repeat whole Section 5 for this operator. The actual proof of Theorem 3.5 is then the same as the proof of Theorem 3.1.

A. Appendix

For a completeness we give here proofs of two standard assertions used in the text (Lemmas A.1, A.2), and a slightly simplified proof of a result given already in [6] (Lemma A.3).

Lemma A.1. *Any linear combination* $\sum_{k=j_0}^{n} a_k e_k, n \in \mathbb{N}$ *of eigenfunctions of* (16)*, where* $j_0 = 2$ *for* $\Gamma_D \neq \emptyset$ *and* $j_0 = 1$ *for* $\Gamma_D = \emptyset$*, changes the sign on the domain* Ω*.*

Proof.
Case $\Gamma_D = \emptyset$**:**
Since e_k is an orthonormal basis (see Remark 4.1) and e_0 is constant, we have

$$\int_\Omega e_0 \sum_{k=j_0}^n a_k e_k \, d\Omega = \sum_{k=j_0}^n a_k \int_\Omega e_0 e_k \, d\Omega = \sum_{k=j_0}^n a_k \int_\Omega (\nabla e_0 \nabla e_k + e_0 e_k) \, d\Omega = \sum_{k=j_0}^n a_k (e_0, e_k)_{W^{1,2}} = 0.$$

Hence, the function $e_0 \sum_{k=j_0}^n a_k e_k$ changes the sign on the domain Ω. Since e_0 is constant, also the function $\sum_{k=j_0}^n a_k e_k$ changes the sign on Ω.
Case $\Gamma_D \neq \emptyset$**:**
We will use the eigenfunction e_1 instead of e_0. Again since e_k is the orthonormal basis, we have

$$\int_\Omega e_1 e_k \, d\Omega = \int_\Omega -\frac{1}{\kappa_1} \Delta e_1 e_k \, d\Omega = \frac{1}{\kappa_1} \int_\Omega \nabla e_1 \nabla e_k \, d\Omega = \frac{1}{\kappa_1} (e_1, e_k)_{H_D^1} = 0 \quad \text{for any } k > 1.$$

The rest is the same as in the case $\Gamma_D = \emptyset$. $\square$

Lemma A.2. *Every bifurcation point* $[d_1, d_2]$ *of* (5), (4) *is also a critical point of* (8), (4)*.*

Proof.
We will show the proof for $\Gamma_D \neq \emptyset$. The proof for $\Gamma_D = \emptyset$ is the same, we only use the system (39) instead of the system (37).

Let $d^0 = [d_1, d_2] \in \mathbb{R}_+^2$ be a bifurcation point of (5), (4). Then there exists a sequence $d^n = [d_1^n, d_2^n]$ such that $d^n \to d^0$ and $W_n = [u_n, v_n] \to 0$ with $\|W_n\| = \|u\|_{H_D^1} + \|v\|_{H_D^1} \neq 0$ and d^n, W_n satisfy (5), (4), i.e. (37). We can assume $\frac{W_n}{\|W_n\|} \rightharpoonup W = [w, z]$. Let's divide the system (37) by $\|W_n\|$. We get

$$
\begin{aligned}
d_1^n \frac{u_n}{\|W_n\|} - b_{1,1} A \frac{u_n}{\|W_n\|} - b_{1,2} A \frac{v_n}{\|W_n\|} - \frac{N_1(u_n, v_n)}{\|W_n\|} + \frac{\tilde{F}(u_n)}{\|W_n\|} &= 0, \\
d_2^n \frac{v_n}{\|W_n\|} - b_{2,1} A \frac{u_n}{\|W_n\|} - b_{2,2} A \frac{v_n}{\|W_n\|} - \frac{N_2(u_n, v_n)}{\|W_n\|} &= 0
\end{aligned}
\tag{64}
$$

due to linearity of A. Due to (24) we have $\frac{N_j(u_n, v_n)}{\|W_n\|} \to 0$ as $n \to +\infty$ for $j = 1, 2$. Since $\frac{u_n}{\|W_n\|} \rightharpoonup w$ and $\frac{v_n}{\|W_n\|} \rightharpoonup z$, using compactness of A and (32) we get $A \frac{u_n}{\|W_n\|} \to Aw$ and $\frac{\tilde{F}(u_n)}{\|W_n\|} \to \beta(w)$, analogously for v_n and z. We have $d_1, d_2 > 0$, therefore it follows from (64) that $\frac{u_n}{\|W_n\|} \to w, \frac{v_n}{\|W_n\|} \to z, \|W\| = 1$ and

$$
\begin{aligned}
d_1^0 w - b_{1,1} Aw - b_{1,2} Az + \beta(w) &= 0, \\
d_2^0 z - b_{2,1} Aw - b_{2,2} Az &= 0.
\end{aligned}
$$

Therefore the point d^0 is a critical point of the system (8), (4). $\square$

Lemma A.3 (see [6])**.** *For any $d_2 > 0$ and $u_0 \in H_D^1$ the operators $P \equiv \beta$ and $P \equiv S_{d_2} - \beta$ satisfy the condition* (51)*.*

Proof.
We will prove (51) for $P = \beta^-$. The proof for β^+ is analogous and for $P \equiv S_{d_2} - \beta$ it will follow by using the definition of β and linearity of S_{d_2}.
Let $u_0, h \in H_D^1$. We will introduce two sets Ω_0 and Ω_{th} such that

$$u_0(\mathbf{x}) < 0 \text{ a.e. on } \Omega_0, \quad u_0(\mathbf{x}) \geq 0 \text{ a.e. on } \Omega \setminus \Omega_0,$$
$$u_0(\mathbf{x}) + th(\mathbf{x}) < 0 \text{ a.e. on } \Omega_{th}, \quad u_0(\mathbf{x}) + th(\mathbf{x}) \geq 0 \text{ a.e. on } \Omega \setminus \Omega_{th}.$$

Then

$$\frac{1}{t}(\beta^-(u_0 + th) - \beta^-(u_0), u_0) - (\beta^-(u_0), h) =$$

$$= \frac{1}{t}\left[\int_\Omega -(u_0 + th)^- u_0 + u_0 u_0^- \, d\Omega\right] - \int_\Omega -u_0^- h \, d\Omega =$$

$$= \frac{1}{t}\left[\int_{\Omega_{th}} (u_0 + th)u_0 \, d\Omega_{th} - \int_{\Omega_0} u_0^2 \, d\Omega_0\right] - \int_{\Omega_0} u_0 h \, d\Omega_0 =$$

$$= \frac{1}{t}\left[\int_{\Omega_{th}} u_0^2 \, d\Omega_{th} - \int_{\Omega_0} u_0^2 \, d\Omega_0\right] + \int_{\Omega_{th}} u_0 h \, d\Omega_{th} - \int_{\Omega_0} u_0 h \, d\Omega_0.$$

We can afford to work with the definition of β^- without s_-, because it is non-negative, i.e. it does not affect the sign of terms under integration.
Let χ_{th} and χ_0 be the characteristic function of Ω_{th} and Ω_0, respectively. We have

$$\lim_{t \to 0} \int_{\Omega_{th}} u_0 h \, d\Omega_{th} = \lim_{t \to 0} \int_\Omega u_0 h \chi_{th} \, d\Omega = \int_\Omega u_0 h \chi_0 \, d\Omega = \int_{\Omega_0} u_0 h \, d\Omega_0$$

by Dominated Convergence theorem. Let's introduce sets $\Omega_{th1}, \Omega_{th2}, \Omega_{th3}$ such that

$$u_0(\mathbf{x}) < -th(\mathbf{x}) \text{ and } u_0(\mathbf{x}) < 0 \text{ almost everywhere on } \Omega_{th1},$$
$$u_0(\mathbf{x}) < -th(\mathbf{x}) \text{ and } u_0(\mathbf{x}) \geq 0 \text{ almost everywhere on } \Omega_{th2},$$
$$u_0(\mathbf{x}) \geq -th(\mathbf{x}) \text{ and } u_0(\mathbf{x}) < 0 \text{ almost everywhere on } \Omega_{th3},$$

with $\Omega_{th} = \Omega_{th1} \cup \Omega_{th2}$ and $\Omega_0 = \Omega_{th1} \cup \Omega_{th3}$. This way we get

$$\int_{\Omega_{th}} u_0^2 \, d\Omega_{th} - \int_{\Omega_0} u_0^2 \, d\Omega_0 = \int_{\Omega_{th1}} u_0^2 \, d\Omega_{th1} + \int_{\Omega_{th2}} u_0^2 \, d\Omega_{th2} - \int_{\Omega_{th1}} u_0^2 \, d\Omega_{th1} - \int_{\Omega_{th3}} u_0^2 \, d\Omega_{th3} =$$

$$= \int_{\Omega_{th2}} u_0^2 \, d\Omega_{th2} - \int_{\Omega_{th3}} u_0^2 \, d\Omega_{th3}.$$

Since $0 \leq u_0 < -th$ a.e. on Ω_{th2} and $0 > u_0 \geq -th$ a.e. on Ω_{th3}, we get

$$\lim_{t \to 0} \frac{1}{t}\left(\int_{\Omega_{th2}} u_0^2 \, d\Omega_{th2} - \int_{\Omega_{th3}} u_0^2 \, d\Omega_{th3}\right) \leq \lim_{t \to 0} \frac{1}{t}\left(\int_{\Omega_{th2}} (th)^2 \, d\Omega_{th2} - \int_{\Omega_{th3}} (th)^2 \, d\Omega_{th3}\right) = 0.$$

Hence, it follows from the discussion above that

$$\lim_{t \to 0} \frac{1}{t}(\beta^-(u_0 + th) - \beta^-(u_0), u_0) - (\beta^-(u_0), h) = 0$$

which proves (51) for β^-. $\qquad\square$

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