THE BEHAVIOUR OF THE NONWANDERING SET OF A PIECEWISE MONOTONIC INTERVAL MAP UNDER SMALL PERTURBATIONS

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Abstract. In this paper piecewise monotonic maps $T\colon [0,1]\to[0,1]$ are considered. Let Q be a finite union of open intervals, and consider the set $R(Q)$ of all points whose orbits omit Q. The influence of small perturbations of the endpoints of the intervals in Q on the dynamical system $(R(Q),T)$ is investigated. The decomposition of the nonwandering set into maximal topologically transitive subsets behaves very unstably. Nonetheless, it is shown that a maximal topologically transitive subset cannot be completely destroyed by arbitrary small perturbations of Q. Furthermore it is shown that every sufficiently "big" maximal topologically transitive subset of a sufficiently small perturbation of $(R(Q),T)$ is "dominated" by a topologically transitive subset of $(R(Q),T)$.

Keywords: piecewise monotonic map, nonwandering set, topologically transitive subset

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Introduction

Let $T\colon [0,1]\to[0,1]$ be a piecewise monotonic map, that means there exists a finite partition $\mathcal{Z}$ of $[0,1]$ of pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}}\overline{Z}=[0,1]$ such that $T|Z$ is continuous and strictly monotonic for all $Z\in\mathcal{Z}$. Fix $K\in\mathbb{N}$, and let $(a_1,a_2)\cup(a_3,a_4)\cup\ldots\cup(a_{2K-1},a_{2K})$ be a finite union of open subintervals of $[0,1]$.

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Define

$$R(a_1, a_2, \ldots, a_{2K}) := \bigcap_{n=0}^{\infty} \overline{[0,1] \setminus T^{-n}\big(\bigcup_{k=1}^{K}(a_{2k-1}, a_{2k})\big)}.$$

The aim of this paper is to investigate the influence of small perturbations of the endpoints of $\bigcup_{k=1}^{K}(a_{2k-1}, a_{2k})$ on the dynamical system $(R(a_1, a_2, \ldots, a_{2K}), T)$.

Such problems were treated in [8], [10] and [11]. If $f\colon [0,1] \to \mathbb{R}$ is a piecewise continuous function, that means $f|Z$ can be extended to a continuous function on $\overline{Z}$ for all $Z \in \mathcal{Z}$, then Theorem 1 in [8] says that the function $(a_1, a_2, \ldots, a_{2K}) \mapsto p(R(a_1, a_2, \ldots, a_{2K}), T, f)$ is continuous, if a certain condition generalizing

$$p(R(a_1, a_2, \ldots, a_{2K}), T, f) > \sup_{x \in R(a_1, a_2, \ldots, a_{2K})} f(x)$$

is satisfied. This implies that the topological entropy is continuous (Corollary 1.1 in [8]). For an expanding T, that means T' is piecewise continuous and there exists an $n \in \mathbb{N}$ with $\inf_{x\in[0,1]} |(T^n)'(x)| > 1$, Theorem 2 in [8] gives that the function $(a_1, a_2, \ldots, a_{2K}) \mapsto \mathrm{HD}\big(R(a_1, a_2, \ldots, a_{2K})\big)$ is continuous. In the case of an expanding C^2-transformation of the circle these results were earlier obtained by Mariusz Urbański ([10] and [11]). All of those results concern certain dynamical invariants, but not the dynamics itself. The aim of this paper is to investigate how the dynamics of $(R(a_1, a_2, \ldots, a_{2K}), T)$ reacts on small perturbations of $(a_1, a_2, \ldots, a_{2K})$. This is done by considering the reaction of these perturbations on the decomposition of the nonwandering set of $(R(a_1, a_2, \ldots, a_{2K}), T)$ into maximal topologically transitive subsets with positive entropy. The results of this paper imply the continuity results mentioned above. However, the results of this paper are not strong enough to obtain stability results for equilibrium states. The behaviour of equilibrium states under small perturbations of $(a_1, a_2, \ldots, a_{2K})$ remains an open problem.

Another stability problem is studied in [4], [7] and [9]. In those papers a closeness relation for piecewise monotonic maps is defined, and small perturbations with respect to this closeness relation are investigated. We get $R(a_1, a_2, \ldots, a_{2K}) = R\Big(T\big|[0,1] \setminus \bigcup_{k=1}^{K}(a_{2k-1}, a_{2k})\Big)$, where $R\Big(T\big|[0,1] \setminus \bigcup_{k=1}^{K}(a_{2k-1}, a_{2k})\Big)$ is defined as in (1.1) of [9]. However, the results of [9] (which are in some sense weaker than the results in this paper) need not be applicable in our case, since $T\big|[0,1] \setminus \bigcup_{k=1}^{K}(\tilde{a}_{2k-1}, \tilde{a}_{2k})$ need not be close to $T\big|[0,1] \setminus \bigcup_{k=1}^{K}(a_{2k-1}, a_{2k})$ in the sense defined in [9], if $|\tilde{a}_j - a_j| < \varepsilon$ for all $j \in \{1, 2, \ldots, 2K\}$ (see p. 39 in Introduction of [8] for a description of this fact).

In [1], [2], [3] and [5] a structure theorem for the nonwandering set of a piecewise monotonic map is shown. It says that

$$\Omega(R(a_1,a_2,\dots,a_{2K}),T)=\bigcup_{i\in I}L_i\cup\bigcup_{j\in J}N_j\cup P\cup W$$

where I is at most countable, J is at most finite, the intersection of two different sets in this decomposition is at most finite, the sets L_i are closed, T-invariant, topologically transitive, and the periodic points of L_i are dense in L_i, the sets N_j are closed, T-invariant, minimal with entropy zero and no periodic points, and they are maximal topologically transitive, the set P is closed, T-invariant, and consists of periodic points, which are contained in nontrivial intervals K, which are mapped into K by T^n for an $n\in\mathbb{N}$, and the elements of W are not contained in $\Omega(\Omega(R(a_1,a_2,\dots,a_{2K}),T),T)$. Furthermore the sets L_i either form a single periodic orbit, or they are maximal topologically transitive subsets with positive entropy. Hence the most interesting part of the dynamics takes place on the at most countably many maximal topologically transitive subsets with positive entropy.

In the first section we give some basic definitions and notation. Our main tool for investigating the structure of the nonwandering set, the Markov diagram (see e.g. [2]), is described in Section 2. Then we describe the structure theorem for the nonwandering set in Section 3. In Section 4 we give an example for the instability of the decomposition of the nonwandering set into maximal topologically transitive subsets with positive entropy. Our main results are contained in Section 5. Theorem 1 says that, if we take a maximal topologically transitive subset L of $(R(a_1,a_2,\dots,a_{2K}),T)$ with positive entropy, and if $(\tilde a_1,\tilde a_2,\dots,\tilde a_{2K})$ is sufficiently close to $(a_1,a_2,\dots,a_{2K})$, then there exists a topologically transitive subset $\tilde L$ of $(R(\tilde a_1,\tilde a_2,\dots,\tilde a_{2K}),T)$ (which in general is not maximal topologically transitive), such that $\tilde L$ and L are close in the Hausdorff metric, and the entropy of $\tilde L$ is close to the entropy of L ("L cannot be completely destroyed"). Furthermore, if $f\colon[0,1]\to\mathbb{R}$ is piecewise continuous and a condition generalizing $p(L,T,f)>\sup_{x\in R(a_1,a_2,\dots,a_{2K})}f(x)$ is satisfied, then also $p(\tilde L,T,f)$ is close to $p(L,T,f)$ and, if T is expanding, then also $\mathrm{HD}(\tilde L)$ is close to $\mathrm{HD}(L)$. If we take a piecewise continuous $f\colon[0,1]\to\mathbb{R}$ and fix an $\alpha>0$, then Theorem 2 gives that for every $(\tilde a_1,\tilde a_2,\dots,\tilde a_{2K})$ which is sufficiently close to $(a_1,a_2,\dots,a_{2K})$ and for every maximal topologically transitive subset $\tilde L$ of $(R(\tilde a_1,\tilde a_2,\dots,\tilde a_{2K}),T)$ which satisfies a condition generalizing $p(\tilde L,T,f)>\sup_{x\in R(a_1,a_2,\dots,a_{2K})}f(x)+\alpha$ ("$\tilde L$ is sufficiently big"), there exist a topologically transitive subset $\tilde L'$ of $(R(\tilde a_1,\tilde a_2,\dots,\tilde a_{2K}),T)$ and a topologically transitive subset L of $(R(a_1,a_2,\dots,a_{2K}),T)$ (in general $\tilde L'$ and L are not maximal topologically transitive), such that $\tilde L'$ and L are close in the Hausdorff metric and are contained in a neighbourhood of $\tilde L$, $p(\tilde L',T,f)$ is close to $p(L,T,f)$, and $p(\tilde{L}, T, f)$ is not bigger than a number close to $p(\tilde{L}', T, f)$ and $p(L, T, f)$ ("$\tilde{L}'$ and L dominate $\tilde{L}$"). A similar result concerning the topological entropy follows immediately from Theorem 2 (Corollary 2.1). For an expanding T a similar result concerning the Hausdorff dimension is given in Theorem 3. The main idea of the proofs is to calculate the decomposition of the nonwandering set into maximal topologically transitive subsets with positive entropy using the Markov diagram as in [2], and to use the results of [8], where the behaviour of the Markov diagram under small perturbations of $(a_1, a_2, \ldots, a_{2K})$ is described.

1. Definitions and notation

We call $\mathcal{Z}$ a *finite partition* of $[0,1]$, if $\mathcal{Z}$ consists of pairwise disjoint open intervals with $\bigcup_{Z \in \mathcal{Z}} \overline{Z} = [0,1]$. A map $T\colon [0,1] \to [0,1]$ is called *piecewise monotonic*, if there exists a finite partition $\mathcal{Z}$ of $[0,1]$ such that $T|Z$ is strictly monotonic and continuous for all $Z \in \mathcal{Z}$. A function $f\colon [0,1] \to \mathbb{R}$ is called *piecewise continuous* with respect to the finite partition $\mathcal{Z}(f)$ of $[0,1]$, if $f|Z$ can be extended to a continuous function on the closure of Z for all $Z \in \mathcal{Z}(f)$. We say that $f\colon [0,1] \to \mathbb{R}$ is *piecewise constant* with respect to the finite partition $\mathcal{Z}(f)$ of $[0,1]$, if $f|Z$ is constant for all $Z \in \mathcal{Z}(f)$. A piecewise monotonic map $T\colon [0,1] \to [0,1]$ is called *expanding*, if there exists a $j \in \mathbb{N}$ such that $(T^j)'$ is a piecewise continuous function and $\inf_{x \in [0,1]} |(T^j)'(x)| > 1$. At this point we want to remark that all results of this paper hold also for the situation considered in [7] and [9], that means $T\colon X \to \mathbb{R}$ is piecewise monotonic, where X is a finite union of closed intervals.

Let $K \in \mathbb{N}$ and suppose that $0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{2K-1} \leqslant a_{2K} \leqslant 1$ with $a_j < a_{j+2}$ for $j \in \{1, 2, \ldots, 2K-2\}$. Set $Q := (a_1, a_2, \ldots, a_{2K-1}, a_{2K})$. Let $\mathcal{Q}_K$ be the set of all such Q's. Now for $Q = (a_1, a_2, \ldots, a_{2K-1}, a_{2K}) \in \mathcal{Q}_K$ define

$$X(Q) := [0,1] \setminus \left(\bigcup_{k=1}^{K} (a_{2k-1}, a_{2k}) \right) \tag{1.1}$$

and

$$R(Q) := \bigcap_{j=0}^{\infty} \overline{[0,1] \setminus T^{-j}\left(\bigcup_{k=1}^{K} (a_{2k-1}, a_{2k}) \right)}\ . \tag{1.2}$$

Let $\mathcal{Z}(Q)$ be the set of all maximal open subintervals of $X(Q) \cap \big(\bigcup_{Z \in \mathcal{Z}} Z \big)$. Observing that the results of [7] and [9] remain true if we allow X to be a finite union of closed intervals and isolated points, we have that $\big(T|X(Q), \mathcal{Z}(Q)\big)$ is a piecewise monotonic map of class R^0 in the sense defined in [7] and [9]. Furthermore we have $R\big(T|X(Q)\big) = R(Q)$, where $R\big(T|X(Q)\big)$ is defined as in (1.1) of [9] (cf. [8]).

As in [8] we define a topology on $\mathcal{Q}_K$. Let $\varepsilon > 0$. Then

$$Q := (a_1, a_2, \dots, a_{2K-1}, a_{2K}) \quad \text{and} \quad \tilde{Q} := (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_{2K-1}, \tilde{a}_{2K})$$

are said to be ε-close, if $|a_j - \tilde{a}_j| < \varepsilon$ for all $j \in \{1, 2, \dots, 2K\}$. Observe that $\big(T|X(Q), \mathcal{Z}(Q)\big)$ and $\big(T|X(\tilde{Q}), \mathcal{Z}(\tilde{Q})\big)$ need not be ε-close with respect to the R^0-topology defined in [7] and [9].

Next we modify $([0,1], T)$ as in [8] in order to get a topological dynamical system. Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to $\mathcal{Z}$, let $K \in \mathbb{N}$, let $Q \in \mathcal{Q}_K$, and let $\mathcal{Y}$ be a finite partition of $[0,1]$ which refines $\mathcal{Z}$. We assume throughout this paper that $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_N\}$ with $Y_1 < Y_2 < \ldots < Y_N$. Set $E := \{\inf Y, \sup Y \colon Y \in \mathcal{Y}\}$. Now define $W := \big(\bigcup_{j=0}^{\infty} T^{-j}(E \setminus \{0,1\})\big) \setminus \{0,1\}$, set $\mathbb{R}_{\mathcal{Y}} := \mathbb{R} \setminus W \cup \{x^-, x^+ \colon x \in W\}$, and define $y < x^- < x^+ < z$, if $y < x < z$ holds in $\mathbb{R}$. This means that we have doubled all endpoints of elements of $\mathcal{Y}$, and we have also doubled all inverse images of doubled points. For $x \in \mathbb{R}_{\mathcal{Y}}$ define $\pi_{\mathcal{Y}}(x) := y$, where $y \in \mathbb{R}$ satisfies either $x = y$ or $y \in W$ and $x \in \{y^-, y^+\}$. We have that $x, y \in \mathbb{R}_{\mathcal{Y}}$, $\pi_{\mathcal{Y}}(x) < \pi_{\mathcal{Y}}(y)$ implies $x < y$. As in [7] we can introduce a metric d on $\mathbb{R}_{\mathcal{Y}}$, which generates the order topology.

Let $\mathcal{Y}(Q)$ be the set of all maximal open subintervals of $X(Q) \cap \big(\bigcup_{Y \in \mathcal{Y}} Y\big)$. Let $X_{\mathcal{Y}}$ be the closure of $[0,1] \setminus W$ in $\mathbb{R}_{\mathcal{Y}}$ and define $X_{\mathcal{Y}}(Q) := \{x \in \mathbb{R}_{\mathcal{Y}} \colon \pi_{\mathcal{Y}}(x) \in X(Q)\}$. Observe that $X_{\mathcal{Y}}$ and $X_{\mathcal{Y}}(Q)$ are compact. For a perfect subset A of $\mathbb{R}$ let $\hat{A}$ be the closure of $A \setminus W$ in $\mathbb{R}_{\mathcal{Y}}$. Now set $\hat{\mathcal{Y}} := \{\hat{Y} \colon Y \in \mathcal{Y}\}$, $\hat{\mathcal{Z}} := \{\hat{Z} \colon Z \in \mathcal{Z}\}$, $\hat{\mathcal{Y}}(Q) := \{\hat{Y} \colon Y \in \mathcal{Y}(Q)\}$ and $\hat{\mathcal{Z}}(Q) := \{\hat{Z} \colon Z \in \mathcal{Z}(Q)\}$. The map $T|[0,1] \setminus (W \cup E)$ can be extended to a unique continuous piecewise monotonic map $T_{\mathcal{Y}} \colon X_{\mathcal{Y}} \to X_{\mathcal{Y}}$. Then $(T_{\mathcal{Y}}, \hat{\mathcal{Z}})$ is a continuous piecewise monotonic map of class R^0 on $X_{\mathcal{Y}}$ in the sense defined in [7]. If there is no confusion we will use the notation $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{Y}(Q)$ and $\mathcal{Z}(Q)$ instead of $\hat{\mathcal{Y}}$, $\hat{\mathcal{Z}}$, $\hat{\mathcal{Y}}(Q)$ and $\hat{\mathcal{Z}}(Q)$. The set $R_{\mathcal{Y}}(Q) := \bigcap_{j=0}^{\infty} T_{\mathcal{Y}}{}^{-j} X_{\mathcal{Y}}(Q)$ satisfies

$$R_{\mathcal{Y}}(Q) = \bigcap_{j=0}^{\infty} \overline{T_{\mathcal{Y}}{}^{-j} X_{\mathcal{Y}}(Q)} = \{x \in \mathbb{R}_{\mathcal{Y}} \colon \pi_{\mathcal{Y}}(x) \in R(Q)\}.$$

If $f\colon [0,1] \to \mathbb{R}$ is piecewise continuous with respect to $\mathcal{Z}$, then there exists a unique continuous function $f_{\mathcal{Y}} \colon X_{\mathcal{Y}} \to \mathbb{R}$ with $f_{\mathcal{Y}}(x) = f(x)$ for all $x \in [0,1] \setminus (W \cup E)$. Finally, we define a map $Y \colon X_{\mathcal{Y}} \to \mathcal{Y}$. If $x \in X_{\mathcal{Y}}$ and $\pi_{\mathcal{Y}}(x) \notin E \setminus \{0,1\}$, then there exists a unique $Y \in \mathcal{Y}$ with $\pi_{\mathcal{Y}}(x) \in \overline{Y}$. Set $Y(x) := Y$ in this case. Otherwise we have either $x = \pi_{\mathcal{Y}}(x)^-$ or $x = \pi_{\mathcal{Y}}(x)^+$, and there exist exactly two $Y^-, Y^+ \in \mathcal{Y}$

with $Y^- < Y^+$ such that $\pi_{\mathcal{Y}}(x) \in \overline{Y^-} \cap \overline{Y^+}$. Now set $Y(x) := Y^-$ if $x = \pi_{\mathcal{Y}}(x)^-$, and $Y(x) := Y^+$ if $x = \pi_{\mathcal{Y}}(x)^+$.

A *topological dynamical system* (X, T) is a continuous map T of a compact metric space X into itself. If $\varepsilon > 0$ and $n \in \mathbb{N}$, then we call a set $E \subseteq X$ (n, ε)-separated, if for every $x \neq y \in E$ there exists a $j \in \{0, 1, \dots, n-1\}$ with $d(T^j x, T^j y) > \varepsilon$. For a continuous function $f\colon X \to \mathbb{R}$ the *topological pressure* $p(X, T, f)$ is defined by

$$p(X, T, f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_E \sum_{x \in E} \exp\Big(\sum_{j=0}^{n-1} f(T^j x)\Big),$$

where the supremum is taken over all (n, ε)-separated subsets E of X.

Then $\big(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}}|R_{\mathcal{Y}}(Q)\big)$ is a topological dynamical system (see [8]). We will use the abbreviation $(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}})$ for $\big(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}}|R_{\mathcal{Y}}(Q)\big)$. As in (1.4) of [8] we define the pressure $p(R(Q), T, f)$ by $p(R(Q), T, f) := p(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}}, f_{\mathcal{Y}})$, and as in (1.5) of [8] we set $S_n(R(Q), f) := \sup\limits_{x \in R_{\mathcal{Y}}(Q)} \sum\limits_{j=0}^{n-1} f_{\mathcal{Y}}(T_{\mathcal{Y}}{}^j x)$, where $n \in \mathbb{N}$. We remark that the condition

$$p(R(Q), T, f) > \lim_{n \to \infty} \frac{1}{n} S_n(R(Q), f),$$

which will be used in this paper, is a generalization of the condition $p(R(Q), T, f) > \sup\limits_{x \in R(Q)} f(x)$. For the definition of the Hausdorff dimension we refer to [8]. We define the nonwandering set $\Omega_{\mathcal{Y}}(R(Q), T)$ of $(R(Q), T)$ by $\Omega_{\mathcal{Y}}(R(Q), T) := \Omega(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}})$, where $\Omega(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}})$ is the nonwandering set of the topological dynamical system $(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}})$ (cf. (2.4) in [9]).

2. The Markov diagram of $(R(Q), T)$

In this section we describe our main tool, the Markov diagram, which was introduced by Franz Hofbauer (see e.g. [2]). This is an at most countable oriented graph which describes the orbit structure of $(R(Q), T)$. We shall also need the notion of a version of the Markov diagram as introduced in [8]. For the convenience of the reader we recall the main steps of this construction. We also recall a stability result for the Markov diagram, which is proved in [8].

Let $T\colon [0, 1] \to [0, 1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0, 1]$, let $K \in \mathbb{N}$, let $Q \in \mathcal{Q}_K$, and let $\mathcal{Y}$ be a finite partition of $[0, 1]$ which refines $\mathcal{Z}$. Let $I_{\mathcal{Y}}$ be the set of all isolated points of $X_{\mathcal{Y}}(Q)$, and set $I_Q := I_{\mathcal{Y}} \cup \big(\{\inf Y, \sup Y \colon Y \in \hat{\mathcal{Y}}\} \cap X_{\mathcal{Y}}(Q)\big)$. Let $Y_0 \in \hat{\mathcal{Y}}(Q)$ and let D be a perfect subinterval of Y_0. A nonempty $C \subseteq X_{\mathcal{Y}}(Q)$ is called a *successor* of D if there exists a $Y \in \hat{\mathcal{Y}}(Q)$ with $C = T_{\mathcal{Y}} D \cap Y$, and we write $D \to C$. We get that every successor

C of D is again a perfect subinterval of an element of $\hat{\mathcal{Y}}(Q)$. Let $\mathcal{D}$ be the smallest set with $\hat{\mathcal{Y}}(Q) \subseteq \mathcal{D}$ and such that $D \in \mathcal{D}$ and $D \to C$ imply $C \in \mathcal{D}$. Then $(\mathcal{D}, \to)$ is called the *Markov diagram* of $(R(Q), T)$ with respect to $\mathcal{Y}$. The set $\mathcal{D}$ is at most countable and its elements are perfect subintervals of elements of $\hat{\mathcal{Y}}(Q)$.

Set $\mathcal{D}_0 := \hat{\mathcal{Y}}(Q)$, and for $r \in \mathbb{N}$ set $\mathcal{D}_r := \mathcal{D}_{r-1} \cup \{\, D \in \mathcal{D} \colon \exists\, C \in \mathcal{D}_{r-1}$ with $C \to D\}$. Then we have $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots$ and $\mathcal{D} = \bigcup_{r=0}^{\infty} \mathcal{D}_r$.

Let $\mathcal{C} \subseteq \mathcal{D}$. For $n \in \mathbb{N}$ we call $C_0 \to C_1 \to \ldots \to C_n$ a *path of length n in $\mathcal{C}$*, if $C_j \in \mathcal{C}$ for $j \in \{0, 1, \ldots, n\}$ and $C_{j-1} \to C_j$ for $j \in \{1, 2, \ldots, n\}$. Furthermore we call $C_0 \to C_1 \to C_2 \to \cdots$ an *infinite path in $\mathcal{C}$*, if $C_j \in \mathcal{C}$ for $j \in \mathbb{N}_0$ and $C_{j-1} \to C_j$ for $j \in \mathbb{N}$. We say that an infinite path $C_0 \to C_1 \to C_2 \to \cdots$ *represents* $x \in R_{\mathcal{Y}}(Q)$, if ${T_{\mathcal{Y}}}^j x \in C_j$ for all $j \in \mathbb{N}_0$. We call $\mathcal{C}$ *irreducible*, if for every $C, D \in \mathcal{C}$ there exists a finite path $C_0 \to C_1 \to \ldots \to C_n$ in $\mathcal{C}$ with $C_0 = C$ and $C_n = D$. If $\mathcal{C}$ is irreducible and finite, then $\mathcal{C}$ is called *finite irreducible*. An irreducible $\mathcal{C}$ is called *maximal irreducible*, if every $\mathcal{C}'$ with $\mathcal{C} \subsetneqq \mathcal{C}' \subseteq \mathcal{D}$ is not irreducible.

If $C \in \mathcal{D}$ and $x \in I_Q$, then we introduce an arrow $C \to \{x\}$, if and only if $x \in T_{\mathcal{Y}} C$. Let $x \in I_Q$. Then we set $j(x) := \min\{j \in \mathbb{N} \colon {T_{\mathcal{Y}}}^j x \notin X_{\mathcal{Y}}(Q)\}$, where we set $j(x) := \infty$ if ${T_{\mathcal{Y}}}^j x \in X_{\mathcal{Y}}(Q)$ for all $j \in \mathbb{N}$. Now define $\mathcal{D}(x) := \big\{\{{T_{\mathcal{Y}}}^j x\} \colon j \in \mathbb{N}_0, j < j(x)\big\}$, define

$$\mathcal{D}_r(x) := \big\{\{{T_{\mathcal{Y}}}^j x\} \colon j \in \mathbb{N}_0, j < \min\{j(x), r+1\}\big\} \text{ for } r \in \mathbb{N}_0,$$

and introduce an arrow $\{{T_{\mathcal{Y}}}^{j-1} x\} \to \{{T_{\mathcal{Y}}}^j x\}$, if $\{{T_{\mathcal{Y}}}^j x\} \in \mathcal{D}(x)$ and $j \in \mathbb{N}$ (there are no other arrows beginning in $\{{T_{\mathcal{Y}}}^{j-1} x\}$). If $B \subseteq I_Q$, then define $\mathcal{D}(B) := \mathcal{D} \cup \bigcup_{x \in B} \mathcal{D}(x)$, and $\mathcal{D}_r(B) := \mathcal{D}_r \cup \bigcup_{x \in B} \mathcal{D}_r(x)$ for $r \in \mathbb{N}_0$. Including these points in the Markov diagram is an important technical tool in our proofs.

The definition of a *version* $(\mathcal{A}, \to)$ *of the Markov diagram* of $(R(Q), T)$ with respect to $\mathcal{Y}$ is given on pp. 43–45 of [8]. We shortly describe its most important properties. If $(\mathcal{A}, \to)$ is a version of the Markov diagram of $(R(Q), T)$ with respect to $\mathcal{Y}$, then there exists a $B \subseteq I_Q$, and there exists a surjective function $A \colon \mathcal{A} \to \mathcal{D}(B)$ such that $c \to d$ in $\mathcal{A}$ implies $A(c) \to A(d)$ in $\mathcal{D}(B)$. Furthermore, for every $c \in \mathcal{A}$ the map A is bijective from $\{d \in \mathcal{A} \colon c \to d\}$ to $\{D \in \mathcal{D}(B) \colon A(c) \to D\}$. We can write $\mathcal{A} = \bigcup_{r=0}^{\infty} \mathcal{A}_r$ with $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ and $A(\mathcal{A}_r) = \mathcal{D}_r(B)$. If $I_{\mathcal{Y}} \subseteq B$, then $(\mathcal{A}, \to)$ is called a *full version of the Markov diagram* of $(R(Q), T)$ with respect to $\mathcal{Y}$. If $\mathcal{C} \subseteq \mathcal{A}$, then we call $c_0 \to c_1 \to c_2 \to \cdots$ an infinite path in $\mathcal{C}$ if $c_j \in \mathcal{C}$ for all $j \in \mathbb{N}_0$ and $c_{j-1} \to c_j$ in $\mathcal{A}$ for all $j \in \mathbb{N}$. We say that an infinite path $c_0 \to c_1 \to c_2 \to \cdots$ represents $x \in R_{\mathcal{Y}}(Q)$, if $A(c_0) \to A(c_1) \to A(c_2) \to \cdots$ represents x.

Now let $T \colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0,1]$, let $f \colon [0,1] \to \mathbb{R}$ be piecewise constant with respect to $\mathcal{Z}$, let $K \in \mathbb{N}$, let $Q \in \mathcal{Q}_K$, and suppose that $\mathcal{Y}$ is a finite partition of $[0,1]$ which refines

$\mathcal{Z}$. Let $(\mathcal{A}, \to)$ be a version of the Markov diagram of $(R(Q), T)$ with respect to $\mathcal{Y}$, and suppose that $\mathcal{C} \subseteq \mathcal{A}$. Then we define the matrix $F_{\mathcal{C}}(f)$ as in Formula (1.8) of [8], that means $F_{\mathcal{C}}(f) := \big(F_{c,d}(f)\big)_{c,d\in\mathcal{C}}$, where $F_{c,d}(f) := e^{f(x)}$ if $c \to d$ in $(\mathcal{A}, \to)$ and $x \in A(c)$, and $F_{c,d}(f) := 0$ otherwise. We denote the norm of the $\ell^1(\mathcal{C})$-operator $u \mapsto uF_{\mathcal{C}}(f)$ by $\|F_{\mathcal{C}}(f)\|$ and its spectral radius by $r\big(F_{\mathcal{C}}(f)\big)$.

Finally we recall Lemma 2 of [8], which describes the stability of the Markov diagram under small perturbations of Q. Roughly spoken this result says that if $\tilde{Q}$ is sufficiently close to Q, then the Markov diagrams of $(R(\tilde{Q}), T)$ and $(R(Q), T)$ have similar initial parts. As we need a bit more than the statement of Lemma 2 in [8] says, we give a full statement of this result (the proof is the same as the proof of Lemma 2 in [8]).

Lemma 1. *Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0,1]$, let $K \in \mathbb{N}$, and let $Q = (a_1, a_2, \ldots, a_{2K-1}, a_{2K}) \in \mathcal{Q}_K$. Suppose that $\mathcal{Y}$ is a finite partition of $[0,1]$ which refines $\mathcal{Z}$, such that $a_j \in \{\inf Y, \sup Y \colon Y \in \mathcal{Y}\}$ for every $j \in \{1, 2, \ldots, 2K-1, 2K\}$. Then for every $r \in \mathbb{N}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\tilde{Q} \in \mathcal{Q}_K$ which is δ-close to Q, there exists a version $(\mathcal{A}, \to)$ of the Markov diagram $(\mathcal{D}, \to)$ of $(R(Q), T)$ with respect to $\mathcal{Y}$, and a full version $(\tilde{\mathcal{A}}, \to)$ of the Markov diagram of $(R(\tilde{Q}), T)$ with respect to $\mathcal{Y}$ with the following properties.*

1. *There exists a function $\varphi\colon \tilde{\mathcal{A}}_r \to \mathcal{A}_r$ such that $\varphi(\tilde{\mathcal{A}}_0) = \mathcal{A}_0$, and for every $c \in \mathcal{A}_r$ we have $\operatorname{card} \varphi^{-1}(c) \leqslant 2$. If $c \in \mathcal{A}_r$ and either $\operatorname{card} \varphi^{-1}(c) > 1$ or $c \notin \varphi(\tilde{\mathcal{A}}_r)$, then $A(c) = \{x\}$ for an $x \in X_{\mathcal{Y}}(Q)$.*
2. *For $c, d \in \tilde{\mathcal{A}}_r$ with $A\big(\varphi(c)\big) \in \mathcal{D}$ the property $c \to d$ in $\tilde{\mathcal{A}}$ implies $\varphi(c) \to \varphi(d)$ in $\mathcal{A}$. Furthermore, $c, d \in \tilde{\mathcal{A}}_r$, $\varphi(c) \to \varphi(d)$ in $\mathcal{A}$ and d is not a successor of c in $\tilde{\mathcal{A}}$ imply that $A\big(\varphi(d)\big) = \{x\}$, where x is contained in $\big\{T_{\mathcal{Y}} \inf A\big(\varphi(c)\big), T_{\mathcal{Y}} \sup A\big(\varphi(c)\big)\big\}$. If $c, d \in \tilde{\mathcal{A}}_r$, $c \to d$ in $\tilde{\mathcal{A}}$, and $\varphi(d)$ is not a successor of $\varphi(c)$ in $\mathcal{A}$, then there exist $c_1, d_1 \in \tilde{\mathcal{A}}_r$ with $c_1 \to d_1$ in $\tilde{\mathcal{A}}$, $\varphi(c_1) = \varphi(c)$, $\varphi(c) \to \varphi(d_1)$ in $\mathcal{A}$, and $A\big(\varphi(d_1)\big) = A\big(\varphi(d)\big)$.*
3. *For every $c \in \tilde{\mathcal{A}}_r$ the set $\tilde{A}(c)$ is ε-close to $A\big(\varphi(c)\big)$ in the Hausdorff metric. Furthermore, if $c \in \tilde{\mathcal{A}}_r$ and $Y \in \mathcal{Y}$ satisfy $Y(x) = Y$ for all $x \in A\big(\varphi(c)\big)$, then $Y(x) = Y$ for all $x \in \tilde{A}(c)$.*
4. *If $c \in \tilde{\mathcal{A}}_0$, and $d_0 = \varphi(c) \to d_1 \to \ldots \to d_r$ is a path of length r in $\mathcal{A}$, then there exist at most $r+1$ different paths $c_0 = c \to c_1 \to \ldots \to c_r$ in $\tilde{\mathcal{A}}$ with $A\big(\varphi(c_j)\big) = A(d_j)$ for $j \in \{1, 2, \ldots, r\}$.*

3. The structure of the nonwandering set of $(R(Q), T)$

In this section we describe a well known result on the structure of the nonwandering set of a piecewise monotonic map.

The definitions of the notions nonwandering set, ω-limit set, topological transitivity and minimality can be found in standard books on dynamical systems (e.g. in [12], see also Section 2 of [9]). We mention here only that we call a set R *topologically transitive*, if there exists an $x \in R$ with $\omega(x) = R$. Furthermore a topologically transitive subset R of a dynamical system X is called *maximal topologically transitive*, if no R' with $R \subsetneqq R' \subseteq X$ is topologically transitive.

The following result describes the structure of the nonwandering set of a piecewise monotonic map, and it is proved in [1], [2], [3] and [5] (see also Section 2 in [9]).

Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0,1]$, let $K \in \mathbb{N}$, $Q \in \mathcal{Q}_K$, and suppose that $\mathcal{Y}$ is a finite partition of $[0,1]$ which refines $\mathcal{Z}$. Then we have

$$\Omega_{\mathcal{Y}}(R(Q), T) = \bigcup_{\mathcal{C} \in \Gamma} L(\mathcal{C}) \cup \bigcup_{j \in J} N_j \cup P \cup W \tag{3.1}$$

where Γ is the at most countable set of maximal irreducible subsets of the Markov diagram $(\mathcal{D}, \to)$ of $(R(Q), T)$ with respect to $\mathcal{Y}$, J is an at most finite index set, and the intersection of two different sets in the decomposition is at most finite. Furthermore we have:

(1) For every $\mathcal{C} \in \Gamma$ the set $L(\mathcal{C})$ is a topologically transitive subset of the dynamical system $(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}})$, and the periodic points of $(L(\mathcal{C}), T_{\mathcal{Y}})$ are dense in $L(\mathcal{C})$. Moreover, either $L(\mathcal{C})$ consists only of one single periodic orbit (in this case for every $C \in \mathcal{C}$ there exists exactly one $D \in \mathcal{C}$ with $C \to D$), or $L(\mathcal{C})$ is an uncountable, maximal topologically transitive subset of $(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}})$ with $h_{\text{top}}(L(\mathcal{C}), T_{\mathcal{Y}}) > 0$ (in this case there exists at least one $C \in \mathcal{C}$ which has more than one successor in $\mathcal{C}$). In the second case we have that every $x \in L(\mathcal{C})$ can be represented by an infinite path in $\mathcal{C}$, and every infinite path in $\mathcal{C}$ represents an $x \in L(\mathcal{C})$.

(2) For every $j \in J$ the set N_j is an uncountable, minimal subset of $(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}})$ which contains no periodic points. Furthermore we have that $h_{\text{top}}(N_j, T_{\mathcal{Y}}) = 0$, there exist only finitely many ergodic, $T_{\mathcal{Y}}$-invariant Borel probability measures on $(N_j, T_{\mathcal{Y}})$, and N_j is a maximal topologically transitive subset of the dynamical system $(R_{\mathcal{Y}}(Q), T_{\mathcal{Y}})$.

(3) The set P is closed and $T_{\mathcal{Y}}$-invariant, and consists of periodic points, which are contained in nontrivial intervals K with the property that ${T_{\mathcal{Y}}}^n$ maps K monotonically into K for an $n \in \mathbb{N}$.

(4) The set W consists of nonperiodic points which are isolated in $\Omega_{\mathcal{Y}}(R(Q),T)$, and therefore are not contained in $\Omega(\Omega_{\mathcal{Y}}(R(Q),T),T_{\mathcal{Y}})$.

Observe that this result implies that the decomposition into maximal topologically transitive subsets, which are not a single periodic orbit, does not depend on the partition $\mathcal{Y}$. More exactly, if $\mathcal{Y}$ and $\mathcal{Y}'$ are two finite partitions refining $\mathcal{Z}$, then there exists a bijective map φ from the set of uncountable maximal topologically transitive subsets of $\Omega_{\mathcal{Y}}(R(Q),T)$ (note that every at most countable maximal topologically transitive subset is a single periodic orbit) to the set of uncountable maximal topologically transitive subsets of $\Omega_{\mathcal{Y}'}(R(Q),T)$ such that $\pi_{\mathcal{Y}}(R)=\pi_{\mathcal{Y}'}\big(\varphi(R)\big)$. Therefore we will speak throughout this paper of uncountable maximal topologically transitive subsets of $(R(Q),T)$, rather than those of $(R_{\mathcal{Y}}(Q),T_{\mathcal{Y}})$.

The most interesting part of the dynamics takes place on the at most countable union of maximal topologically transitive subsets with positive entropy. In the next sections we shall investigate the influence of small perturbations of Q on these sets. Set

$$\begin{aligned}\mathcal{M}(R(Q),T) := \{L\colon L &\text{ is a maximal topologically transitive} \\ &\text{subset of } (R(Q),T) \text{ with } h_{\mathrm{top}}(L,T)>0\},\end{aligned} \tag{3.2}$$

and define

$$N(R(Q),T) := \operatorname{card} \mathcal{M}(R(Q),T). \tag{3.3}$$

Hence $N(R(Q),T)\in\mathbb{N}\cup\{0,\infty\}$. By (2.8), (2.9) and (2.10) of [9] we get

$$h_{\mathrm{top}}(R(Q),T) = \sup_{L\in\mathcal{M}(R(Q),T)} h_{\mathrm{top}}(L,T) \tag{3.4}$$

whenever $h_{\mathrm{top}}(R(Q),T)>0$, and

$$p(R(Q),T,f) = \sup_{L\in\mathcal{M}(R(Q),T)} p(L,T,f) \tag{3.5}$$

whenever $f\colon[0,1]\to\mathbb{R}$ is a piecewise continuous function with $p(R(Q),T,f) > \lim_{n\to\infty}\frac{1}{n}S_n(R(Q),f)$, and if T is additionally expanding and $h_{\mathrm{top}}(R(Q),T)>0$ (which is equivalent to $\mathrm{HD}\big(R(Q)\big)>0$) then

$$\mathrm{HD}\big(R(Q)\big) = \sup_{L\in\mathcal{M}(R(Q),T)} \mathrm{HD}(L). \tag{3.6}$$

By the Structure Theorem it suffices to find the maximal irreducible subsets $\mathcal{C}$ of the Markov diagram, where there is at least one $C\in\mathcal{C}$ which has more than one successor in $\mathcal{C}$, if one wants to find the maximal topologically transitive subsets of $(R(Q),T)$ with positive entropy.

4. An example of merging maximal topologically transitive subsets

In this section we give an example of an expanding piecewise monotonic map $T\colon [0,1] \to [0,1]$, and a $Q \in \mathcal{Q}_2$ with $N(R(Q),T) = 2$, such that for every $\varepsilon > 0$ there exists a $\tilde{Q} \in \mathcal{Q}_2$ which is ε-close to Q, with $N(R(\tilde{Q}),T) = 1$. This example is the example given in (3.1) of [9] adapted to our situation.

Set $\mathcal{Z} := \{(0,\frac{1}{3}),(\frac{1}{3},\frac{2}{3}),(\frac{2}{3},1)\}$. We define a map $T\colon [0,1] \to [0,1]$ by

$$Tx := \begin{cases} 1-3x & \text{for } x \in [0,\frac{1}{3}], \\ 3x-1 & \text{for } x \in [\frac{1}{3},\frac{2}{3}], \\ 3-3x & \text{for } x \in [\frac{2}{3},1]. \end{cases} \tag{4.1}$$

For $s \in [0,\frac{1}{6})$ define $Q_s := (0,\frac{1}{6}-s,\frac{5}{6}+s,1)$. Observe that $Q_s \in \mathcal{Q}_2$, and that given $\varepsilon > 0$, then Q_s is ε-close to Q_0 if $s < \varepsilon$.

Define $M := [\frac{1}{3},\frac{2}{3}]$, $A_0 := [\frac{1}{6}-s,\frac{1}{3}]$, $B_0 := [\frac{2}{3},\frac{5}{6}+s]$, and for $n \in \mathbb{N}$ define $A_n := [\frac{1}{3},\frac{1}{2}+3^n s]$ and $B_n := [\frac{1}{2}-3^n s,\frac{2}{3}]$.

In the case $s = 0$ the Markov diagram $(\mathcal{D},\to)$ of $(R(Q_0),T)$ is

$$\mathcal{D} = \{M, A_0, A_1, B_0, B_1\}$$

with the arrows $A_j \to A_k$ and $B_j \to B_k$ for $j,k \in \{0,1\}$, $M \to A_0$, $M \to B_0$ and $M \to M$. Hence the maximal irreducible subsets of $(\mathcal{D},\to)$ are $\mathcal{C}_1 := \{A_0,A_1\}$, $\mathcal{C}_2 := \{B_0,B_1\}$ and $\{M\}$. Therefore the maximal topologically transitive subsets of $(R(Q_0),T)$ with positive entropy are $L_1 := L(\mathcal{C}_1)$ and $L_2 := L(\mathcal{C}_2)$, hence $\mathcal{M}(R(Q_0),T) = \{L_1,L_2\}$ and $N(R(Q_0),T) = 2$.

Now let $N \in \mathbb{N}$, and set $s := \frac{1}{2}\frac{1}{3^{N+2}}$. Then the Markov diagram $(\mathcal{D},\to)$ of $(R(Q_s),T)$ is $\mathcal{D} = \{M, A_0, A_1, \dots, A_N, B_0, B_1, \dots, B_N\}$ with the arrows $A_j \to A_0$ and $B_j \to B_0$ for $j \in \{0,1,\dots,N\}$, $A_j \to A_{j+1}$ and $B_j \to B_{j+1}$ for $j \in \{0,1,\dots, N-1\}$, $A_N \to M$ and $B_N \to M$, $M \to A_0$, $M \to B_0$ and $M \to M$. Hence $(\mathcal{D},\to)$ is irreducible, and therefore the only maximal topologically transitive subset of $(R(Q_s),T)$ is $L(\mathcal{D}) = R(Q_s)$. This gives $\mathcal{M}(R(Q_s),T) = \{L(\mathcal{D})\}$ and $N(R(Q_s),T) = 1$.

5. Stability results for maximal topologically transitive subsets with positive entropy

The example of the previous section shows that the decomposition of the nonwandering set into maximal topologically transitive subsets with positive entropy behaves very unstably. The number $N(R(Q),T)$ does not depend continuously on Q. Hence we cannot expect general stability results for $\mathcal{M}(R(Q),T)$. However, there are stability results for the elements of $\mathcal{M}(R(Q),T)$. In this section it will be shown that a maximal topologically transitive subset with positive entropy cannot be destroyed completely by an arbitrary small perturbation of Q (Theorem 1). Furthermore, if $\tilde{Q}$ is sufficiently close to Q, then to each element $\tilde{L}$ of $\mathcal{M}(R(\tilde{Q}),T)$ which is sufficiently "big", there can be assigned "close" (in general not maximal) topologically transitive subsets of $(R(Q),T)$ and $(R(\tilde{Q}),T)$, which are "bigger" than $\tilde{L}$ (Theorem 2, Corollary 2.1 and Theorem 3).

Theorem 1. *Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0,1]$, let $K \in \mathbb{N}$, and let $Q \in \mathcal{Q}_K$. Furthermore let $k \in \mathbb{N}$, and for $j \in \{1,2,\dots,k\}$ let $f_j\colon [0,1] \to \mathbb{R}$ be a piecewise continuous function with respect to $\mathcal{Z}$. Let L be a maximal topologically transitive subset of $(R(Q),T)$ with $h_{\mathrm{top}}(L,T) > 0$, and suppose that $p(L,T,f_j) > \lim_{n\to\infty} \frac{1}{n} S_n(L,f_j)$ for $j \in \{1,2,\dots,k\}$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $\tilde{Q} \in \mathcal{Q}_K$ which is δ-close to Q, there exists a topologically transitive subset $\tilde{L}$ of $(R(\tilde{Q}),T)$ which satisfies*

(5.1) $\tilde{L}$ *and* L *are ε-close in the Hausdorff metric,*

$$|h_{\mathrm{top}}(\tilde{L},T) - h_{\mathrm{top}}(L,T)| < \varepsilon, \tag{5.2}$$

and

$$|p(\tilde{L},T,f_j) - p(L,T,f_j)| < \varepsilon \quad \text{for } j \in \{1,2,\dots,k\}. \tag{5.3}$$

If in addition T is expanding, then $\delta > 0$ can be chosen such that besides (5.1), (5.2) *and* (5.3), *we also have*

$$|\,\mathrm{HD}(\tilde{L}) - \mathrm{HD}(L)| < \varepsilon. \tag{5.4}$$

Proof. Let $Q := (a_1, a_2, \dots, a_{2K-1}, a_{2K})$ and $\tilde{Q} := (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_{2K-1}, \tilde{a}_{2K})$. Set $J_0 := \{j \in \{1,2,\dots,K-1\} \colon a_{2j} = a_{2j+1}\}$. Now define

$$\tilde{a}'_{2j-1} := \min\{a_{2j-1}, \tilde{a}_{2j-1}\}$$

for $j \in \{1,2,\dots,K\}$ with $j-1 \notin J_0$, and $\tilde{a}'_{2j} := \max\{a_{2j}, \tilde{a}_{2j}\}$ for $j \in \{1,2,\dots,K\}$ with $j \notin J_0$. Set $L := K - \operatorname{card} J_0$. For $j \in \{1,2,\dots,L\}$ define $b_{2j-1} := a_{2r-1}$, $\tilde{b}'_{2j-1} := \tilde{a}'_{2r-1}$, $b_{2j} := a_{2s}$, and $\tilde{b}'_{2j} := \tilde{a}'_{2s}$, where r is the unique number such that $r-1 \notin J_0$ and $\operatorname{card}(\{0,1,\dots,r-1\} \setminus J_0) = j$, and s is the unique number such that $s \notin J_0$ and $\operatorname{card}(\{1,2,\dots,s\} \setminus J_0) = j$. Set $Q' := (b_1, b_2, \dots, b_{2L-1}, b_{2L})$ and

$\tilde{Q}' := (\tilde{b}'_1, \tilde{b}'_2, \ldots, \tilde{b}'_{2L-1}, \tilde{b}'_{2L})$. Then $Q' \in \mathcal{Q}_L$, and there exists a $\delta_1 > 0$ such that $\tilde{Q}$ is δ-close to Q for a $\delta \leqslant \delta_1$ implies that $\tilde{Q}' \in \mathcal{Q}_L$ and $\tilde{Q}'$ is δ-close to Q'. By (1.2) we get $R(\tilde{Q}') \subseteq R(\tilde{Q})$.

Using (1.2) we see that $R(Q) = R(Q') \cup R_0$, where R_0 is an at most countable set by the definition of Q'. Therefore the Structure Theorem described in Section 3 implies that L is a maximal topologically transitive subset of $(R(Q'), T)$. Furthermore, using (1.1) we get by the definition of $\tilde{Q}'$ and Q' that $X(\tilde{Q}') \subseteq X(Q')$, and hence $\big(T|X(\tilde{Q}'), \mathcal{Z}(\tilde{Q}')\big)$ is δ-close to $\big(T|X(Q'), \mathcal{Z}(Q')\big)$ in the R^0-topology defined in [7] and [9]. Now the first part of the theorem follows from Theorem 2 in [9].

It remains to show in the case of an expanding piecewise monotonic map T that δ can be chosen small enough, so that also (5.4) holds. To this end observe that $\big(T|X(\tilde{Q}'), \mathcal{Z}(\tilde{Q}')\big)$ is δ-close to $\big(T|X(Q'), \mathcal{Z}(Q')\big)$ in the R^1-topology defined in [7] and [9], since $X(\tilde{Q}') \subseteq X(Q')$. Now the desired result follows from Theorem 3 of [9]. □

Another proof of this result can be given, analogous to the proofs of Theorem 2 and Theorem 3 in [9] but using Lemma 1 of this paper instead of Lemma 6 of [7]. This proof would be a bit simpler than the proofs of Theorem 2 and Theorem 3 in [9].

Theorem 2. *Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0,1]$, let $K \in \mathbb{N}$, and let $Q \in \mathcal{Q}_K$. Furthermore let $f\colon [0,1] \to \mathbb{R}$ be a piecewise continuous function with respect to $\mathcal{Z}$. Then for every $\varepsilon > 0$ and for every $\alpha > 0$ there exists a $\delta > 0$ such that for every $\tilde{Q} \in \mathcal{Q}_K$ which is δ-close to Q, the following holds. If $\tilde{L}$ is a maximal topologically transitive subset of $(R(\tilde{Q}), T)$ with $p(\tilde{L}, T, f) > \lim\limits_{n\to\infty} \frac{1}{n} S_n(R(Q), f) + \alpha$, then there exists a topologically transitive subset $\tilde{L}'$ of $(R(\tilde{Q}), T)$ and a topologically transitive subset L of $(R(Q), T)$ with the following properties:*

$$\begin{aligned} &\text{for every } x \in \tilde{L}' \text{ there is a } y \in \tilde{L} \text{ with } |x - y| < \varepsilon \\ &\text{for every } x \in L \text{ there is a } y \in \tilde{L} \text{ with } |x - y| < \varepsilon \\ &\text{the sets } \tilde{L}' \text{ and } L \text{ are } \varepsilon\text{-close in the Hausdorff metric,} \end{aligned} \tag{5.5}$$

$$\begin{aligned} p(\tilde{L}, T, f) &< p(\tilde{L}', T, f) + \varepsilon, \\ p(\tilde{L}, T, f) &< p(L, T, f) + \varepsilon, \text{ and} \\ |p(\tilde{L}', T, f) - p(L, T, f)| &< \varepsilon. \end{aligned} \tag{5.6}$$

P r o o f. Let $\varepsilon > 0$ and $\alpha > 0$. Let $Q = (a_1, a_2, \ldots, a_{2K-1}, a_{2K})$. We can assume that ε is small enough to ensure $\varepsilon \leqslant \alpha$.

By the piecewise continuity of f there exists a finite partition $\mathcal{Y}$ of $[0,1]$ refining $\mathcal{Z}$ with $a_j \in \{\inf Y, \sup Y \colon Y \in \mathcal{Y}\}$ for all $j \in \{1,2,\ldots,2K-1,2K\}$, such that

$$\begin{aligned} \sup_{Y\in\mathcal{Y}} \operatorname{diam} Y < \tfrac{1}{4}\varepsilon \ \text{ and} \\ \sup_{Y\in\mathcal{Y}} \sup_{x,y\in Y} |f(x)-f(y)| < \tfrac{1}{4}\varepsilon. \end{aligned} \tag{5.7}$$

If $x \in Y$ for a $Y \in \mathcal{Y}$, then define

$$f_1(x) := \sup_{y\in Y} f(y).$$

Then $f_1 \colon [0,1] \to \mathbb{R}$ is a piecewise constant function with respect to $\mathcal{Y}$. By (5.7) we have

$$\begin{aligned} f(x) \leqslant f_1(x) < f(x) + \tfrac{1}{4}\varepsilon \qquad \text{and} \\ p(R,T,f) \leqslant p(R,T,f_1) < p(R,T,f) + \tfrac{1}{4}\varepsilon \end{aligned} \tag{5.8}$$

for every closed, T-invariant $R \subseteq [0,1]$.

Let $(\mathcal{D},\to)$ be the Markov diagram of $(R(Q),T)$ with respect to $\mathcal{Y}$. Let $\mathcal{K} \subseteq \mathcal{Y}$ be nonempty. Set $\mathcal{K}_0 := \mathcal{K}(Q)$, and for $r \in \mathbb{N}$ set

$$\mathcal{K}_r := \mathcal{K}_{r-1} \cup \{\, D \in \mathcal{D} \colon \exists\, C \in \mathcal{K}_{r-1},\, \exists\, Y \in \mathcal{K} \text{ with } C \to D \text{ and } D \subseteq Y\}.$$

Finally define $\mathcal{D}_{\mathcal{K}} := \bigcup_{r=0}^{\infty} \mathcal{K}_r$. Then Lemma 4 in [7] gives

$$r\big(F_{\mathcal{D}_{\mathcal{K}}}(f_1)\big) = \lim_{r\to\infty} \|F_{\mathcal{K}_r}(f_1)^r\|^{\frac{1}{r}} = \inf_{r\in\mathbb{N}} \|F_{\mathcal{K}_r}(f_1)^r\|^{\frac{1}{r}}. \tag{5.9}$$

Analogously to the proofs of Theorem 7 and Corollary 1 of Theorem 9 in [2], and to the proof of Lemma 6 in [6] (cf. also the proof of Lemma 2 in [9]) we get using Lemma 1 of [8] that there exists an $r_{\mathcal{K}} \in \mathbb{N}$ such that for every maximal irreducible $\mathcal{C} \subseteq \mathcal{D}_{\mathcal{K}}$ with

$$\log r\big(F_{\mathcal{C}}(f_1)\big) > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q), f_1) + \tfrac{1}{4}\varepsilon,$$

and for every version $(\mathcal{A},\to)$ of the Markov diagram of $(R(Q),T)$ with respect to $\mathcal{Y}$, there exists a finite irreducible $\mathcal{C}' \subseteq \mathcal{A}_{r_{\mathcal{K}}}$ with $\{A(c) \colon c \in \mathcal{C}'\} \subseteq \mathcal{C} \cap \mathcal{K}_{r_{\mathcal{K}}}$ and

$$\log r\big(F_{\mathcal{C}}(f_1)\big) - \tfrac{1}{4}\varepsilon \leqslant \log r\big(F_{\mathcal{C}'}(f_1)\big) \leqslant \log r\big(F_{\mathcal{C}}(f_1)\big). \tag{5.10}$$

By (5.9) we can choose this $r_{\mathcal{K}}$ such that we have also

$$\log\big((r+1)\|F_{\mathcal{K}_r}(f_1)^r\|\big)^{\frac{1}{r}} \leqslant \log r\big(F_{\mathcal{D}_{\mathcal{K}}}(f_1)\big) + \tfrac{1}{4}\varepsilon \tag{5.11}$$

for every $r \geqslant r_{\mathcal{K}}$.

There exists an $r_0 \in \mathbb{N}$, such that $\frac{1}{r}S_r(R(Q), f_1) < \lim_{n\to\infty} \frac{1}{n}S_n(R(Q), f_1) + \frac{1}{4}\varepsilon$ for every $r \geqslant r_0$. Now choose an $r \in \mathbb{N}$ with $r \geqslant r_0$ and $r \geqslant r_{\mathcal{K}}$ for every nonempty $\mathcal{K} \subseteq \mathcal{Y}$ (such an r exists, since there exist only finitely many nonempty $\mathcal{K} \subseteq \mathcal{Y}$). Then (5.8) and the definition of $S_n(R(Q), f)$ ((1.5) of [8]) give

$$\frac{1}{r}S_r(R(Q), f_1) < \lim_{n\to\infty} \frac{1}{n}S_n(R(Q), f) + \frac{1}{2}\varepsilon. \tag{5.12}$$

Fix this r for the rest of this proof. By Lemma 1 there exists a $\delta \in (0, \frac{1}{4}\varepsilon)$ such that the conclusions of Lemma 1 hold with ε replaced by $\frac{1}{4}\varepsilon$.

Let $\tilde{Q} \in \mathcal{Q}_K$ be δ-close to Q, and let $\tilde{L}$ be a maximal topologically transitive subset of $(R(\tilde{Q}), T)$ with $p(\tilde{L}, T, f) > \lim_{n\to\infty} \frac{1}{n}S_n(R(Q), f) + \alpha$. Let $(\mathcal{A}, \to)$ be the version of the Markov diagram of $(R(Q), T)$ with respect to $\mathcal{Y}$, let $(\tilde{\mathcal{A}}, \to)$ be the version of the Markov diagram $(\tilde{\mathcal{D}}, \to)$ of $(R(\tilde{Q}), T)$ with respect to $\mathcal{Y}$ occuring in the conclusions of Lemma 1, and let $\varphi\colon \tilde{\mathcal{A}}_r \to \mathcal{A}_r$ be the function described in the conclusions of Lemma 1.

First we show that $h_{\mathrm{top}}(\tilde{L}, T) > 0$ and $p(\tilde{L}, T, f_1) > \lim_{n\to\infty} \frac{1}{n}S_n(R(\tilde{Q}), f_1)$. By (2) of Lemma 1 we get that $c, d \in \tilde{\mathcal{A}}_r$ and $c \to d$ in $\tilde{\mathcal{A}}$ imply $A\big(\varphi(c)\big) \to A\big(\varphi(d)\big)$ in $\mathcal{D}$. Hence the definition of $S_n(R(Q), f)$ gives $S_r(R(\tilde{Q}), f_1) \leqslant S_r(R(Q), f_1)$. By (5.8) and (5.12) this implies

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}S_n(R(\tilde{Q}), f) &\leqslant \lim_{n\to\infty} \frac{1}{n}S_n(R(\tilde{Q}), f_1) \leqslant \frac{1}{r}S_r(R(\tilde{Q}), f_1)\\ &\leqslant \frac{1}{r}S_r(R(Q), f_1) < \lim_{n\to\infty} \frac{1}{n}S_n(R(Q), f) + \frac{1}{2}\varepsilon.\end{aligned}$$

Since $p(\tilde{L}, T, f) > \lim_{n\to\infty} \frac{1}{n}S_n(R(Q), f) + \alpha$ and $\varepsilon \leqslant \alpha$, using (5.8) this gives $p(\tilde{L}, T, f) > \lim_{n\to\infty} \frac{1}{n}S_n(R(\tilde{Q}), f)$ and $p(\tilde{L}, T, f_1) > \lim_{n\to\infty} \frac{1}{n}S_n(R(\tilde{Q}), f_1)$, which implies $h_{\mathrm{top}}(\tilde{L}, T) > 0$.

Hence the Structure Theorem described in Section 3 gives that there exists a maximal irreducible $\mathcal{C} \subseteq \tilde{\mathcal{A}}$ with $\tilde{L} = L(\{\tilde{A}(c)\colon c \in \mathcal{C}\})$ and $\tilde{A}(c) \in \tilde{\mathcal{D}}$ for every $c \in \mathcal{C}$. Set $\mathcal{K} := \{Y \in \mathcal{Y}\colon \exists\, c \in \mathcal{C} \text{ with } \tilde{A}(c) \subseteq Y\}$. As above define $\tilde{\mathcal{K}}_0 := \mathcal{K}(\tilde{Q})$, $\tilde{\mathcal{K}}_n := \tilde{\mathcal{K}}_{n-1} \cup \{\, D \in \tilde{\mathcal{D}}\colon \exists\, C \in \tilde{\mathcal{K}}_{n-1},\ \exists\, Y \in \mathcal{K} \text{ with } C \to D \text{ and } D \subseteq Y\}$ for $n \in \mathbb{N}$, and $\tilde{\mathcal{D}}_{\mathcal{K}} := \bigcup_{n=0}^{\infty} \tilde{\mathcal{K}}_n$. Then using Lemma 6 in [6] (cf. also the proof of Theorem 7 in [2]), Lemma 4 in [7] and the proof of Lemma 3 in [7] we get that

$$\begin{aligned}p(\tilde{L}, T, f_1) &= \log r\big(F_{\mathcal{C}}(f_1)\big)\\ &\leqslant \log r\big(F_{\tilde{\mathcal{D}}_{\mathcal{K}}}(f_1)\big) \leqslant \log \|F_{\tilde{\mathcal{K}}_r}(f_1)^r\|^{\frac{1}{r}}.\end{aligned} \tag{5.13}$$

Using Lemma 4 of [7] we get by (3) and (4) of Lemma 1 that

$$\|F_{\tilde{\mathcal{K}}_r}(f_1)^r\| \leqslant (r+1)\|F_{\mathcal{K}_r}(f_1)^r\|.$$

Hence (5.8), (5.11) and (5.13) imply

$$p(\tilde{L},T,f) \leqslant p(\tilde{L},T,f_1) \leqslant \log r\big(F_{\mathcal{D}_\mathcal{K}}(f_1)\big) + \tfrac{1}{4}\varepsilon. \tag{5.14}$$

Using (5.8) and $\varepsilon \leqslant \alpha$ we arrive at $\log r\big(F_{\mathcal{D}_\mathcal{K}}(f_1)\big) > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q),f_1) + \frac{1}{4}\varepsilon$. Using the proof of Lemma 6 in [6] (cf. also the proof of Theorem 7 in [2]) we get by (3.5) and (5.14) that there exists a maximal irreducible $\mathcal{E} \subseteq \mathcal{D}_\mathcal{K}$ with $\log r\big(F_\mathcal{E}(f_1)\big) > \lim_{n\to\infty} \frac{1}{n} S_n(R(Q),f_1) + \frac{1}{4}\varepsilon$ and $\log r\big(F_\mathcal{E}(f_1)\big) > p(\tilde{L},T,f) - \frac{\varepsilon}{2}$. Now (5.10) implies that there exists an irreducible $\mathcal{E}' \subseteq \mathcal{A}_r$ with $\{A(c)\colon c \in \mathcal{E}'\} \subseteq \mathcal{E} \cap \mathcal{K}_r$ and

$$\log r\big(F_{\mathcal{E}'}(f_1)\big) > p(\tilde{L},T,f) - \tfrac{3}{4}\varepsilon. \tag{5.15}$$

Set $\tilde{\mathcal{E}}' := \varphi^{-1}(\mathcal{E}')$. By (1) and (2) of Lemma 1, $\varphi\colon \tilde{\mathcal{E}}' \to \mathcal{E}'$ is bijective and $c \to d$ in $\tilde{\mathcal{E}}'$ is equivalent to $\varphi(c) \to \varphi(d)$ in $\mathcal{E}'$.

We get that $\{A(c)\colon c \in \mathcal{E}'\}$ is contained in a maximal irreducible $\mathcal{E}_1 \subseteq \mathcal{D}$, and $\{\tilde{A}(c)\colon c \in \tilde{\mathcal{E}}'\}$ is contained in a maximal irreducible $\mathcal{E}_2 \subseteq \tilde{\mathcal{D}}$. Now define

$$L := \{x \in L(\mathcal{E}_1)\colon x \text{ is represented by an infinite path in } \mathcal{E}'\}$$

and

$$\tilde{L}' := \{x \in L(\mathcal{E}_2)\colon x \text{ is represented by an infinite path in } \tilde{\mathcal{E}}'\}.$$

The proof of Theorem 4 in [2] shows that L and $\tilde{L}'$ are topologically transitive.

It follows from (3) of Lemma 1 and from (5.7) that L and $\tilde{L}'$ are $\frac{1}{4}\varepsilon$-close (and therefore ε-close) in the Hausdorff metric. Let $x \in L$. Then there exists a $c \in \mathcal{E}'$ with $x \in A(c)$ and $A(c) \in \mathcal{K}_r$. Hence there is a $Y \in \mathcal{K}$ with $A(c) \subseteq Y$. By the definition of $\mathcal{K}$ there is a $d \in \mathcal{C}$ with $\tilde{A}(d) \subseteq Y$. Therefore there exists a $y \in \tilde{L}$ with $y \in Y$, and by (5.7) we get $|x-y| < \frac{1}{4}\varepsilon < \varepsilon$. If $x \in \tilde{L}'$, then there exist a $y_1 \in L$ and a $y \in \tilde{L}$ with $|x-y_1| < \frac{1}{4}\varepsilon$ and $|y_1 - y| < \frac{1}{4}\varepsilon$, which gives $|x-y| < \frac{\varepsilon}{2} < \varepsilon$. This shows (5.5).

By Lemma 6 in [6] (cf. the proof of Theorem 7 in [2]) we get

$$p(L,T,f_1) = \log r\big(F_{\mathcal{E}'}(f_1)\big) \quad \text{and} \quad p(\tilde{L}',T,f_1) = \log r\big(F_{\tilde{\mathcal{E}}'}(f_1)\big).$$

Using (1.9) and (1.10) of [8] we get that $r\big(F_{\tilde{\mathcal{E}}'}(f_1)\big) = r\big(F_{\mathcal{E}'}(f_1)\big)$. Hence (5.8) gives $|p(\tilde{L}',T,f) - p(L,T,f)| < \frac{1}{4}\varepsilon < \varepsilon$. Using (5.8) and (5.15) we get $p(L,T,f) > p(\tilde{L},T,f) - \varepsilon$ and $p(\tilde{L}',T,f) > p(\tilde{L},T,f) - \varepsilon$, which completes the proof. □

If we set $f = 0$ in Theorem 2 we get the following result concerning the topological entropy of $\tilde{L}$, L and $\tilde{L}'$.

Corollary 2.1. *Let $T\colon [0,1] \to [0,1]$ be a piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0,1]$, let $K \in \mathbb{N}$, and let $Q \in \mathcal{Q}_K$. Then for every $\varepsilon > 0$ and for every $\alpha > 0$ there exists a $\delta > 0$ such that for every $\tilde{Q} \in \mathcal{Q}_K$ which is δ-close to Q, the following holds. If $\tilde{L}$ is a maximal topologically transitive subset of $(R(\tilde{Q}), T)$ with $h_{\mathrm{top}}(\tilde{L}, T) > \alpha$, then there exists a topologically transitive subset $\tilde{L}'$ of $(R(\tilde{Q}), T)$ and a topologically transitive subset L of $(R(Q), T)$ such that (5.5) is true and the following holds:*

$$\begin{aligned} h_{\mathrm{top}}(\tilde{L}, T) &< h_{\mathrm{top}}(\tilde{L}', T) + \varepsilon, \\ h_{\mathrm{top}}(\tilde{L}, T) &< h_{\mathrm{top}}(L, T) + \varepsilon, \text{ and} \\ |h_{\mathrm{top}}(\tilde{L}', T) - h_{\mathrm{top}}(L, T)| &< \varepsilon. \end{aligned} \tag{5.16}$$

Using (3.4) and (3.5) we can easily deduce from Theorem 1 and Theorem 2 that the pressure and the topological entropy depend continuously on Q (see Theorem 1 and Corollary 1.1 of [8]).

Finally, we prove that for an expanding T a result analogous to Theorem 2 concerning the Hausdorff dimension of $\tilde{L}$, L and $\tilde{L}'$ is true.

Theorem 3. *Let $T\colon [0,1] \to [0,1]$ be an expanding piecewise monotonic map with respect to the finite partition $\mathcal{Z}$ of $[0,1]$, let $K \in \mathbb{N}$, and $Q \in \mathcal{Q}_K$. Then for every $\varepsilon > 0$ and for every $\alpha > 0$ there exists a $\delta > 0$ such that for every $\tilde{Q} \in \mathcal{Q}_K$ which is δ-close to Q, the following holds. If $\tilde{L}$ is a maximal topologically transitive subset of $(R(\tilde{Q}), T)$ with $\mathrm{HD}(\tilde{L}) > \alpha$, then there exists a topologically transitive subset $\tilde{L}'$ of $(R(\tilde{Q}), T)$ and a topologically transitive subset L of $(R(Q), T)$ such that (5.5) and the following hold:*

$$\begin{aligned} \mathrm{HD}(\tilde{L}) &< \mathrm{HD}(\tilde{L}') + \varepsilon, \\ \mathrm{HD}(\tilde{L}) &< \mathrm{HD}(L) + \varepsilon, \text{ and} \\ |\,\mathrm{HD}(\tilde{L}') - \mathrm{HD}(L)| &< \varepsilon. \end{aligned} \tag{5.17}$$

Proof. By the proof of Lemma 3 in [6] we can choose an $\eta > 0$ such that $2\eta \leqslant \varepsilon < \alpha$, $1 - (1 + \frac{\eta}{B})^{-1} < \frac{\varepsilon}{2}$, $t_x := (x - \varepsilon)(1 + \frac{\eta}{B}) < x$ for all $x \in [\alpha, 1]$, and, whenever $x \in [\alpha, 1]$ and $R \subseteq [0,1]$ is closed and T-invariant with $p(R, T, -x \log |T'|) = 0$, then $p(R, T, -t_x \log |T'|) > \eta$, where B is as in Lemma 3 of [9]. Again using the proof of

Lemma 3 in [6] we can assume that for every closed and T-invariant $R \subseteq [0,1]$ and for every closed and T-invariant $R' \subseteq [0,1]$ the property

$$\left|p(R,T,-t\log|T'|) - p(R',T,-t\log|T'|)\right| < \eta \text{ for all } t \in [0,1]$$

implies $|t_R - t_{R'}| < \frac{\varepsilon}{2}$, where t_R is the unique zero of $t \mapsto p(R,T,-t\log|T'|)$, and $t_{R'}$ is the unique zero of $t \mapsto p(R',T,-t\log|T'|)$.

The proof of Theorem 2 shows that there exists a finite partition $\mathcal{Y}$ of $[0,1]$ refining $\mathcal{Z}$, such that $\sup_{Y\in\mathcal{Y}} \sup_{x,y\in Y} |\log|T'|(x) - \log|T'|(y)| < \eta$, and there exists a $\delta > 0$ such that for every $\tilde{Q} \in \mathcal{Q}_K$ which is δ-close to Q, the conclusions of Theorem 2 hold with ε and α replaced by η. Let $\tilde{Q} \in \mathcal{Q}_K$ be δ-close to Q and let $\tilde{L}$ be a maximal topologically transitive subset of $(R(\tilde{Q}),T)$ with $\mathrm{HD}(\tilde{L}) > \alpha$. Then $t_1 := (\mathrm{HD}(\tilde{L}) - \varepsilon)(1 + \frac{\eta}{B}) < \mathrm{HD}(\tilde{L})$. As in the proof of Theorem 3 in [7] we get $\lim_{n\to\infty} \frac{1}{n} S_n(R(Q), -t\log|T'|) < 0$ for all $t > 0$, hence by Theorem 2, Lemma 3 and Lemma 9 of [6] we get

$$\lim_{n\to\infty} \frac{1}{n} S_n(R(Q), -t_1\log|T'|) < 0 < \eta < p(\tilde{L},T,-t_1\log|T'|).$$

Now choose sets $\tilde{L}'$ and L as in the proof of Theorem 2 with f replaced by $-t_1\log|T'|$. Then we get that (5.5) is true and

$$|p(\tilde{L}',T,-t\log|T'|) - p(L,T,-t\log|T'|)| < \eta$$

for all $t \in [0,1]$. Hence Lemma 3 in [9] gives $|\mathrm{HD}(\tilde{L}') - \mathrm{HD}(L)| < \varepsilon$. By Theorem 2 we get $p(\tilde{L}',T,-t_1\log|T'|) > 0$ and $p(L,T,-t_1\log|T'|) > 0$. Therefore Lemma 3 in [9] implies $\mathrm{HD}(\tilde{L}') > \mathrm{HD}(\tilde{L}) - \varepsilon$ and $\mathrm{HD}(L) > \mathrm{HD}(\tilde{L}) - \varepsilon$, which completes the proof. □

From (3.6), Theorem 1 and Theorem 3 we can easily deduce that the Hausdorff dimension depends continuously on Q (see Theorem 2 in [8]).

References

[1] *F. Hofbauer*: The structure of piecewise monotonic transformations. Ergodic Theory Dynam. Systems *1* (1981), 159–178.

[2] *F. Hofbauer*: Piecewise invertible dynamical systems. Probab. Theory Related Fields *72* (1986), 359–386.

[3] *F. Hofbauer, P. Raith*: Topologically transitive subsets of piecewise monotonic maps, which contain no periodic points. Monatsh. Math. *107* (1989), 217–239.

[4] *M. Misiurewicz, S. V. Shlyachkov*: Entropy of piecewise continuous interval maps. European conference on iteration theory (ECIT 89), Batschuns, 1989 (Ch. Mira, N. Netzer, C. Simó, Gy. Targoński, eds.). World Scientific, Singapore, 1991, pp. 239–245.

[5] *Z. Nitecki*: Topological dynamics on the interval. Ergodic theory and dynamical systems, Vol. 2, Proceedings of the Special Year at the University of Maryland, 1979/1980 (A. Katok, ed.). Progress in Mathematics 21, Birkhäuser, Boston, 1982, pp. 1–73.
[6] *P. Raith*: Hausdorff dimension for piecewise monotonic maps. Studia Math. *94* (1989), 17–33.
[7] *P. Raith*: Continuity of the Hausdorff dimension for piecewise monotonic maps. Israel J. Math. *80* (1992), 97–133.
[8] *P. Raith*: Continuity of the Hausdorff dimension for invariant subsets of interval maps. Acta Math. Univ. Comenian. *63* (1994), 39–53.
[9] *P. Raith*: The dynamics of piecewise monotonic maps under small perturbations. Preprint, Warwick, 1994.
[10] *M. Urbański*: Hausdorff dimension of invariant sets for expanding maps of a circle. Ergodic Theory Dynam. Systems *6* (1986), 295–309.
[11] *M. Urbański*: Invariant subsets of expanding mappings of the circle. Ergodic Theory Dynam. Systems *7* (1987), 627–645.
[12] *P. Walters*: An introduction to ergodic theory. Graduate Texts in Mathematics 79, Springer, New York, 1982.

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