AN APPLICATION OF EIGENFUNCTIONS OF p-LAPLACIANS TO DOMAIN SEPARATION

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Dedicated to Professor J. Nečas on the occasion of his 70th birthday

Abstract. We are interested in algorithms for constructing surfaces Γ of possibly small measure that separate a given domain Ω into two regions of equal measure. Using the integral formula for the total gradient variation, we show that such separators can be constructed approximatively by means of sign changing eigenfunctions of the p-Laplacians, $p \to 1$, under homogeneous Neumann boundary conditions. These eigenfunctions turn out to be limits of steepest descent methods applied to suitable norm quotients.

Keywords: perimeter, relative isoperimetric inequality, p-Laplacian, eigenfunctions, steepest decent method

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be an open, bounded, connected Lipschitzian domain. We denote by C_0^1, L^p, $H^{1,p}$ and $(H^{1,p})' = H^{-1,p'}$, $1 \leqslant p \leqslant 2$, $p' = \frac{p}{p-1}$, the usual spaces of functions defined on Ω (cf. [13]); $(\cdot,\cdot)$ means the pairing between spaces and their duals, $\|\cdot\|_p$ is the norm in L^p. Further, BV denotes the space of functions with bounded variation on Ω [11] and

$$\int_\Omega |Du| = \sup_g \Big(\int_\Omega u \nabla \cdot g \,\mathrm{d}x \Big),\ g \in C_0^1(\Omega, \mathbb{R}^n),\ |g(x)| \leqslant 1,\ x \in \Omega.$$

(Note that $\int_\Omega |Du| = \|\nabla u\|_1$, provided $u \in H^{1,1}$.) Let finally

$$V_p = \begin{cases} \Big\{ u \in H^{1,p},\ \displaystyle\int_\Omega |u|^{p-2} u \,\mathrm{d}x = 0 \Big\}, & \text{if } p > 1, \\ \Big\{ u \in BV,\ \displaystyle\int_\Omega \operatorname{sign} u \,\mathrm{d}x = 0 \Big\}, & \text{if } p = 1. \end{cases}$$

Based on a joint work with K. Gärtner [8]

There is a practical interest [12] in algorithms for constructing surfaces Γ of possibly small measure $|\Gamma|$ which separate Ω into two regions of equal measure, i.e., in solving the minimum problem

$$\varphi_1(E) = \frac{P_\Omega(E)}{|E|} \to \min,\ E \subset \Omega,\ |E| = \frac{|\Omega|}{2}, \tag{1}$$

where $P_\Omega(E) = |\Gamma|$ is the perimeter of E relative to Ω and $|E|$ is the measure of E.

In this contribution we show how to solve the geometrical problem (1) by analytical tools. Roughly speaking, we look for approximative solutions of the form $E = \{x \in \Omega,\ u(x) > 0\}$, where u minimizes

$$F_1(u) = \frac{\int_\Omega |Du|}{\|u\|_1} \to \min,\ u \in V_1. \tag{2}$$

The key idea for this approach is Federer's observation (cf. [5]) that the infimum of the functional

$$\varphi(E) = \frac{P_\Omega(E)}{\min(|E|^{\frac{1}{p^*}}, |\Omega \setminus E|^{\frac{1}{p^*}})} \to \min,\ E \subset \Omega,\ p^* = \frac{n}{n-1}, \tag{3}$$

coincides with that of

$$\varphi(u) = \frac{\int_\Omega |Du|}{\|u - t_0(u)\|_{p^*}} \to \min,\ u \in BV, \tag{4}$$

where the functional t_0 is defined by

$$t_0(u) = \sup\{t,\ |E_t| \geqslant |\Omega \setminus E_t|\},\ E_t = \{x \in \Omega, u(x) > t\}. \tag{5}$$

To specify the connection between (3) and (4) we quote some basic facts from [5], [6]:

(i) Let u be locally integrable on Ω. Then

$$\int_\Omega |Du| = \int_{-\infty}^{\infty} P_\Omega(E_t)\,\mathrm{d}t. \tag{6}$$

(ii) Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected Lipschitzian domain. Then Ω satisfies a relative isoperimetric inequality, i.e., there exists a constant $Q = Q(\Omega)$ such that

$$\min(|E|^{\frac{1}{p^*}}, |\Omega - E|^{\frac{1}{p^*}}) \leqslant Q P_\Omega(E). \tag{7}$$

(iii) Let Ω, Q be as in (ii) and let u be as in (i). Then

$$\|u - t_0(u)\|_{p^*} \leqslant Q \int_\Omega |Du|. \tag{8}$$

A special case of (i) is

$$\int_\Omega |D\chi_E| = P_\Omega(E), \tag{9}$$

where χ is the characteristic function. Hence the map $E \to \chi_E - \chi_{\Omega\setminus E}$ directly connects (1) and (2). The inverse direction may be indicated by the map $u \to E_u$ with

$$E_u = \{x \in \Omega,\ u(x) > 0\}.$$

The functional F_1 still is unpleasant from the algorithmical point of view. Therefore we shall approximate F_1 by (apart from zero) differentiable functionals

$$F_p(u) = \frac{\|\nabla u\|_p^p}{\|u\|_p^p},\ 0 \neq u \in V_p,\ p \in (1,2]. \tag{10}$$

The next section clarifies the relation between φ, φ_1 and F_1. In Section 3 we establish convergence of minimizers of F_p, $p \to 1$, to minimizers of F_1. Section 4 is devoted to a convergence result concerning a steepest descent method for F_p. Here each iteration $u_{p,i}$ has to be calculated as a (unique) solution of a nonlinear elliptic boundary value problem under homogeneous Neumann conditions. It is shown that $F_p(u_{p,i})$ for $i \to \infty$ tends monotonically decreasing to $F_p(u_p)$, where u_p is a sign changing eigenfunction of the p-Laplacian.

Proofs of these results can be found in [8].

2. Relations between φ_1 and F_1

In this section we want to justify the transition from (1) to (2). We start with an adaptation of inequality (8), which will be more convenient for our purposes.

Lemma 1. *Let Q be the relative isoperimetric constant from* (7). *Then*

$$\begin{aligned} &\|u\|_1 \leqslant \Big(\frac{|\Omega|}{2}\Big)^{\frac{1}{n}} Q \int_\Omega |Du|,\ u \in V_1, \\ &\|u\|_p \leqslant 2^{\frac{p-1}{p}} \Big(\frac{|\Omega|}{2}\Big)^{\frac{1}{n}} Q \|\nabla u\|_p,\ u \in H^{1,p} \cap V_1,\ p \in \Big[1, p^* = \frac{n}{n-1}\Big]. \end{aligned} \tag{11}$$

R e m a r k 1 . The inequality (11) specifies the constant in Poincaré's inequality. For $p=1$, (11) is sharp. Indeed, suppose equality is attained in (7) for a set E with $|E|=\frac{|\Omega|}{2}$, as for example in the case of convex domains $\Omega\subset\mathbb{R}^2$ (cf. [2]). Then $u=\chi_E-\chi_{\Omega\setminus E}\in V_1$ and

$$\|u\|_1=|\Omega|=2\Big(\frac{|\Omega|}{2}\Big)^{\frac{1}{n}}\Big(\frac{|\Omega|}{2}\Big)^{\frac{1}{p^*}}=2\Big(\frac{|\Omega|}{2}\Big)^{\frac{1}{n}}QP_\Omega(E)=2^0\Big(\frac{|\Omega|}{2}\Big)^{\frac{1}{n}}Q\int_\Omega|Du|.$$

For convex domains Ω another specification is well known [10]:

$$\|u\|_p\leqslant\Big(\frac{|\Omega|}{\omega_n}\Big)^{\frac{1-n}{n}}d^n\|\nabla u\|_p,\;\; u\in H^{1,p},\;\;\int_\Omega u\,\mathrm{d}x=0,$$

where ω_n is the volume of the unit sphere in $\mathbb{R}^n$ and d is the diameter of Ω.

The minimum problems (3) and (2) are equivalent in the following sense:

Proposition 1. *A set $E_1\subset\Omega$ with $|E_1|=\frac{|\Omega|}{2}$ is a minimizer of φ if and only if $u_1=\chi_{E_1}-\chi_{\Omega\setminus E_1}\in V_1$ is a minimizer of F_1.*

R e m a r k 2 . Evidently, each minimizer E of φ with $|E|=\frac{|\Omega|}{2}$ is a solution of the minimum problem (1). For convex domains $\Omega\subset\mathbb{R}^2$ the existence of such minimizers is proved in [2].

On the basis of the next result we will replace (1) by (2).

Theorem 1. (i) *Let $u_1\in V_1$ be a minimizer of F_1 and let $E_1=\{x\in\Omega,$ $u_1(x)>0\}$. Then*

$$\varphi_1(E_1)\leqslant\varphi_1(E)\quad\text{for all } E\subset\Omega \text{ with } |E|=\frac{|\Omega|}{2}. \tag{12}$$

(ii) *Let in addition $|\{x\in\Omega,u_1(x)=0\}|=0$. Then E_1 is a solution of* (1).

3. The functionals F_p and the limit $p \to 1$

In this section we will justify the transition from the minimum problem (2) to the regularized minimum problems

$$F_p(u) = \frac{\|\nabla u\|_p^p}{\|u\|_p^p} \to \min,\ 0 \neq u \in V_p,\ p \in (1,2]. \tag{13}$$

R e m a r k 3 . Since F_p is homogeneous, (13) is equivalent to

$$G_p(u) = \|\nabla u\|_p \to \min,\ u \in V_p,\ \|u\|_p = 1,\ p \in (1,2].$$

Proposition 2. *Let*

$$d = \inf_{u \in V_p} F_p(u).$$

Then there exists a (minimizer) $u \in V_p$ such that $F_p(u) = d$.

Minimizers of $u \in H^{1,p}$ satisfy necessarily the Euler Lagrange equations, i.e., the nonlinear eigenvalue problem (cf. [4])

$$A_p u = F_p(u) B_p u, \tag{14}$$

where the operators $A_p, B_p \in (H^{1,p}) \to (H^{-1,p'})$ are defined by

$$\begin{aligned} &(A_p u, h) = (|\nabla u|^{p-2} \nabla u, \nabla h),\ \forall h \in H^{1,p}, \\ &(B_p u, h) = (b_p(u), h),\ b_p(u) = |u|^{p-2} u. \end{aligned} \tag{15}$$

R e m a r k . (14), (15) can be seen as a weak formulation (comp. [9]) of the nonlinear eigenvalue problem

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = F_p(u) |u|^{p-2} u \text{ in } \Omega,\ \nu \cdot \nabla u = 0 \text{ on } \partial\Omega,$$

where ν is the outer unit normal on $\partial\Omega$.

The minimum problem (13) approximates (2) in the following sense:

Theorem 2. *Let $u_p \in H^{1,p}$, $1 < p \leqslant 2$, be a minimizer for F_p in (13), such that*

$$\|u_p\|_p = 1. \tag{16}$$

Then

(i) *there exists a sequence* $p_i \to 1$ *and a function* $u \in BV$ *such that*

$$u_i := u_{p_i} \to u \text{ in } L^q,\ q \in (1, p^*),\ F_{p_i}(u_i) \to \lambda \geqslant F_1(u);$$

(ii) u *is a minimizer of* F_1*;*

(iii)

$$B_i u_i \rightharpoonup z \text{ in } L^q,\ z \in Su,\ \int_\Omega z \,\mathrm{d}x = 0,$$

where S *is the maximal monotone operator generated by the (multivalued) function*

$$\operatorname{Sign} s = \begin{cases} \operatorname{sign} s & \text{if } s \neq 0, \\ [-1, 1] & \text{if } s = 0. \end{cases}$$

4. Steepest descent method

Due to Theorems 1, 2 the original minimum problem (1) is approximatively reduced to the construction of minimizers u_p of the functional F_p for suitable p near 1. In this section we fix $p \in (1, 2]$ and establish a steepest descent method for solving iteratively the corresponding Euler Lagrange equations, i.e., the nonlinear eigenvalue problems (14):

$$B_p u_i + \tau A_p u_i = B_p u_{i-1} + \tau F_p(u_{i-1}) B_p u_i,\ i = 1, 2, \ldots, u_0 \in V_p,\ u_0 \neq 0, \tag{17}$$

where τ is a relaxation parameter, which may be interpreted as a time step.

Theorem 3. *Let* $\tau p F_p(u_0) < 1$.
Then

(i) *for each* i (17) *has a unique solution* $u_i \in V_p$*;*

(ii) *the sequence* $(F_p(u_i))$ *is decreasing,* $F_p(u_i) \to \lambda > 0$*;*

(iii) *the sequence* $(\|u_i\|_p)$ *is bounded, moreover,*

$$\|u_0\|_p^p \leqslant \|u_i\|_p^p \leqslant c \colon = \frac{1}{1 - \tau p F(u_0)} \|u_0\|_p^p,\ \|B_p u_i - B_p u_{i-1}\|_1 \to 0;$$

(iv) *there exist a subsequence* $(u_{i_j}) \subset (u_i)$ *and a function* $u \in V_p$ *such that* u *is a nontrivial solution of the nonlinear eigenvalue problem* (14) *and*

$$u_{i_j} \to u \quad \text{in} \quad H^{1,p},\ F_p(u) = \lambda,\ \int_\Omega B_p u \,\mathrm{d}x = 0.$$

Corollary 1. *The nonlinear eigenvalue problem* (14) *has a solution* $u_p \in H^{1,p}$ *for* $p \in (1,2]$ *such that*

$$\|u_p\|_p = 1, \ \int_\Omega |u_p|^{p-2} u_p \, \mathrm{d}x = 0.$$

u_p *is in* $H^{1,p}$ *the strong limit of the iteration sequence* $(u_{p,i})$ *defined by* (17). *Moreover,*

$$F_p(u_{p,i}) \downarrow_{i\to\infty} F_p(u_p).$$

References

[1] *H. W. Alt, S. Luckhaus*: Quasilinear elliptic-parabolic differential equations. Math. Z. *183* (1983), 311–341.
[2] *A. Cianchi*: On relative isoperimetric inequalities in the plane. Bollettino U.M.I. *7* (1989), 3–13.
[3] *F. Di Benedetto*: Degenerate Parabolic Equations. Springer, Basel, 1993.
[4] *P. Drábek, A. Kufner, F. Nicolosi*: Quasilinear Elliptic Equations with Degenerations and Singularities. Walter de Gruyter, Berlin, 1997.
[5] *H. Federer, W. H. Flemming*: Normal and integral currents. Ann. Math. *72* (1960), 458–520.
[6] *W. H. Flemming, R. Rishel*: An integral formula for total gradient variation. Arch. Math. *11* (1960), 218–222.
[7] *H. Gajewski, K. Gärtner*: On the discretization of van Roosbroeck's equations with magnetic field. Z. Angew. Math. Mech. 76 (1996), 247–264.
[8] *H. Gajewski, K. Gärtner*: Domain separation by means of sign changing eigenfunctions of p-Laplacians. Preprint No. 526, Weierstraß Institute, Berlin, 1999.
[9] *H. Gajewski, K. Gröger, K. Zacharias*: Nichtlineare Operatorgleichungen ond Operatordifferentialgleichungen. Akademie, Berlin, 1974.
[10] *D. Gilbarg, N. S. Trudinger*: Elliptic Partial Differential Equations of Second Order. Springer, 1983.
[11] *E. Giusti*: Minimal Surfaces and Functions of Bounded Variation. Birkhäuser, Basel, 1984.
[12] *O. Schenk, W. Fichtner, K. Gärtner*: ETH-Zürich. Technical Report No. 97/17.
[13] *E. Zeidler*: Nonlinear functional Analysis and Its Applications II/B. Springer, 1983.

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