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SOLVABILITY PROBLEM FOR STRONG-NONLINEAR NONDIAGONAL PARABOLIC SYSTEM

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Abstract. A class of q-nonlinear parabolic systems with a nondiagonal principal matrix and strong nonlinearities in the gradient is considered.We discuss the global in time solvability results of the classical initial boundary value problems in the case of two spatial variables. The systems with nonlinearities $q \in (1,2)$, $q = 2$, $q > 2$, are analyzed.

Keywords: boundary value problems, nonlinear parabolic systems, solvability

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Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with a sufficiently smooth boundary. For a fixed $T > 0$ and $Q = \Omega \times (0,T)$, we consider a solution $u\colon Q \to \mathbb{R}^N$, $u = (u^1, \ldots, u^N)$, $N > 1$, of the parabolic system

$$(1) \qquad u^k_t - \frac{\mathrm{d}}{\mathrm{d}x_\alpha} a^k_\alpha(z,u,u_x) + b^k(z,u,u_x) = 0, \quad z = (x,t) \in Q, \quad k = 1, \ldots, N.$$

We define the set $D = \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN}$ and assume that

a) the functions $a = \{a^k_\alpha\}^{k \leqslant N}_{\alpha \leqslant n}$ and $b = \{b^k\}^{k \leqslant N}$ are sufficiently smooth on D;

b) for a fixed $q > 1$, $a(\cdot,\cdot,p) \sim |p|^{q-1}$, $b(\cdot,\cdot,p) \sim |p|^q$, $|p| \gg 1$, all derivatives of a and b that we need have the natural growth with respect to the gradient;

c) the nondiagonal principal matrix $\bigl\{\frac{\partial a^k_\alpha}{\partial p^l_\beta}\bigr\}^{\alpha,\beta \leqslant n}_{k,l \leqslant N}$ satisfies the following assumptions on D:

$$(2) \quad \frac{\partial a^k_\alpha(z,u,p)}{\partial p^l_\beta} \xi^k_\alpha \xi^l_\beta \geqslant \nu(1+|p|)^{q-2}|\xi|^2, \qquad \left|\frac{\partial a(\ldots)}{\partial p}\right| \leqslant \mu(1+|p|)^{q-2}, \quad \forall \xi \in \mathbb{R}^{nN};$$

d) the strongly nonlinear term b satisfies the condition

$$(3) \qquad |b(z,u,p)| \leqslant b_0(1+|p|)^q, \quad (z,u,p) \in D.$$

Here $\nu, \mu, b_0 = \mathrm{const} > 0$.

We investigate the solvability of the Cauchy-Dirichlet problem

$$u|_{\partial' Q} = \varphi, \tag{4}$$

where $\partial' Q$ is the parabolic boundary of Q and φ is a given smooth function.

First of all, we recall some known results.

We fix the class

$$V = L_q\big((0,T), W_q^1(\Omega)\big) \cap L_\infty(Q)$$

and note that the global solvability of (1), (4) in V was stated for the scalar situation ($N = 1$) in the following sense. Assume that for a fixed $M > 0$ an apriori estimate

$$\|u\|_{\infty,Q} \leqslant M \tag{5}$$

can be derived. Then there exists a solution $u \in V \cap \mathbb{C}^\alpha(\overline{Q})$ with some $\alpha \in (0,1)$. Further regularity of the solution follows provided all the data are smooth enough [1].

In some sense this result is also valid for a class of quasilinear *diagonal* systems ($N > 1$, $q = 2$). More precisely, if estimate (5) and the "smallness" condition $b_0 M < \nu$ hold then a solution u of (1), (4) exists in $V \cap \mathbb{C}^\alpha(\overline{Q})$.

It should be remarked that due to the maximum principle we are able to formulate sufficient conditions which provide estimate (5) in the cases mentioned above.

Now, let us consider a parabolic (elliptic) system with a *nondiagonal* principal matrix. In this situation, the following questions arise: i) How to guarantee estimate (5)? ii) Is the class V suitable for proving global solvability of (1), (4)?

Under the conditions a)–d), the global solvability problem for (1), (4) has not been solved yet.

Certainly, we cannot expect classical global solvability of this problem. As is known, there are counterexamples of the regularity for quasilinear nondiagonal systems even if $b \equiv 0$ ($q = 2$, $n > 2$) [2]. On the other hand, for systems (1) whose main part is the heat operator, but the term $b(z,u,p)$ is non-zero and satisfies (3), singularities can appear in Q at some time. The heat flow of harmonic maps provides an example of such a situation (see, for example, [3], [4]).

From the above, it follows that there are two reasons that cause nonsmoothness of solutions of the problem under consideration.

During the recent years, the author has investigated the global solvability for (1), (4) under assumptions a)–d) in the following particular case.

We define the functional

$$E[u] = \int\limits_\Omega f(x,u,u_x)\,\mathrm{d}x, \qquad u = (u^1,\dots,u^N), \quad N > 1, \tag{6}$$

and denote by $L = \{L^k\}^{k\leqslant N}$ the Euler operator of E:

$$L^k u = -\frac{\mathrm{d}}{\mathrm{d}x_\alpha} f_{p^k_\alpha} + f_{u^k}.$$

Then system (1) is the gradient flow for the functional E. Consider the problem

$$\begin{aligned} &u^k_t - \frac{\mathrm{d}}{\mathrm{d}x_\alpha} f_{p^k_\alpha}(x,u,u_x) + f_{u^k}(x,u,u_x) = 0, \quad (x,t)\in Q, \quad k\leqslant N,\\ &u\big|_\Gamma = 0, \qquad u\big|_{t=0} = \varphi_0(x), \end{aligned} \tag{7}$$

where $\Gamma = \partial\Omega\times(0,T)$.

The variational structure of system (7) provides an apriori estimate of the solution u:

$$\|u_t\|^2_{2,Q} + \sup_{(0,T)} \|u_x(\cdot,t)\|^q_{q,\Omega} \leqslant e_0, \tag{8}$$

where $e_0 = \mathrm{const}$ depends on the data only.

Moreover, this structure also ensures monotonicity of the global energy

$$E[u(\cdot,t_1)] \leqslant E[u(\cdot,t_2)], \qquad \forall\, t_1 > t_2,$$

and a local energy estimate

$$\|u_t\|^2_{2,\mathcal{P}_R(z^0)} + \sup_{\lambda_R(t^0)} \|u_x(\cdot,t)\|^q_{q,\Omega_R(x^0)} \leqslant \frac{c}{R^q} \int\limits_{\mathcal{P}_{2R}(z^0)} (1+|u_x|)^q\,\mathrm{d}P. \tag{9}$$

In (9) and below, we denote

$$\begin{aligned} &\mathcal{P}_R(z^0) = Q_R(z^0)\cap Q, \quad Q_R(z^0) = B_R(x^0)\times\lambda_R(t^0),\\ &B_R(x^0) = \big\{x\in\mathbb{R}^n;\ |x-x^0| < R\big\}, \quad \lambda_R(t^0) = (t^0 - R^q, t^0 + R^q),\\ &\Omega_R(x^0) = B_R(x^0)\cap\Omega. \end{aligned}$$

We say that $Q_R(z^0)$ is a q-parabolic cylinder and denote by

$$\delta_q(z^1,z^2) = \sup\big\{|x^1-x^2|, |t^1-t^2|^{1/q}\big\}, \qquad \forall\, z^1,z^2\in\mathbb{R}^{n+1}, \tag{10}$$

the q-parabolic distance in $\mathbb{R}^{n+1}$.

To introduce an example of system (7), we put

$$f(x,u,p) = \langle A(x,u)p,\ p\rangle\,(1+|p|)^{q-2}, \qquad q>1, \tag{11}$$

in the definition (6) of E. We assume that $A(\cdot,\cdot)$ is a nondiagonal positive definite and smooth matrix on $\overline{\Omega} \times \mathbb{R}^N$, and, in addition, $A^{\alpha\beta}_{kl} = A^{\beta\alpha}_{lk}$. Generated by the function (11) the system (7) satisfies conditions a)–d), in particular, $f_u(\cdot,\cdot,p) \sim |p|^q$, $|p| \gg 1$.

Let us now proceed to discussing some solvability results recently proved by the author.

We stated some solvability results for problem (7) in the case of two spatial variables.

First, we considered problem (7) with $n = q = 2$. We analyzed it with quasilinear and nonlinear operators under the Dirichlet or Neumann type conditions ([5]–[8]). For all these situations the following result was proved.

Theorem 1. *For a fixed number $T > 0$, there exists a global solution of (7), which is almost everywhere smooth in $\overline{Q}$. The singular set consists of at most finitely many points. The solution u has a finite norm (8), and it is a weak solution in the sense of distributions.*

This result was proved with help of the continuability theorem of smooth solutions from a semiclosed time interval. We essentially exploited the imbedding theorems for two dimensional domains and the fact that the "local normalized energy" $\frac{1}{R^{n-q}} \int_{B_R(x^0)} |u_x(x,t)|^q \,\mathrm{d}x$ is a monotonic function of R if $n = q = 2$.

For the case $n = 2$ and $q > 2$, we have the following result.

Theorem 2. *Let $q > 2$, $n = 2$, and let T be a positive fixed number. There exists a smooth solution of problem (7) in $\overline{Q}$ if all the data are sufficiently smooth.*

Now, we give a sketch of the proof of this result.We start by deriving some apriori estimates for solutions u of (7) smooth on the time interval $[0,T)$.

First of all, from (8) one can deduce an apriori estimate

$$\|u\|_{C^\gamma(\overline{Q},\delta_q)} \leqslant \mathrm{const} \tag{12}$$

where γ is a number in $(0,1)$.

Estimate (12) allows us to derive apriori estimates for stronger norms of u in $\overline{Q}$.

From this point, we study problem (7) in a local setting. Let $v(y,t) = u(x(y),t)$ be a solution of the problem

$$\begin{aligned} &v_t - \frac{\mathrm{d}}{\mathrm{d}y_\alpha} A^k_\alpha(y,v,v_y) + \mathbb{B}^k(y,v,v_y) = 0, \quad (y,t) \in Q_2^+, \quad k,\dots,N,\\ &v\big|_{t=0} = \varphi_0(x(y)), \quad v\big|_{\Gamma_2^+} = 0, \end{aligned} \tag{13}$$

where $Q_R^+ = B_R^+(0) \times (0,T)$, $B_R^+(0) = B_R(0) \cap \{y_2 > 0\}$, $\Gamma_R^+ = \gamma_R(0) \times (0,T)$ and $\gamma_R(0) = B_R(0) \cap \{y_2 = 0\}$. On the set $D = Q_2^+ \times \mathbb{R}^N \times \mathbb{R}^{2N}$, functions A_α^k and $\mathbb{B}^k$ satisfy conditions (2) and (3) with some other constants.

With help of (12), one can derive the inequality

$$\begin{aligned}\sup_{\lambda_R(t^0)} \int\limits_{\Omega_R(y^0)} |v_y(y,t)|^2 \,\mathrm{d}y + \int\limits_{\mathcal{P}_R(z_0)} \big[(1+|v_y|)^{q-2}|v_{yy}|^2 \\ + (1+|v_y|)^{q+2}\big] \,\mathrm{d}\mathcal{P} \leqslant cR^{2\alpha}, \quad \forall\, R \leqslant R_0, \quad z^0 = (y^0,t^0) \in Q_{3/2}^+,\end{aligned} \tag{14}$$

with $\alpha \in (0,1)$ and $R_0 > 0$.

Next, we state that for some $s \in (0,1)$ we have

$$\sup_{(0,T)} \|v_t(\cdot,t)\|_{2+2s, B_1^+(0)} \leqslant c_1. \tag{15}$$

Here and below, c_i, $i = 1, \dots, 4$, are positive constants depending on the data only. To derive (15), we use (14) for estimating the strongly nonlinear terms generated by functions $\mathbb{B}^k$. After that, we are able to look at our problem as at an elliptic one for a fixed $t \in (0,T)$.

The reverse Hölder inequalities hold for

$$V(\cdot,t) = (1+|v_y(\cdot,t)|)^{\frac{q-2}{2}} |v_{yy}(\cdot,t)|$$

in $B_{\frac12}^+(0)$.

Due to the Gehring Lemma, we have the estimate of $\|V(\cdot,t)\|_{p,B_{\frac12}^+(0)}$ with some $p > 2$. As a consequence, we arrive at the estimates

$$\sup_{(0,T)} \|v_{yy}(\cdot,t)\|_{p,B_{\frac12}^+(0)} \leqslant c_2, \quad \sup_{(0,T)} \|v_y(\cdot,t)\|_{\mathbb{C}^\beta\,(\overline{B_{\frac12}^+(0)})} \leqslant c_3, \ \beta = 1 - 2/p > 0. \tag{16}$$

Estimates (12), (16) guarantee that

$$\|u_x\|_{\mathbb{C}^{\beta_0}\,(\overline{Q};\delta_q)} \leqslant c_4$$

with some $\beta_0 > 0$.

Apriori estimates of stronger norms of u up to $t = T$ follow from the linear theory. It means that u can be extended as a smooth function up to $t = T$.

Due to the known solvability results, there exists a smooth solution u of (7) on some time interval $[0,T_0)$. Let T_0 define the maximal interval of existence of a smooth solution u. Suppose that $T_0 < T$. As was explained above, u can be extended as a smooth function up to $t = T_0$. Thus, one obtains a contradiction with the definition of T_0. From all that was said above, it follows that $T_0 \geqslant T$. Theorem 2 is proved.

A more complicated case is $n = 2$, $q \in (1,2)$. To prove the solvability of (7), we introduce approximate problems

$$
\begin{gathered}
u_t + Lu - \varepsilon \Delta u = 0 \quad \text{in } Q, \\
u\big|_{\Gamma} = 0, \qquad u\big|_{t=0} = \varphi^{\varepsilon}, \qquad \varepsilon \in (0,1].
\end{gathered} \tag{17}
$$

Here φ^{ε} is an approximation of φ, φ^{ε} satisfies the compatibility conditions for system (17), φ^{ε} tends to φ in the strong sense if $\varepsilon \to 0$.

For a fixed $\varepsilon > 0$, we prove global classical solvability of (17) in the space $\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})$ with Hölder exponent $\alpha \in (0,1)$ (the definition of this space see in [1], Ch. I, §1). Certainly, the norm of the solutions u^{ε} of (17) in this space tends to infinity if $\varepsilon \to 0$. We are able to estimate different norms of u^{ε} due to the fact that for a fixed $\varepsilon > 0$ the Laplace operator forms the main part of the elliptic operator of system (17), and functions $f_u(x,u,u_x)$ are not the strongly nonliner terms with respect to the Laplace operator. Also, it is worth noting that all estimates of u^{ε} are derived in the standard cylinders ($q = 2$ in the definition of Q_R).

The sequence u^{ε} tends in some sense to the limit function u if $\varepsilon \to 0$, and u is a solution of (7). More exactly, the following fact was proved.

Theorem 3. *Let $q \in (1,2)$ and let T be a fixed positive number. There exists a solution u of problem (7), which is almost everywhere smooth in $\overline{Q}$; $u \in L_{\infty}((0,T); \overset{\circ}{W}{}^1_q(\Omega))$, and $u_t \in L_2(Q)$. The closed singular set Σ of u has $\dim_{q-\mathcal{H}} \Sigma \leqslant 2$ ($\dim_{\mathcal{H}} \Sigma \leqslant 4 - q$). Moreover, $\dim_H \Sigma^{\tau} \leqslant 2 - q$, $\forall\, \tau > 0$, where $\Sigma^{\tau} = \Sigma \cap \{t = \tau\}$.*

In the statement of Theorem 3, the estimate $\dim_{q-\mathcal{H}} \Sigma \leqslant 2$ means that for all $\eta > 0$ we have $\mathcal{H}_{2-q+\eta}(\Sigma; \delta_q) = 0$, where the δ_q-parabolic metric is defined in (10).

Now, we explain the main steps of the proof of Theorem 3.

Lemma 1. *There exists a number ω_0 depending on the data only, such that if*

$$
\omega^{\varepsilon_j}_{R_0}(z^0) \equiv \operatorname*{osc}_{\mathcal{P}_{R_0}(z^0)} u^{\varepsilon_j} \leqslant \omega_0 \tag{18}
$$

for a point $z^0 \in \overline{Q}$ with some $R_0 > 0$ and a sequence $\varepsilon_j \to 0$, then

$$
\|u^{\varepsilon_j}\|_{\mathbb{C}^{\gamma_1}(\overline{\mathcal{P}_{R_*}(z^0)};\delta_q)} + \|u^{\varepsilon_j}_x\|_{C^{\gamma_2}(\overline{\mathcal{P}_{R_*}(z^0)};\delta_q)} \leqslant c_0 \tag{19}
$$

with some $\gamma_1, \gamma_2 \in (0,1)$ and $R_* = R_*(R_0, \omega_0) < R_0$, and u^{ε_j} is a solution of (17).

It should be remarked that, in general (in the case of nondiagonal matrix and condition (3)), the smallness of the oscillation of a solution does not guarantee an estimate of the Hölder norm of the solution.

It is evident that condition (18) provides smoothness of the solution u at the point z_0.

The next step is to introduce an integral description of a regular point of u.

Lemma 2. *Suppose that for a point $z^0 \in \overline{Q}$ there exist numbers $\mathcal{K} > 0$, $\beta > 1$, $R_0 > 0$ and a sequence $\varepsilon_j \to 0$ such that*

$$(20) \qquad \sup_{\hat{z} \in \mathcal{P}_{R_0}(z^0)} \sup_{R \leqslant R_0} \frac{\left(\log_2 \dfrac{2R_0}{R}\right)^{\frac{\beta q^2}{2(q-1)}}}{R^2} \int\limits_{\mathcal{P}_R(\hat{z})} H^2_{\varepsilon_j}\, \mathrm{d}z \leqslant \mathcal{K},$$

where $H^2_\varepsilon = (1 + |u^\varepsilon_x|)^q + \varepsilon |u^\varepsilon_x|^2$. Then estimate (18) *holds in $\mathcal{P}_{R_1}(z^0)$ with some $R_1 < R_0$.*

To derive (18) from (20), we exploit a local energy estimate for solutions u^ε of (17). We also use a certain condition for functions from the Sobolev space $W^1_q(\Omega)$, $n = 2$, that makes it possible to estimate their oscillation. To prove (18), we analyze both cases $\varepsilon_j < \frac{R^{2-q}}{\chi(R)}$ and $\varepsilon_j \geqslant \frac{R^{2-q}}{\chi(R)}$, where $\chi(R) = \frac{\mathcal{K}^{\frac{2-q}{q}}}{(\log_2 \frac{2R_0}{R})^\gamma}$, $\gamma = \frac{q(2-q)\beta}{2(q-1)}$, and $R \leqslant R_0$. Next, we denote by $\mathcal{R}$ the set of all points z_0 in $\overline{Q}$, where (20) holds with some parameters $\mathcal{K}$, β, R_0, and $\{\varepsilon_j\}_{j \in N}$, $\varepsilon_j \to 0$, and put $\Sigma = \overline{Q} \setminus \mathcal{R}$.

We have the following description of Σ:

A point z^0 belongs to Σ if for all $M_k \to \infty$ and $R_k \to 0$, there exist sequences of points $\xi_k \in \mathcal{P}_{R_k}(z^0)$ and numbers $\varrho_k \leqslant R_k$ such that

$$(21) \qquad \varliminf_{\varepsilon \to 0} \frac{\left(\log_2 \dfrac{2R_k}{\varrho_k}\right)^\gamma}{\varrho_k^2} \int\limits_{\mathcal{P}_{\varrho_k}(\xi_k)} H^2_\varepsilon\, \mathrm{d}z > M_k,$$

where $\gamma = \frac{\beta q^2}{2(q-1)}$.

The relation (21) does not allow us to estimate the Hausdorff measure of Σ and, therefore, instead of (21), we prove that one can exploit the following description of $z^0 \in \Sigma$:

for arbitrary sequences $M_k \to \infty$ and $R_k \to 0$, there exist a sequence of numbers $r_k \leqslant 2R_k$ and an absolute number $c_* > 1$ such that

$$\varliminf_{\varepsilon\to 0} \frac{\left(\log_2 \frac{4R_k}{r_k}\right)^\gamma}{r_k^2} \int\limits_{\mathcal{P}_{r_k}(z^0)} H_\varepsilon^2 \,\mathrm{d}z > \frac{M_k}{c_*}. \tag{22}$$

From (22), we deduce an estimate of the Hausdorff dimension of the set Σ as was pointed in Theorem 3.

It should be noted that the presence of the logarithmic multiplier in (22) does not allow us to assert that

$$\begin{aligned} \mathcal{H}_2(\Sigma;\delta_q) < +\infty \quad (\mathcal{H}_{4-q}(\Sigma;\delta) <\ +\infty,\ \delta=\delta_2) \\ \text{and}\ \ H_{2-q}(\Sigma^\tau) < +\infty,\ \forall\,\tau>0. \end{aligned} \tag{23}$$

If (23) were proved, then we could pass to the limit in the integral identity corresponding to problem (17) and state that the limit function u is a weak solution of (22) in the sense of distributions.

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