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# QUALITATIVE THEORY OF HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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*Dedicated to the memory of Prof. Árpád Elbert*

*Abstract.* Some recent results concerning properties of solutions of the half-linear second order differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad p > 1, \tag{$*$}$$

are presented. A particular attention is paid to the oscillation theory of $(*)$. Related problems are also discussed.

*Keywords*: half-linear equation, Picone's identity, scalar $p$-Laplacian, variational method, Riccati technique, principal solution

*MSC 2000*: 34C10

## 1. Introduction

In this contribution we deal with the oscillatory properties and related problems concerning the half-linear second order differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad p > 1, \tag{1}$$

where $r, c$ are continuous functions and $r(t) > 0$. During the recent years it was shown that solutions of (1) behave in many aspects like those of the Sturm-Liouville equation

$$(r(t)x')' + c(t)x = 0 \tag{2}$$

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which is the special case $p = 2$ of (1). The aim of this paper is to present some results of this investigation and also to point out situations where the properties of (1) and (2) (considerably) differ. Note that the term half-linear equations is motivated by the fact that the solution space of (1) has just one half of the properties which characterize linearity, namely homogeneity (but not additivity).

The investigation of qualitative properties of nonlinear second order differential equations has a long history. Recall here only the papers of Emden [26], Fowler [27], Thomas [40], and the book of Sansone [39] containing the survey of the results achieved in the first half of the last century. In the fifties and later decades the number of papers devoted to nonlinear second order differential equations increased rapidly, so we mention here only treatments directly associated with (1). Even if some ideas concerning the properties of solutions of (1) can be already found in the papers of Bihari [4], [5], Elbert and Mirzov with their papers [21], [36] are the ones usually regarded as pioneers of the qualitative theory of (1). In later years, in particular in the nineties, the striking similarity between oscillatory properties of (1) and (2) was revealed. On the other hand, in some aspects, e.g. the Fredholm-type alternative for solutions of boundary value problems associated with (1), it turned out that the situation is completely different in the linear and half-linear case and the absence of additivity of the solution space of (1) brings completely new phenomena.

The paper is organized as follows. In the next section we present a brief survey of basic properties of solutions of (1). Section 3 is devoted to the oscillation theory of (1) and in the last section we discuss some other problems associated with (1).

## 2. Basic properties of (1)

Consider a special equation of the form (1)

$$(\Phi(x'))' + (p-1)\Phi(x) = 0$$

and denote by $S = S(t)$ its solution given by the initial condition $S(0) = 0$, $S'(0) = 1$. In [21] it is shown that the behaviour of this solution is very similar to that of the classical sine function. In particular, this function is odd, periodic with the half-period $\pi_p := \frac{2\pi}{p\sin(\pi/p)}$, and satisfies the generalized Pythagorian identity $|S(t)|^p + |S'(t)|^p = 1$.

Using this function, one can introduce the generalized Prüfer transformation as follows. Let $x$ be a nontrivial solution of (1). There exist differentiable functions $\varrho, \varphi$ such that

$$x(t) = \varrho(t)S(\varphi(t)), \quad r^{q-1}(t)x'(t) = \varrho(t)S'(\varphi(t)),$$

where $q$ is the conjugate number of $p$, i.e., $\frac{1}{p}+\frac{1}{q}=1$, and the functions $\varrho,\varphi$ satisfy a certain first order system with a Lipschitzian right-hand side, which is, in turn, uniquely solvable. Hence, the same holds for (1): given $t_0\in\mathbb{R}$ and $A,B\in\mathbb{R}$, there exists a unique solution of (1) satisfying $x(t_0)=A$, $x'(t_0)=B$ which is extensible over the whole interval where $r,c$ are continuous and $r(t)>0$.

Other important objects associated with (1) are the $p$-degree functional

$$\mathcal{F}(y;a,b):=\int_a^b\left[r(t)|y'|^p-c(t)|y|^p\right]\,\mathrm{d}t \tag{3}$$

(equation (1) is the Euler-Lagrange equation of $\mathcal{F}$ considered in the class of functions satisfying the zero boundary condition $y(a)=0=y(b)$) and the generalized Riccati equation

$$w'+c(t)+(p-1)r^{1-q}(t)|w|^q=0,\quad q=\frac{p}{p-1}, \tag{4}$$

which is related to (1) by the substitution $w=r\Phi(x')/\Phi(x)$.

Functional (3) and equation (4) are related by the half-linear version of Picone's identity

$$\mathcal{F}(y;a,b)=w(t)|y||_a^b+p\int_a^b r^{1-q}(t)P(r^{q-1}(t)y',w(t)\Phi(y))\,\mathrm{d}t,$$

where

$$P(u,v):=\frac{|u|^p}{p}-uv+\frac{|v|^q}{q}\geqslant 0$$

for all $u,v\in\mathbb{R}$ with the equality if and only if $v=\Phi(u)$, and $w$ is a solution of (4) defined on the whole interval $[a,b]$. This identity is the main tool in the proof of the next statement which summarizes the basic oscillatory properties of (1). This statement is usually referred as the Roundabout Theorem.

**Proposition 1.** *The following statements are equivalent:*

(i) *Equation* (1) *is disconjugate on an interval* $I=[a,b]$, *i.e., any nontrivial solution of* (1) *has at most one zero in* $I$.

(ii) *There exists a solution of* (1) *having no zero in* $[a,b]$.

(iii) *There exists a solution* $w$ *of the generalized Riccati equation* (4) *which is defined on the whole interval* $[a,b]$.

(iv) *The* $p$*-degree functional* $\mathcal{F}(y;a,b)$ *is positive for every* $0\not\equiv y\in W_0^{1,p}(a,b)$.

Observe that Proposition 1 implies that Sturm separation and comparison theorems extend verbatim to (1). Indeed, the separation theorem is essentially the equivalence (i) $\iff$ (ii), while the comparison theorem is hidden in the equivalence (i) $\iff$ (iv). In particular, similarly as in the linear case, equation (1) can be classified as oscillatory or nonoscillatory according to whether any nontrivial solution has or has not infinitely many zeros tending to $\infty$.

Finally, let us mention at least two differences in the basic properties of solutions of (1) (in addition to the already mentioned absence of additivity of the solution space). If $x_1$, $x_2$ is a pair of linearly independent solutions of (1), we have no analogue of the Wronskian-type identity $r(x_1'x_2 - x_1x_2') = \text{const}$ which holds for (2), see [22]. We have also no half-linear analogue of the linear transformation identity with $x = h(t)y$,

$$h(t)[(r(t)x')' + c(t)x] = (r(t)h^2(t)y')' + h(t)[(r(t)h'(t))' + c(y)h(t)]y, \tag{5}$$

which is the starting point of the transformation theory of (2), see [6].

## 3. Oscillation theory

The equivalences given in the Roundabout Theorem (Proposition 1) suggest two main methods of the oscillation theory of (1). The basic ideas and results based on them are briefly explained in this section.

3.1. *Variational method*. This method is based on the equivalence (i) $\iff$ (iv) in Proposition 1. According to this equivalence, to prove that (1) is oscillatory it suffices to construct (for any $T \in \mathbb{R}$) a nontrivial function $y \in W_0^{1,p}(T,\infty)$ such that

$$\mathcal{F}(y;T,\infty) := \int_T^\infty [r(t)|y'|^p - c(t)|y|^p] \,\mathrm{d}t \leqslant 0. \tag{6}$$

On the other hand, for nonoscillation of (1) we need to show the existence of $T \in \mathbb{R}$ such that $\mathcal{F}(y;T,\infty) > 0$ for every $0 \not\equiv y \in W_0^{1,p}(T,\infty)$.

In oscillation criteria, a typical construction of a function $y$ for which (6) holds reads as follows. Let $T$ be arbitrary, $T < t_0 < t_1 < t_2 < t_3$, and let

$$y(t) = \begin{cases} 0, & t \in [T,t_0], \\ f(t), & t \in [t_0,t_1], \\ 1, & t \in [t_1,t_2], \\ g(t), & t \in [t_2,t_3], \\ 0, & t \in [t_3,\infty), \end{cases} \tag{7}$$

where $f, g$ are solutions of the one-term equation

$$(r(t)\Phi(x'))' = 0 \tag{8}$$

satisfying $f(t_0)=0$, $f(t_1)=1$, $g(t_2)=1$, $g(t_3)=0$, i.e.

$$f(t)=\left(\int_{t_0}^{t} r^{1-q}(s)\,\mathrm{d}s\right)\left(\int_{t_0}^{t_1} r^{1-q}(s)\,\mathrm{d}s\right)^{-1},$$
$$g(t)=\left(\int_{t}^{t_3} r^{1-q}(s)\,\mathrm{d}s\right)\left(\int_{t_2}^{t_3} r^{1-q}(s)\,\mathrm{d}s\right)^{-1}.$$

By direct computation, using the second mean value theorem of integral calculus applied to the integrals $\int_{t_0}^{t_1} r|f|^p$, $\int_{t_2}^{t_3} r|g|^p$ (see e.g. [14]) we have

$$\mathcal{F}(y;T,\infty)=\left(\int_{t_0}^{t_1} r^{1-q}(s)\,\mathrm{d}s\right)^{1-p}+\left(\int_{t_2}^{t_3} r^{1-q}(s)\,\mathrm{d}s\right)^{1-p}+\int_{s_1}^{s_2} c(t)\,\mathrm{d}t, \tag{9}$$

where $s_1\in(t_0,t_1)$, $s_2\in(t_2,t_3)$. Using this computation we can now easily prove the following oscillation criteria.

**Theorem 1.** *Suppose that $\int^{\infty} r^{1-q}(t)\,\mathrm{d}t=\infty$. Then each of the following conditions is sufficient for oscillation of* (1)*:*

(i) (*Leighton-Wintner-type criterion* [36]).

$$\lim_{b\to\infty}\int^{b} c(t)=\infty. \tag{10}$$

(ii) (*Nehari-type criterion* [14]). *The integral $\int^{\infty} c(t)\,\mathrm{d}t$ is convergent and*

$$\liminf_{t\to\infty}\left(\int^{t} r^{1-q}(s)\,\mathrm{d}s\right)^{p-1}\left(\int_{t}^{\infty} c(s)\,\mathrm{d}s\right)>1. \tag{11}$$

Proof. A short computation and (9) show that each of conditions (i), (ii) implies that the points $t_i$, $i=0,\dots,3$, can be chosen in such a way that $\mathcal{F}(y;T,\infty)<0$. $\square$

Concerning the nonoscillation criteria proved via the variational method, their proofs are usually based on the Wirtinger-type inequality

$$\int_{T}^{\infty}|M'(t)||y|^p\,\mathrm{d}t\leqslant p^p\int_{T}^{\infty}\frac{M^p(t)}{|M'(t)|^{p-1}}|y'|^p\,\mathrm{d}t \tag{12}$$

where $M$ is a differentiable function such that $M'(t)\neq 0$ on $[T,\infty)$, which holds for every $0\not\equiv y\in W_0^{1,p}(T,\infty)$, see e.g. [12]. Using (12) we can prove the following statement.

**Theorem 2.** *Suppose that* $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$ *and*

$$\limsup_{t\to\infty} \left(\int^t r^{1-q}(s)\,\mathrm{d}s\right)^{p-1} \left(\int_t^\infty c_+(s)\,\mathrm{d}s\right) < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \tag{13}$$

*where* $c_+(t) = \max\{0, c(t)\}$. *Then* (1) *is nonoscillatory.*

P r o o f. Setting $M(t) := (\int^t r^{1-q}(s)\,\mathrm{d}s)^{1-p}$ and using (13) one can show that $\mathcal{F}(y; T, \infty) < 0$. $\square$

The previous two statements are typical examples of the application of the variational method in the oscillation theory of half-linear equations.

3.2. R i c c a t i  t e c h n i q u e. In this subsection we briefly sketch how the equivalence (i) $\iff$ (iii) can be used to derive (non)oscillation criteria. We illustrate the application of this method in the criteria which are similar to those presented in the previous subsection.

We start with an improvement of the statement given in Theorem 1 (ii).

**Theorem 3.** *Suppose that* $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$, *the integral* $\int^\infty c(t)\,\mathrm{d}t$ *is convergent and* $\liminf$ *in* (11) *is* $> \frac{1}{p}(\frac{p-1}{p})^{p-1}$. *Then* (1) *is oscillatory.*

P r o o f. By contradiction, suppose that (1) is nonoscillatory. Then the solutions of the associated Riccati equation (4) satisfy the integral equation

$$w(t) = \int_t^\infty c(t)\,\mathrm{d}t + (p-1)\int_t^\infty r^{1-q}(s)|w|^q\,\mathrm{d}s,$$

see [34]. Multiplying this equation by $\left(\int^t r^{1-q}(s)\,\mathrm{d}s\right)^{p-1}$, using (13) and supposing that $\limsup_{t\to\infty} \left(\int^t r^{1-q}(s)\,\mathrm{d}s\right)^{p-1} w(t) =: \mu < \infty$ (in the case $\mu = \infty$, to get a contradiction is even easier than in our case $\mu < \infty$) we find an $\varepsilon > 0$ such that $\mu$ satisfies the inequality

$$\mu > \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} + \varepsilon + |\mu|^q.$$

Since $|t|^q - t + \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \geqslant 0$ for $t \in \mathbb{R}$, we have the required contradiction. $\square$

Nonoscillation criteria based on the Riccati technique are usually proved using a slightly different method than just the equivalence mentioned in Proposition 1. We look for a solution of the *inequality*

$$v' + c(t) + (p-1)r^{1-q}(t)|v|^q \leqslant 0 \tag{14}$$

instead of (4), since it essentially means that a certain Sturmian majorant of (1) is nonoscillatory and hence (1) is nonoscillatory as well.

**Theorem 4.** *Suppose that* $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$ *and* $\int^\infty c(t)\,\mathrm{d}t$ *converges. If*

$$\limsup_{t\to\infty} \left(\int^t r^{1-q}(s)\,\mathrm{d}s\right)^{p-1} \left(\int_t^\infty c(s)\,\mathrm{d}s\right) < \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \tag{15}$$

$$\liminf_{t\to\infty} \left(\int^t r^{1-q}(s)\,\mathrm{d}s\right)^{p-1} \left(\int_t^\infty c(s)\,\mathrm{d}s\right) > -\frac{2p-1}{p}\left(\frac{p-1}{p}\right)^{p-1}, \tag{16}$$

*then* (1) *is nonoscillatory.*

P r o o f. Let $v(t) = \beta\big(\int^t r^{1-q}(s)\,\mathrm{d}s\big)^{1-p}$, $\beta := \frac{1}{p}(\frac{p-1}{p})^{p-1}$. Using (15) and (16) it is not difficult to verify that this function really satisfies (14), see [14]. $\square$

All criteria presented by Theorems 1–4 contain the assumption $\int^\infty r^{1-q} = \infty$. A slight modification of the proofs of these statements enables us to find their counterparts in the case $\int^\infty r^{1-q} < \infty$, see [14].

3.3. P e r t u r b a t i o n p r i n c i p l e a n d p r i n c i p a l s o l u t i o n. In the previous criteria, equation (1) has been regarded as a perturbation of the (nonoscillatory) one-term differential equation (8). It was shown that if the "perturbation" function $c$ in (1) is "sufficiently positive" ("not too positive") then (1) becomes oscillatory (remains nonoscillatory). The exact quantitative characterization of these vague concepts is just the content of Theorems 1–4.

From this point of view, it is a natural idea to consider (1) not as a perturbation of the one-term equation (8), but as a perturbation of the the general two-term nonoscillatory equation

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0, \tag{17}$$

where $\tilde{c}$ is a continuous function, and (non)oscillation criteria for (1) are formulated in terms of the difference $c-\tilde{c}$. Note that this idea applied to *linear* equation (2) does not produce essentially new criteria, due to the transformation identity (5). Indeed, this identity applied to (2) written in the form $(rx')'+cx = (rx')'+\tilde{c}x+(c-\tilde{c})x = 0$ and with the transformation function $h$ which a solution of $(rx')' + \tilde{c}x = 0$ gives

$$h(t)\big[(r(t)x')' + \tilde{c}(t)x + (c(t) - \tilde{c}(t))x\big] = (r(t)h^2(t)y')' + h^2(t)[c(t) - \tilde{c}(t)]y,$$

so the resulting equation is again an equation of the form (2). One can then investigate it as the perturbation of the one-term equation $(r(t)h^2(t)y')' = 0$ and to transform the results obtained "back" to the original equation (2).

Concerning the half-linear equation (1), as mentioned in Section 2, we have no half-linear version of (5), so the idea to investigate (1) as a perturbation of two-terms equation (17) brings really qualitatively new criteria. This approach was (implicitly) used for the first time by Elbert [23], who proved that (1) with $r \equiv 1$ is oscillatory provided

$$\lim_{b\to\infty} \int^b \Big(c(t) - \frac{\gamma}{t^p}\Big) t^{p-1}\,\mathrm{d}t = \infty, \quad \gamma = \Big(\frac{p-1}{p}\Big)^p. \tag{18}$$

In this setting equation (1) with $r \equiv 1$ is viewed as a perturbation of the generalized Euler equation with the critical constant

$$(\Phi(x'))' + \frac{\gamma}{t^p}\Phi(x) = 0, \tag{19}$$

whose one solution $x(t) = t^{\frac{p-1}{p}}$ can be computed explicitly and the other solutions behave asymptotically as $t^{\frac{p-1}{p}} \lg^{\frac{2}{p}} t$, see [24].

A natural question is why it is just the power $t^{p-1}$ which appears by the difference $c(t) - \tilde{\gamma} t^{-p}$ in (18). In the linear case $p = 2$ the answer is that $h = t^{1/2}$ is the so called *principal solution* of the Euler equation $x'' + \frac{1}{4t^2}x = 0$ (principal solutions of (2) and (1) are discussed in more detail later in this section). More precisely, the transformation $x = t^{1/2}y$ transforms (2) with $r \equiv 1$ into the equation $(ty')' + \big(c - \frac{1}{4t^2}\big)\, ty = 0$.

The concept of the principal solution of (1) was introduced by Mirzov [37] and in [13], [25] it was shown that this solution has many of the properties of the principal solution of (2). Using this concept we can now prove the following generalization of the Elbert criterion (18), see [14], [17].

**Theorem 5.** *Suppose that* (17) *is nonoscillatory and $h$ is its (positive) principal solution. If*

$$\lim_{b\to\infty} \int^b (c(t) - \tilde{c}(t))\, h^p(t)\,\mathrm{d}t = \infty \tag{20}$$

*then* (1) *is oscillatory.*

P r o o f. The proof is similar to that of Theorem 1. We modify the test function (7) as follows. We let $y = h$ for $t \in [t_1, t_2]$. Then $f, g$ are solutions of (17) (instead of (8)) satisfying $f(t_1) = h(t_1)$, $g(t_2) = h(t_2)$. Using (20) and the properties of principal solutions of (1) one can show that the points $t_i$, $i = 0, \dots, 3$, can be chosen in such a way that $\mathcal{F}(y; T, \infty) < 0$, see [17]. □

Observe that in the case $\tilde{c} \equiv 0$ Theorem 5 reduces to Theorem 1 (i). Indeed, if $\int^\infty r^{1-q} = \infty$ then $h \equiv 1$ is the principal solution of the one term equation (8) and (20) is the same as (10). Note also that Theorems 2–4 can be extended along the line treated in this subsection as well, we refer to [12], [17] for more details.

We conclude this part with some remarks and open problems concerning the principal solution of (1). In the linear case the principal solution $\tilde{x}$ of (2) is (equivalently) defined as a solution for which

$$\lim_{t\to\infty} \frac{\tilde{x}(t)}{x(t)} = 0 \quad \Longleftrightarrow \quad \int^\infty \frac{\mathrm{d}t}{r(t)\tilde{x}^2(t)} = \infty, \tag{21}$$

where $x$ is any solution of (2) linearly independent of $\tilde{x}$. Since both these characterizations are based on the linearity of the solutions space of (2), they do not extend (directly) to (1). Mirzov [37] in his construction of the principal solution of (1) used the fact that if this equation is nonoscillatory, then among all solutions of the associated Riccati equation (4) there exists a minimal one $\widetilde{w}$, in the sense that any other solution $w$ of this equation satisfies $w(t) > \widetilde{w}(t)$ eventually. The principal solution of (1) is then defined by

$$\tilde{x}(t) = C \exp\left\{ \int^t r^{1-q}(s)\Phi^{-1}(\widetilde{w}(s))\,\mathrm{d}s \right\},$$

$\Phi^{-1}$ being the inverse function of $\Phi$, i.e. the solution $\tilde{x}$ which is determined by $\widetilde{w} = r\Phi(\tilde{x}')/\Phi(\tilde{x})$. In the linear case this construction is equivalent to (21), see [28].

In [16] we have tried to find an integral characterization of the principal solution of (1) in such a way that in the linear case it reduces to the second expression in (21). The main results of [16] are summarized in the next statement.

**Theorem 6.** *Suppose that equation* (1) *is nonoscillatory and $\tilde{x}$ is its solution such that $\tilde{x}'(t) \neq 0$ for large $t$.*

(i) *Let $p \in (1,2)$. If*

$$\int^\infty \frac{\mathrm{d}t}{r(t)x^2(t)|x'(t)|^{p-2}} = \infty, \tag{22}$$

*then $\tilde{x}$ is the principal solution.*

(ii) *Let $p > 2$. If $\tilde{x}$ is the principal solution then* (22) *holds.*

(iii) *Suppose that $\int^\infty r^{1-q}(t)\,\mathrm{d}t = \infty$, the function $\gamma(t) := \int_t^\infty c(s)\,\mathrm{d}s$ exists and $\gamma(t) \geqslant 0$, but $\gamma(t) \not\equiv 0$ eventually. Then $\tilde{x}(t)$ is the principal solution if and only if* (22) *holds.*

This theorem shows that the equivalent integral characterization of the principal solution of (1) is known only in some particular cases. The subject of the present investigation is whether (22) is really a good characterization. Another intensively studied problem is the limit characterization of the principal solution of (1), i.e. the extension of the first relation of (21) to solutions of (1). For some results of this effort see [8], [15], [19].

## 4. Related topics

In this section we very briefly present some selected results related to the oscillation theory of (1).

4.1. Asymptotics of nonoscillatory solutions. In this subsection we suppose that the function $c$ in (1) is positive or negative for large $t$, in the former case we suppose in addition that (1) is nonoscillatory. Since the solution space of (1) is homogeneous, we can restrict our attention to positive solutions. These solutions can be divided into two main classes according to their behaviour for large $t$,

$$\mathbb{M}^+ = \{x\colon\ x'(t) > 0\}, \quad \mathbb{M}^- = \{x\colon\ x'(t) < 0\},$$

and each class $\mathbb{M}^+, \mathbb{M}^-$ is the union of two subclasses

$$\begin{aligned} \mathbb{M}^+ &= \mathbb{M}^+_\infty \cup \mathbb{M}^+_B, \quad \mathbb{M}^+_\infty = \{x\colon\ x(t) \to \infty\},\ \mathbb{M}^+_B = \{x\colon\ x(t) \to L < \infty\}, \\ \mathbb{M}^- &= \mathbb{M}^-_B \cup \mathbb{M}^-_0, \quad \mathbb{M}^-_B = \{x\colon\ x(t) \to L > 0\},\ \mathbb{M}^-_0 = \{x\colon\ x(t) \to 0\}. \end{aligned}$$

The investigation of the classification of nonoscillatory solutions of linear equations (2) along this line was initiated in [35]. Afterward, several papers extending the results of this paper have appeared, see e.g. in [7], [9] and the references given therein. An important role is played by the integrals

$$\begin{aligned} J_1 &= \lim_{T\to\infty} \int^T r^{1-q}(t)\Phi^{-1}\left(\int^t c(s)\,\mathrm{d}s\right)\mathrm{d}t, \\ J_2 &= \lim_{T\to\infty} \int^T r^{1-q}(t)\Phi^{-1}\left(\int_t^T c(s)\,\mathrm{d}s\right)\mathrm{d}t \end{aligned}$$

and using these intgrals the following results have been established in [7].

**Theorem 7.** *Suppose that $c(t) < 0$ for large $t$, then*
(a) $J_1 = -\infty$, $J_2 = -\infty \Longrightarrow \mathbb{M}^- = \mathbb{M}^-_0 \neq \emptyset$,
(b) $J_1 = -\infty$, $J_2 > -\infty \Longrightarrow \mathbb{M}^- = \mathbb{M}^-_B \neq \emptyset$,

(c) $J_1 > -\infty$, $J_2 > -\infty \Longrightarrow \mathbb{M}_B^- \neq \emptyset$, $\mathbb{M}_0^- \neq \emptyset$,
(d) $J_1 = -\infty \Longrightarrow \mathbb{M}^+ = \mathbb{M}_\infty^+$,
(e) $J_1 > -\infty \Longrightarrow \mathbb{M}^+ = \mathbb{M}_B^+$.

4.2. BVP's associated with (1). Consider the Dirichlet boundary value problem

$$(\Phi(x'))' + \lambda\Phi(x) + g(t,x) = f(t), \quad x(0) = 0 = x(\pi_p), \tag{23}$$

where $\lambda$ is a real-valued parameter, the number $\pi_p$ is defined in Section 2 and the "nonhalf-linearity" $g$ and the forcing term $f$ satisfy certain additional assumptions. The literature dealing with solvability of (23) (not only with the Dirichlet boundary condition) is very voluminous, see e.g. [10] and the references given therein. The situation is similar to the linear case in some aspects, but in some cases one meets completely different phenomena. Here we mention one of them, the Fredholm-type alternative for solvability of the BVP

$$(\Phi(x'))' + \lambda\Phi(x) = f(t), \quad x(0) = 0 = x(\pi_p). \tag{24}$$

Note that $\lambda_1 = (p-1)$ is the principal eigenvalue of the unforced equation (24).

Using the detailed analysis of the geometry of the functional associated with (24)

$$\mathcal{J}_f(y) = \frac{1}{p}\int_0^{\pi_p} \{|y'|^p - \lambda|y|^p\}\,\mathrm{d}t - \int_0^{\pi_p} f(t)y\,\mathrm{d}t$$

whose critical points over $W_0^{1,p}(0,\pi_p)$ are solutions of (24), one can prove the following statements, see [20] and the references given therein.

**Theorem 8.** *Fredholm's alternative for $p$-Laplacian with $p \neq 2$.*

(i) *If $\lambda < 0$, the functional $J_f$ has a unique minimum over $W_0^{1,p}(0,\pi_p)$, it is coercive and (24) has a unique solution.*
(ii) *If $0 < \lambda < \lambda_1$, the functional $J_f$ is still coercive, but there exists $f \in C[0,\pi_p]$ such that $J_f$ has at least two critical points (one of them is the global minimum over $W_0^{1,p}(0,\pi_p)$, the other one is of saddle type).*
(iii) *If $\lambda = \lambda_1$ (and similarly for higher eigenvalues $\lambda_n = (p-1)^n$), the condition*

$$\int_0^{\pi_p} f(t)S(t)\,\mathrm{d}t = 0,$$

*where $S(t)$ is the generalized sine function given in Section 2, i.e. the solution of (24) with $f \equiv 0$, is only sufficient but not necessary for solvability of (24). More* *precisely, there exists an open cone $\mathcal{C} \subset C[0, \pi_p]$ such that* (24) *has at least two solutions for every $f \in \mathcal{C}$ and*

$$\int_0^{\pi_p} f(t)S(t)\,\mathrm{d}t \neq 0.$$

4.3. Half-linear difference equations. In recent years, considerable attention has been paid to the oscillation theory of various *difference equations.* In particular, the oscillation theory of the Sturm-Louville difference equation

$$\Delta(r_k \Delta x_k) + c_k x_{k+1} = 0, \quad \Delta x_k = x_{k+1} - x_k,$$

has been deeply developed, see [1], [31]. The discrete counterpart of (1) is the difference equation

$$\Delta(r_k \Phi(\Delta x_k)) + c_k \Phi(x_{k+1}) = 0, \quad r_k \neq 0. \tag{25}$$

The basic oscillatory properties of (25) are established in [38] and the main result of that paper is the discrete version of the Roundabout Theorem (Proposition 1).

**Theorem 9.** *The following statements are equivalent*

(i) *Equation* (25) *is disconjugate on the discrete interval* $[0, N]$, *i.e. the solution $x$ given by the initial condition $x_0 = 0$, $x_1 \neq 0$ has no generalized zero in* $(0, N+1]$, *i.e.*

$$r_k x_k x_{k+1} > 0, \quad k = 1, \ldots, N.$$

(ii) *There exists a solution $x$ of* (25) *having no generalized zero in* $[0, N+1]$.

(iii) *There exists a solution $w = \{w_k\}_{k=0}^{N+1}$ of the Riccati-type equation*

$$\Delta w_k + c_k + \left(1 - \frac{r_k}{\Phi(\Phi^{-1}(r_k) + \Phi^{-1}(w_k))}\right) w_k = 0, \quad w = \frac{r\Phi(\Delta x)}{\Phi(x)}$$

*such that $r_k + w_k > 0$, $k = 1, \ldots, N$.*

(iv) *We have*

$$\mathcal{F}_d(y; 0, N) = \sum_{k=0}^{N} \{r_k |\Delta y_k|^p - c_k |y_{k+1}|^p\} > 0$$

*for every nontrivial $y = \{y_k\}_{k=0}^{N+1}$ satisfying $y_0 = 0 = y_{N+1}$.*

4.4. Scalar methods for $p$-Laplacian. Several physical phenomena can be described by the partial differential equation with the so-called $p$-Laplacian

$$\operatorname{div}\left(\|\nabla u\|^{p-2} \nabla u\right) + c(x)\Phi(u) = 0, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{26}$$

see e.g. [11]. It can be shown that the $p$-degree functional

$$\mathcal{F}(u;\Omega) = \int_\Omega \{\|\nabla u\|^p - c(x)|u|^p\}\,\mathrm{d}x,$$

the Riccati type equation

$$\operatorname{div} w + c(x) + (p-1)\|w\|^q = 0, \quad w = \frac{\nabla u}{u} \tag{27}$$

and Picone's-type identity

$$\begin{aligned}\mathcal{F}(y;\Omega) &= \int_{\partial\Omega} w|y|^p\,\mathrm{d}S + p\int_\Omega P(\nabla y, \Phi(y)w)\,\mathrm{d}x,\\ P(u,v) &= \frac{\|u\|^p}{p} - \|u,v\| + \frac{\|v\|^q}{q} \geqslant 0,\end{aligned}$$

where $w$ is a solution of (27) defined in the whole domain $\overline{\Omega}$, can be used to establish the Roundabout Theorem, the Sturmian theory and to a certain extent also the oscillation theory similarly as for the ordinary differential equation (1) and for partial differential equations with the "normal" Laplacian, see [2], [3], [18], [29].

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