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ON THE STRUCTURE OF RANDOM HYPERGRAPHS

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Abstract. Let $\mathcal{H}_n$ be a countable random n-uniform hypergraph for $n > 2$ and let $\mathbb{P}(\mathcal{H}_n) = \{f[\mathcal{H}_n] : f : \mathcal{H}_n \to \mathcal{H}_n \text{ is an embedding}\}$. We prove that a linear order L is isomorphic to the maximal chain in the partial order $\langle \mathbb{P}(\mathcal{H}_n) \cup \{\emptyset\}, \subset\rangle$ if and only if L is isomorphic to the order type of a compact set of reals whose minimal element is non-isolated.

1. Introduction

1.1. **Background and the statement of the result.** The purpose of this note is to completely characterize chains of isomorphic substructures of the Fraïssé limit of finite n-uniform hypergraphs for each $n > 1$, thus generalizing some results from [10] to higher dimensions. Fraïssé theory, the systematic study of ultrahomogeneous universal structures, was initiated in the mid 1950's by Roland Fraïssé [3]. Typical examples of Fraïssé limits are the rational line $\langle \mathbb{Q}, <\rangle$ and the countable random graph (i.e. Rado graph). A particularly active research area is the investigation of the automorphism groups of these structures (see [5] for the most notable example). Besides that, there has been great interest in considering the embeddings of an ultrahomogeneous structure into itself (for a relational structure $\mathbb{X}$, denote $\mathrm{Emb}(\mathbb{X}) = \{f : \mathbb{X} \to \mathbb{X} : f \text{ is an embedding}\}$). For example, see [2] for some results on the self-embeddings of ultrahomogeneous n-uniform hypergraphs or [13] for one of the most prominent result concerning self-embeddings of ultrahomogeneous structures. In this context, one usually investigates the set of isomorphic substructures of a structure $\mathbb{X}$, denoted $\mathbb{P}(\mathbb{X}) = \{f[\mathbb{X}] : f \in \mathrm{Emb}(\mathbb{X})\} = \{A \subset \mathbb{X} : A \cong \mathbb{X}\}$.

The set $\mathbb{P}(\mathbb{X})$ is naturally ordered by inclusion, and we will be interested in order types of chains in these partial orders where $\mathbb{X}$ is the countable random n-uniform hypergraph (for all $n \geq 2$). By a well-known Hausdorff maximal principle (also known as the Kuratowski lemma, one of the equivalents of the AC), each chain is contained in a maximal one, so the characterization of maximal chains will give a complete answer. Maximal chains in various partial orders were extensively investigated in the literature. The first result

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related to ours is a theorem of Kuratowski [7] from 1921. which states that if κ is a regular cardinal, then a linear order L is isomorphic to a maximal chain in $P(\kappa)$ if and only if it is isomorphic to the order of all initial segments of some linear order of size κ. This result of Kuratowski was followed by results of Day [1], Koppelberg [6], Monk [12] and others. Besides in [10], some recent results related to the ones in this paper can be found in [9, 11]. The main result of this paper is the following.

Theorem 1.1. *Let $\mathcal{H}_n$, $n > 1$, be a countable random n-uniform hypergraph. Then a linear order L is isomorphic to a maximal chain in the partial order $\langle \mathbb{P}(\mathcal{H}_n) \cup \{\emptyset\}, \subset\rangle$ if and only if it is isomorphic to the order type of a compact set of reals whose minimum is non-isolated.*

Note that results in [10] claim that the same characterization of maximal chains of isomorphic substructures holds for Henson graphs, while for disjoint unions of complete graphs L must be isomorphic to a compact nowhere dense set of reals with minimum non-isolated. Also, we remark that we in fact investigate chains in the poset $\langle [\omega]^\omega, \subset\rangle$, and that already mentioned Kuratowski's result is the first result of this sort and that it claims that there are no 'continuous' maximal chains in $\langle [\omega]^\omega, \subset\rangle$. This precisely means that each maximal chain in $\langle [\omega]^\omega, \subset\rangle$ must have dense jumps while on the other hand, our result shows that when we add the structure of random n-uniform hypergraph to the countable set then there are 'continuous' maximal chains in $\langle \mathbb{P}(\mathcal{H}_n) \cup \{\emptyset\}, \subset\rangle$, for example there is a maximal chain of type $[0, 1]$.

1.2. **Preliminaries.** In this paper n will be reserved for natural numbers and $|X|$ denotes the cardinality of a set X, in particular ω is the cardinality of a countably infinite set. For a set X and $n \geq 1$, by $[X]^n$ we denote the set of all n-element subsets of X, i.e. $[X]^n = \{y \subset X : |y| = n\}$. Also, $[X]^{<\omega}$ denotes the set of all finite subsets of X. If f maps A into B, then $f[A] = \{f(x) : x \in A\}$. The power set of X is denoted by $P(X)$

A *relational structure* $\mathbb{X} = \langle X, \{\rho_i : i \in I\}\rangle$ consists of a set X and relations ρ_i $(i \in I)$. Often, when there can be no confusion, we do not make distinction between denoting the structure $\mathbb{X}$ and the underlying set X. We say that a structure $\mathbb{Y} = \{Y, \{\sigma_i : i \in I\}\}$ is a *substructure* of $\mathbb{X}$ if and only if $Y \subset X$ and for each $i \in I$ we have $\sigma_i = Y^{\mathrm{ar}(\rho_i)} \cap \rho_i$. A mapping $f : \mathbb{X} \to \mathbb{Y}$ is an *embedding* of a relational structure $\mathbb{X}$ into relational structure $\mathbb{Y}$ if and only if f is 1-1 and it holds $(k_i = \mathrm{ar}(\rho_i))$

$$\forall i \in I\ \forall \langle a_1, \dots, a_{k_i}\rangle \in X^{k_i} \quad \langle a_1, \dots, a_{k_i}\rangle \in \rho_i \Leftrightarrow \langle f(a_1), \dots, f(a_{k_i})\rangle \in \sigma_i.$$

Notice that we make a distinction between embedding and a homomorphism for relational structures (in this article, we will only be concerned with embeddings).

We say that a relational structure $\mathbb{X}$ is *ultrahomogeneous* if and only if any isomorphism ϕ between finite substructures of $\mathbb{X}$ can be extended to an automorphism of $\mathbb{X}$. Further, we say that a relational structure $\mathbb{X}$ is *universal* for a class of structures $\mathcal{K}$ if and only if for each $\mathbb{K} \in \mathcal{K}$ there is an embedding

$f : \mathbb{K} \to \mathbb{X}$. We use the following characterization of ultrahomogeneity (see [4, Theorem 12.1.2.]).

Lemma 1.2. *Let $\mathbb{X}$ be a countable relational structure. Then $\mathbb{X}$ is ultrahomogenoeus if and only if for any finite substructure F of $\mathbb{X}$ and any embedding $f : F \to \mathbb{X}$, and for any element $a \in \mathbb{X} \setminus F$ there exists an embedding $g : F \cup \{a\} \to \mathbb{X}$ which is an extension of f.*

Now we mention a few notions related to order theory. We say that a linear order is *complete* if and only if it is Dedekind-complete and has minimum and maximum (the reader may find this definition of completeness non-standard, but we use it in order to shorten some statements). We say that a linear order L is *boolean* if and only if it is complete and has dense jumps, i.e. complete and for any $x, y \in L$ if $x < y$ then there are $s, t \in L$ such that $x \leq s < t \leq y$ and $(s, t)_L = \emptyset$.

We will also need the notions of a filter and a set dense in a partial order. Let $\langle P, \leq \rangle$ be a partial order, a set $D \subset P$ is *dense* in P if for any $p \in P$ there is $q \in D$ such that $q \leq p$. A set $G \subset P$ is a *filter* in P if and only if for all $x, y \in G$ there is $q \in G$ such that $q \leq x, y$ (i.e. elements of G are pairwise compatible in G) and for any $x \in G$ if $y > x$, then also $y \in G$. The following is a well-known fact.

Lemma 1.3 (Rasiowa-Sikorski)**.** *Let $\langle P, \leq \rangle$ be a partially ordered set and $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ a countable family of sets dense in P. Then there is a filter G in P such that $G \cap D_n \neq \emptyset$ for all $n \in \mathbb{N}$.*

1.3. **Maximal chains.** First note that the linear order L is isomorphic to the order type of a compact (nowhere dense compact) set of reals whose minimum is non-isolated if and only if it is complete (boolean), $\mathbb{R}$ embeddable and has a non-isolated minimum. For a proof of this fact see [8].

Recall that a positive family on a countable set X is a family $\mathcal{P} \subset P(X)$ satisfying (see also [8]):

(P1) $\emptyset \notin \mathcal{P}$;
(P2) $A \in \mathcal{P} \wedge B \in [A]^{<\omega} \Rightarrow A \setminus B \in \mathcal{P}$;
(P3) $A \in \mathcal{P} \wedge A \subset B \subset X \Rightarrow B \in \mathcal{P}$;
(P4) $\exists A \in \mathcal{P}\ \ |X \setminus A| = \omega$.

For example, each non-principal ultrafilter on ω is a positive family on ω. Also, the family of all dense subsets of the rational line $\mathbb{Q}$ is a positive family on $\mathbb{Q}$. Positive families play an important role in investigation of maximal chains in the posets of the form $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$. Namely, Theorem 2.2. in [11] states that if there is a positive family $\mathcal{P}$ on $\mathbb{X}$ such that $\mathcal{P} \subset \mathbb{P}(\mathbb{X})$ then for each countable, complete, $\mathbb{R}$-embeddable linear order L whose minimum is non-isolated, there is a maximal chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$ isomorphic to L. This allows us to reformulate Theorem 3.2. from [10] in the following slightly weaker manner.

Theorem 1.4. *Let $\mathbb{X}$ be a countable relational structure and $\langle \mathbb{Q}, < \rangle$ the rational line. If there exists a partition $\{J_m : m \in \omega\}$ of $\mathbb{Q}$ and a structure with the domain $\mathbb{Q}$ of the same signature as $\mathbb{X}$ such that:*

(i) *J_0 is a dense subset of $\langle \mathbb{Q}, < \rangle$,*
(ii) *J_m ($m \in \omega$) are coinitial subsets of $\langle \mathbb{Q}, < \rangle$,*
(iii) *$(-\infty, x)_{J_0} \subset A \subset (\infty, x)_{\mathbb{Q}}$ implies $A \cong \mathbb{X}$ for $x \in \mathbb{R} \cup \{\infty\}$,*
(iv) *$(-\infty, q]_{J_0} \subset C \subset (\infty, q]_{\mathbb{Q}}$ implies $C \ncong \mathbb{X}$ for $q \in J_0$,*
(v) *there is a positive family $\mathcal{P}$ on $\mathbb{X}$ such that $\mathcal{P} \subset \mathbb{P}(\mathbb{X})$,*

then for each $\mathbb{R}$-embeddable complete linear order L with $\min L$ non-isolated there is a maximal chain in $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$ isomorphic to L.

Next result, proved in [11], shows that ultrahomogeneous structures provide a nice framework for investigating maximal chains of their isomorphic substructures.

Theorem 1.5. *Let $\mathbb{X}$ be a countable ultrahomogeneous structure of an at most countable relational language which contains at least one non-trivial isomorphic substructure, i.e. $\mathbb{P}(\mathbb{X}) \neq \{X\}$. Then for each linear order L the implication (1)⇒(2) is true, where*

(1) *L is isomorphic to a maximal chain in the poset $\langle \mathbb{P}(\mathbb{X}) \cup \{\emptyset\}, \subset \rangle$;*
(2) *L is a complete $\mathbb{R}$-embeddable linear order with $\min L$ non-isolated.*

2. Random hypergraphs

For $n \geq 2$, a n-uniform hypergraph is a relational structure $\langle X, \rho \rangle$, satisfying $\mathrm{ar}(\rho) = n$ and such that $\langle x_0, \dots, x_{n-1} \rangle \in \rho$ implies $x_i \neq x_j$ for all $i \neq j$ in n and $\langle x_{\pi(0)}, \dots, x_{\pi(n-1)} \rangle \in \rho$ for all permutations π of n (see [4]). Note that this is equivalent to saying that n-uniform hypergraph is a pair $\langle X, \rho \rangle$ where X is a set and $\rho \subset [X]^n$, so we will sometimes refer to the first formulation, and sometimes, when it is more convenient, to the second.

Recall that the class of countably many (up to isomorphism) finite structures is a Fraïssé class (see [4]) if it is hereditary, satisfies joint embedding and amalgamation property and contains structures of arbitrary large finite cardinality. It is well known that the class $\mathcal{K}_n$ of finite n-uniform hypergraphs ($n \geq 2$) is a Fraïssé class, hence the famous Fraïssé's theorem states there there is a unique up to isomorphism countable ultrahomogeneous relational structure whose age is exactly $\mathcal{K}_n$ (the age of a relational structure is the class of all of its finitely generated substructures).

Definition 2.1. For $n \geq 2$, the countable ultrahomogeneous n-uniform hypergraph universal for all finite n-uniform hypergraphs is called the countable random n-uniform hypergraph.

The following lemma gives a useful reformulation of the definition of the countable random n-uniform hypergraphs. Note also that Fraïssé's theorem states that the countable random n-uniform hypergraph is universal even for the class of all countable n-uniform hypergraphs.

Lemma 2.2. *Suppose that $n > 1$ and that a countable n-uniform hypergraph $\langle \mathcal{H}_n, \Gamma_n \rangle$ satisfies the following condition: for any $A \in [\mathcal{H}_n]^{<\omega} \setminus \bigcup_{i<n-1} [\mathcal{H}_n]^i$ and any $B \subset [A]^{n-1}$ there exists $a \in \mathcal{H}_n \setminus A$ such that for all $b \in B$ we have $\{a\} \cup b \in \Gamma_n$, while for all $b \in [A]^{n-1} \setminus B$ it holds $\{a\} \cup b \notin \Gamma_n$. Then $\mathcal{H}_n$ is isomorphic to the countable random n-uniform hypergraph.*

Proof. We have to prove that the n-uniform hypergraph $\langle \mathcal{H}_n, \Gamma_n \rangle$ satisfying the conditions of the lemma is ultrahomogeneous and universal for all finite n-uniform hypergraphs. So let F be a finite substructure of $\mathcal{H}_n$ and $f : F \to \mathcal{H}_n$ an embedding. Pick an arbitrary $a \in \mathcal{H}_n \setminus F$. If $|F| < n-1$ then any 1-1 map $g : F \cup \{a\} \to \mathcal{H}_n$ with $g \restriction F = f$ is an embedding because in that case $\Gamma_n \cap [F \cup \{a\}]^n = \emptyset$ (hence the conclusion of Lemma 1.2 is fulfilled). If $|F| \geq n-1$, consider the set $f[F]$, and define $B \subset [f[F]]^{n-1}$ in the following way:

$$b \in B \iff \{a\} \cup f^{-1}[b] \in \Gamma_n. \tag{2.1}$$

Next, pick an element $x \in \mathcal{H}_n \setminus f[F]$ such that $\forall b \in B \;\; \{x\} \cup b \in \Gamma_n$ and that $\forall b \in [f[F]]^{n-1} \setminus B \;\; (\{x\} \cup b \notin \Gamma_n)$. Note that the existence of x follows from the assumption of the lemma. Finally, the mapping $g : F \cup \{a\} \to \mathcal{H}_n$ given by $g(y) = f(y)$ $(y \in F)$ and $g(a) = x$ is an embedding by construction (namely, the condition (2.1) ensures that g is an embedding) so the conclusion of Lemma 1.2 is fulfilled.

In order to finish the proof it will be enough to prove that $\mathcal{H}_n$ is universal for all countable n-uniform hypergraphs. Let $\mathbb{A} = \langle \{a_1, a_2, \dots\}, \rho \rangle$ be an arbitrary countable n-uniform hypergraph. Hence, $\rho \subset [A]^n$. If $|A| < n$ then any 1-1 mapping $h : A \to \mathcal{H}_n$ is an embedding because in that case $[A]^n = \emptyset$, and that implies $\rho \cap [A]^n = \Gamma_n \cap [h[A]]^n = \emptyset$. If $|A| \geq n$, then we define the embedding f recursively. First, pick any elements $x_1, \dots, x_{n-1} \in \mathcal{H}_n$ and define $f_{n-1}(a_i) = x_i$ for $1 \leq i \leq n-1$ (f_{n-1} is an embedding according to the previous considerations in this paragraph). Assume that an embedding $f_l : \{a_1, \dots, a_l\} \to \mathcal{H}_n$ $(n-1 \leq l)$ is given. Define the set $B \subset [f_l[\{a_1, \dots, a_l\}]]^{n-1}$ in the following way:

$$b \in B \iff \{a_{l+1}\} \cup f_l^{-1}[b] \in \rho. \tag{2.2}$$

Pick an element $x_{l+1} \in \mathcal{H}_n \setminus f_l[\{a_1, \dots, a_l\}]$ such that $\forall b \in B \;\; \{x_{l+1}\} \cup b \in \Gamma_n$ and that $\forall b \in [f[\{a_1, \dots, a_l\}]]^{n-1} \setminus B \;\; (\{x_{l+1}\} \cup b \notin \Gamma_n)$. Then, the mapping $f_{l+1} : \{a_1, \dots, a_{l+1}\} \to \{x_1, \dots, x_{l+1}\}$, given by $f_{l+1}(y) = f_l(y)$ $(y \in \{a_1, \dots, a_l\})$ and $f_{l+1}(a_{l+1}) = x_{l+1}$, is clearly an embedding which is an extension of f_l (again, the condition (2.2) ensures that f_{l+1} is an embedding). If we proceed in the same way for all $l \geq n-1$, then $f = \bigcup_{l \geq n-1} f_l : A \to \mathcal{H}_n$ is an embedding and the lemma is proved. □

For the rest of the paper, we will denote the countable random n-uniform hypergraph by $\mathcal{H}_n$.

3. Main theorem

In this section we prove the central result of this article by constructing the specific representation of $\mathcal{H}_n$ in order to easily locate its isomorphic substructures. We essentially plan to use Theorem 1.4 so pick any partition $(0,1)\cap\mathbb{Q} = \bigcup_{m\in\omega} J'_m$ into countably many pairwise disjoint dense sets. Now define the sets $J_m = J'_m + \mathbb{Z}$ for every $m \in \mathbb{Z}$. It is clear that the family $\{J_m : m \in \mathbb{Z}\}$ is a partition of the rational line into pairwise disjoint dense sets such that if $x \in J_0$, then $x + k \in J_0$ for any $k \in \mathbb{Z}$.

Let $\mathbb{P}$ be the partial order of all finite k-uniform hypergraphs $p = \langle H_p, \Gamma_p\rangle$ (i.e. H_p is a set and $\Gamma_p \subset [H_p]^n$) such that $H_p \subset \mathbb{Q}$ ($\mathbb{Q}$ is the rational line) and that for all $a, a-1, \dots, a-n+1, b \in \mathbb{Q}$ it holds:

$$\forall A \in [\{a, a-1, \dots, a-n+1\}]^{n-1} \quad \{b\} \cup A \in \Gamma_p \Rightarrow b > a. \tag{3.1}$$

For p, q in $\mathbb{P}$, we put

$$p \leq q \iff H_p \supset H_q \ \wedge\ \Gamma_p \cap [H_q]^n = \Gamma_q. \tag{3.2}$$

Hence, $p \leq q$ if and only if q is a substructure of p.

Lemma 3.1. *The set $\mathbb{P}$ with the relation $\leq$ on $\mathbb{P}$ is a partially ordered set.*

Proof. The reflexivity is clear. For transitivity notice that if $p \leq q$ and $q \leq r$ we have $H_r \subset H_q \subset H_p$ and $\Gamma_p \cap [H_q]^n = \Gamma_q$ and $\Gamma_q \cap [H_r]^n = \Gamma_r$, and it is easy to see that $\Gamma_p \cap [H_r]^n = \Gamma_r$. To see that $\leq$ is antisymmetric notice that if $p \leq q$ and $q \leq p$, then from $H_p \subset H_q \subset H_p$ follows $H_p = H_q$ and then $\Gamma_p = \Gamma_p \cap [H_p]^n = \Gamma_p \cap [H_q]^n = \Gamma_q$, or equivalently $p = q$. □

Lemma 3.2. *Let $m \in \mathbb{N}$, $A \in [\mathbb{Q}]^{<\omega} \setminus \bigcup_{i<n-1}[\mathbb{Q}]^i$ and $B \subset [A]^{n-1}$. Then*

$$\begin{aligned} D_B^{A,m} = \ & \{p \in \mathbb{P} : \exists q \in (\max A, \max A + \tfrac{1}{m}) \cap J_0 \\ & \forall b \in B\ (\{q\} \cup b \in \Gamma_p)\ \forall b \in ([A]^{n-1} \setminus B)\ (\{q\} \cup b \notin \Gamma_p)\} \end{aligned}$$

is a set dense in $\mathbb{P}$.

Proof. Take any $p \in \mathbb{P}$ and assume that $A \subset H_p$ (if not, define $H_{p_2} = H_p \cup A$ and $\Gamma_{p_2} = \Gamma_p$ and continue with p_2 instead p). Because $\mathbb{Q}$ is a dense linear ordering there is:

$$q \in \big((\max A, \max A + \tfrac{1}{m}) \cap \mathbb{Q}\big) \setminus \bigcup_{a\in H_p} \bigcup_{k\in(-n,n)\cap\mathbb{Z}} \{a + k\}\,. \tag{3.3}$$

Define p_1 in the following way:

- $H_{p_1} = H_p \cup \{q\}$;
- $\Gamma_{p_1} = \Gamma_p \cup \{\{q\} \cup b : b \in B\}$.

It is clear that if $p_1 \in \mathbb{P}$, then $p_1 \in D_B^{A,m}$ and $p_1 \leq p$. Now we prove that $p_1 \in \mathbb{P}$. Assume the contrary, i.e. that for some $a, a-1, \dots, a-n+1, b \in H_{p_1}$ we have:

$$\exists A \in [\{a, a-1, \dots, a-n+1\}]^{n-1} \quad \{b\} \cup A \in \Gamma_p \wedge b \leq a. \tag{3.4}$$

There are three possibilities (regarding the position of q with respect to b and A which exists by (3.4)):

(1) $q = a$ which is not possible because in that case $q = (a-1)+1$ with $a-1 \in A \subset H_p$. Contradiction with the choice of q.
(2) $q = a-k$ $(0 < k < n)$ which is impossible because in this case $q = (a-(k-1))-1$ with $a-(k-1) \in H_p$. Again contradiction with the choice of q.
(3) $q = b$ (so $q \leq a$). In this case we have that $a \in A$, but $q > \max A$ which is a contradiction.

Hence, $p_1 \in \mathbb{P}$ and the lemma is proved. □

Since there are only countably many positive integers and only countably many finite subsets of the rational line, there are countably many sets $D_B^{A,m}$, and according to Lemma 1.3 there is a filter G in $\mathbb{P}$ such that $G \cap D_B^{A,m} \neq \emptyset$ for each $A \in [\mathbb{Q}]^{<\omega} \setminus \bigcup_{i<n-1}[\mathbb{Q}]^i, B \subset [A]^{n-1}, m \in \mathbb{N}$. Define $\Gamma = \bigcup_{p\in G}\Gamma_p$. Because $\Gamma_p \subset [\mathbb{Q}]^n$ for all $p \in G$, we have that $\Gamma \subset [\mathbb{Q}]^n$ so $\langle \mathbb{Q}, \Gamma\rangle$ is a countable n-uniform hypergraph. Notice also that for each $p \in G$ we have that:

$$\Gamma \cap [H_p]^n = \Gamma_p. \tag{3.5}$$

It is clear that $\Gamma_p \subset [H_p]^n \cap \Gamma$ (from the definition of Γ), so assume that for some $p \in G$ there is some $a = \{a_1, \ldots, a_n\} \in (\Gamma \cap [H]^n) \setminus \Gamma_p$. Because $a \in \Gamma$ there is some $r \in G$ such that $a \in \Gamma_r$. Since G is a filter, there is some $t \in G$ such that $t \leq p, r$, i.e. t is an extension of both p and r. Because $a \notin \Gamma_p$, from (3.2) we conclude that $a \notin \Gamma_t$. However, because $a \in \Gamma_r$, again from (3.2) we conclude that $a \in \Gamma_t$ which is a contradiction so (3.5) holds.

Now, using Lemma 2.2, we prove that $\langle \mathbb{Q}, \Gamma\rangle$ is isomorphic to the countable random n-uniform hypergraph $\mathcal{H}_n$. Take any finite $A \subset \mathbb{Q}$ such that $|A| \geq n-1$ and $B \subset [A]^{n-1}$. The set $D_B^{A,1}$ is dense in $\mathbb{P}$ so there is some $p \in G \cap D_B^{A,1}$. According to the definition of Γ we have that $\Gamma_p \subset \Gamma$, hence there is some $q > \max A$ (which implies $q \notin A$) such that for all $b \in B$ we have $\{q\} \cup b \in \Gamma_p$ (which implies $\{q\} \cup b \in \Gamma$) and that for all $b \in [A]^{n-1} \setminus B$ we have $\{q\} \cup b \notin \Gamma_p$ (which, according to (3.5), implies $\{q\} \cup b \notin \Gamma$). So by Lemma 2.2 we have $\langle \mathbb{Q}, \Gamma\rangle \cong \mathcal{H}_n$.

Lemma 3.3. *If $\mathcal{H}_n$, $n > 1$, is the countable random n-uniform hypergraph, then there exists a positive family $\mathcal{P}$ on $\mathcal{H}_n$ such that $\mathcal{P} \subset \mathbb{P}(\mathcal{H}_n)$.*

Proof. We will prove that

$$\mathcal{P} = \left\{\mathbb{Q} \setminus \bigcup_{m\in\mathbb{Z}} F_m : \forall m \in \mathbb{Z} \;\; F_m \in [[m, m+1)]^{<\omega}\right\}$$

is a positive family in $\mathbb{P}(\mathbb{Q}, \Gamma)$ (note that each element of $\mathcal{P}$ is given by different choice of the collection $\{F_m : m \in \mathbb{Z}\}$). Take any $X \in \mathcal{P}$. We will show that X satisfies the conditions of Lemma 2.2. Take any finite $A \subset X$ such that $|A| \geq n-1$ and any $B \subset [A]^{n-1}$. First we find $m_0 \in \mathbb{Z}$ such that $\max A \in [m_0, m_0+1)_{\mathbb{Q}}$. This m_0 clearly exists because A is a finite set. Also, because F_{m_0} is a finite set and $A \cap F_{m_0} = \emptyset$, there is an $m \in \mathbb{N}$ such that $(\max A, \max A + \frac{1}{m}) \cap F_{m_0} = \emptyset$, i.e. $(\max A, \max A + \frac{1}{m}) \cap \mathbb{Q} \subset X$. Now, because the set $D_B^{A,m}$ is dense in $\mathbb{P}$ there is some $p \in G \cap D_B^{A,m}$. Lemma 3.2 states that there is some $q \in X$ such that $\forall b \in B$ $\{q\} \cup b \in \Gamma_p \subset \Gamma$ and $\forall b \in ([A]^{n-1} \setminus B)$ $\{q\} \cup b \notin \Gamma_p = \Gamma \cap [H_p]^n$. Hence for each $X \in \mathcal{P}$ we have that $X \cong \mathcal{H}_n$.

To conclude the proof we should still show that $\mathcal{P}$ is a positive family on $\mathbb{Q}$. The condition (P1) is clearly satisfied because only finitely many points are removed from each bounded interval in $\mathbb{Q}$ to obtain elements of $\mathcal{P}$. For the same reason (P2) and (P3) are also satisfied. The set $\mathbb{Q} \setminus \bigcup_{m_0 \in \mathbb{Z}} \{m_0\}$ is in $\mathcal{P}$ and witnesses that the condition (P4) is true. □

In order to apply Theorem 1.4, we have to show that open intervals are copies of $\mathcal{H}_n$ while closed intervals are not.

Lemma 3.4. *It holds:*

1. $(-\infty, x)_{J_0} \subset A \subset (\infty, x)_{\mathbb{Q}}$ *implies* $\langle A, \Gamma\rangle \cong \langle \mathbb{Q}, \Gamma\rangle$ *for* $x \in \mathbb{R} \cup \{\infty\}$*;*
2. $(-\infty, q]_{J_0} \subset C \subset (\infty, q]_{\mathbb{Q}}$ *implies* $\langle C, \Gamma\rangle \ncong \langle \mathbb{Q}, \Gamma\rangle$ *for* $q \in J_0$.

Proof. To prove (1) take any finite $X \subset A$ such that $|X| \geq n-1$ and take $B \subset [X]^{n-1}$. There is some $m \in \mathbb{N}$ such that $\max X + \frac{1}{m} < \sup A = x$ (this can be done because of the choice of A and J_0). Now, because the set $D_B^{X,m}$ is dense in $\mathbb{P}$, there is some $p \in G \cap D_B^{X,m}$. In p there is some $q \in (\max X, \max X + \frac{1}{m}) \cap J_0 \subset A$ such that $\forall b \in B$ $(\{q\} \cup b \in \Gamma_p \subset \Gamma)$ and that $\forall b \in [X]^{n-1} \setminus B$ $(\{q\} \cup b \notin \Gamma_p = \Gamma \cap [H_p]^n)$. So according to Lemma 2.2, A is isomorphic to $\mathcal{H}_n$.

To prove (2) consider the set $Y = \{q, q-1, \dots, q-n+1\} \subset C$ (we have that $Y \subset C$ because of the choice of the partition $\{J_m : m \in \omega\}$). Now, if $\langle C, \Gamma\rangle$ is isomorphic to $\mathcal{H}_n$, then there is an element $b \in C$ such that $\forall X \in [Y]^{n-1}$ $(\{b\} \cup X \in \Gamma)$. According to the definition of Γ, for each $X \in [Y]^{n-1}$ there is some $p_X \in G$ such that $\{b\} \cup X \subset H_{p_X}$. Because G is a filter there is some $p \leq p_X$ for all $X \in [Y]^{n-1}$. In this p we have that $\forall X \in [Y]^{n-1}$ $(\{b\} \cup X \in \Gamma)$ yet $b \leq \max C = q$. But, this is a contradiction to the definition of $\mathbb{P}$ (condition (3.1)). □

Now we can prove the main result of this article.

Theorem 3.5. *For a linear order L, the following conditions are equivalent.*

1. *L is a complete, $\mathbb{R}$-embeddable linear order with* $\min L$ *non-isolated;*
2. *L is isomorphic to a maximal chain in the poset* $\langle \mathbb{P}(\mathcal{H}_n) \cup \{\emptyset\}, \subset\rangle$*;*
3. *L is isomorphic to a compact set K of reals such that* $\min K \in K'$.

Proof. The equivalence of (1) and (3) was shown in [9], while the implication (2)⇒(1) follows from Theorem 1.5.

To prove implication (1)⇒(2) note that from the choice of partition $\{J_m : m \in \omega\}$ and according to Lemma 3.4, conditions (i)-(iv) of Theorem 1.4 are satisfied. Also, Lemma 3.3 proves that the condition (v) of Theorem 1.4 is satisfied. Hence, Theorem 1.4 implies that (1)⇒(2) is proved. □

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