On a certain generalization of first-countable spaces

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Motivation - products of Baire spaces

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 $X, Y \ Baire \ spaces \Rightarrow X \times Y \ Baire ???$

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3 / 17

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 - \bigcirc If X is separable then $\{X\}$ is a rich family in X \bigcirc

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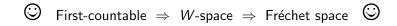
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• A *W*-space which is not first-countable:

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W-spaces \widetilde{W} -spaces

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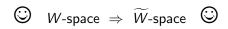
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The point x is an accumulation point of this sequence.



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Proof.

A certain Σ -product of uncountably many copies of the space $\beta\mathbb{N}$ works.



Further applications of W-spaces

Theorem

Let $f: X \times Y \to Z$ be separately continuous. Suppose that X is a Baire space, Z is regular, and $y_0 \in Y$ is a \widetilde{W} -point. Then f is quasi-continuous at each point of $X \times \{y_0\}$.

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(separately continuous \equiv separately continuous in each coordinate,

f is quasi-continuous at $p \equiv$ for every open sets $U \ni p$ and $W \ni f(p)$ there is an open set $\emptyset \neq V \subseteq U$ such that $f(V) \subseteq W$)

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Corollary

Let G be a semitopological group. Suppose that G is a regular Baire \widetilde{W} -space and a Δ -Baire space. Then G is a topological group.

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Let $f: X \times Y \to Z$ be separately continuous. Suppose that X is a Baire space, Y is a \widetilde{W} -space which possesses a rich family of Baire spaces, and Z is a regular space that is fragmented by some metric whose topology contains the topology of Z. Then f is continuous at the points of a dense G_{δ} -subset of $X \times Y$.

The End

17 / 17