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GENERIC REPRESENTATIONS OF COUNTABLE GROUPS

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Abstract. The paper is devoted to a study of generic representations (homomorphisms) of discrete countable groups Γ in Polish groups G, i.e. those elements in the Polish space $\mathrm{Rep}(\Gamma, G)$ of all representations of Γ in G, whose orbit under the conjugation action of G on $\mathrm{Rep}(\Gamma, G)$ is comeager. We investigate finite approximability, a strictly related notion, for actions on countable structures such as tournaments, triangle-free graphs, and, more generally, K_n-free graphs, and we show how it is related to the Ribes-Zalesski-like properties of the acting groups. We prove that every finitely generated abelian group has a generic representation in the automorphism group of the random tournament; in particular, that there is a comeager conjugacy class in this group. We also provide a simpler proof of a recent result of Glasner, Kitroser and Melleray characterizing groups having a generic permutation representation.

Then we investigate representations of infinite groups Γ in automorphism groups of metric structures such as the isometry group $\mathrm{Iso}(\mathbb{U})$ of the Urysohn space, isometry group $\mathrm{Iso}(\mathbb{U}_1)$ of the Urysohn sphere, or the linear isometry group $\mathrm{LIso}(\mathbb{G})$ of the Gurarii space. We show that the conjugation action of $\mathrm{Iso}(\mathbb{U})$ on $\mathrm{Rep}(\Gamma, \mathrm{Iso}(\mathbb{U}))$ is generically turbulent.

Introduction

A *representation* of a countable, discrete group Γ in a Polish (i.e., separable and completely metrizable) topological group G is a homomorphism of Γ into G. The most frequently studied representations are finite-dimensional representations, i.e. homomorphisms into the matrix groups $\mathrm{GL}(n, \mathbb{K})$, where $n \in \mathbb{N}$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and then unitary representations, i.e., homomorphisms into the unitary group $U(H)$ of separable Hilbert spaces H.

Especially within descriptive set theory, other interesting cases have been recently considered as well, e.g., representations in the isometry group $\mathrm{Iso}(\mathbb{U})$ of the Urysohn space (see [18]) or representations in the symmetric group S_∞ of a countable set (see [6].) As a matter of fact, representations in automorphism groups of certain structures are nothing but actions on these

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structures, as is the case with the above mentioned examples, where representations are actions on finite-dimensional vector spaces, Hilbert spaces, the Urysohn space $\mathbb{U}$ or a countable set (with no structure.)

Instead of a single representation, one can also investigate the space of all representations $\mathrm{Rep}(\Gamma, G)$, which can be equipped with a natural Polish topology. The action of G on $\mathrm{Rep}(\Gamma, G)$ by conjugation leads to the concept of *generic representation*, i.e., a representation whose orbit is comeager in $\mathrm{Rep}(\Gamma, G)$. As it turns out, such a global approach can provide insight both into the structure of Γ and the structure of G, and it reveals interesting connections between the theory of countable groups, the theory of Polish groups, and model theory.

An example of such a connection is the notion of *ample generics.* A Polish group G has ample generics if every free group F_n on n generators has a generic representation in G. Groups with ample generics satisfy certain very strong properties, e.g., the automatic continuity property, which means that all (abstract) group homomorphisms from these groups into separable groups are continuous. On the other hand, ample generics are related to the *Hrushovski property* that has been extensively studied in the context of Fraïssé theory. Recall that a Fraïssé class of structures $\mathcal{K}$ has the Hrushovski property if for every $A \in \mathcal{K}$ there is $B \in \mathcal{K}$ containing A, and such that every partial automorphism of A can be extended to an automorphism of B. It turns out that for a Fraïssé class $\mathcal{K}$ with sufficiently free amalgamation, the Hrushovski property implies the existence of ample generics in the automorphism group $\mathrm{Aut}(M)$ of its limit M, i.e., implies that F_n has a generic representation in $\mathrm{Aut}(M)$ for every $n \geq 1$.

Another aspect of this phenomenon has been revealed in the works of Herwig and Lascar [9] who showed that the Hrushovski property is strictly related to the Ribes-Zalesskii property for free groups, which, in turn, is tied up with the pro-finite structure of countable groups. Later Rosendal [22] proved that a countable group Γ has the Ribes-Zalesskii property if and only if every action of Γ on a metric space is *finitely approximable*, i.e., every action of Γ on a metric space X can be approximated by actions of Γ on finite metric spaces. It is not hard to see that for free groups the latter statement is equivalent to the Hrushovski property for metric spaces.

In the present paper, we continue this line of research. Although we work with representations in automorphism groups of structures from several areas of mathematics such as graphs, metric spaces, Banach spaces, etc., the paper is not just a collection of separate results. There is a unified methodology behind all our results. That is, whenever we construct some generic representation of some countable group Γ, it is a Fraïssé limit of some 'simple actions' of Γ. On the other hand, whenever we show that some group Γ does not have a generic representation in an automorphism group of a structure of a given type, it is essentially by showing that there are too many actions of Γ on structures of this given type, i.e. one cannot construct a Fraïssé limit of actions which would be dense. It is possible that this approach could be formalized and we partially do so for generic actions with finite orbits in the second section.

Let us present the main results of the paper. First of all, we study finite approximability for several classes of countable structures, namely tournaments, and K_n-free graphs. We prove (Theorem 1.1) that all actions of finitely generated abelian groups on tournaments are finitely approximable. We also formulate a property closely related to the Ribes-Zalesskii property (Definition 1.7), and prove (Theorem 1.8) that it implies finite approximability for tournaments. We leave it open whether finitely generated free groups have this property, which would imply the Hrushovski property for tournaments and ample generics for the automorphism group of the random tournament. Then we turn to triangle-free, and more generally K_n-free graphs, $n \geq 3$. We show (Theorem 1.11, and Theorem 1.10) that the 2-Ribes-Zalesskii property, and the 3-Ribes-Zalesskii property, which are weak versions of the Ribes-Zalesskii property, form the lower and the upper 'group-theoretic bounds' for finite approximability of actions on triangle-free graphs (resp. K_n-free graphs).

We also give in that section a simpler Fraïssé -theoretic proof of the main result from [6] that says that a countable, discrete group Γ has a generic representation in S_∞ if and only if it is solitary.

In the next section, we generalize known results relating, for finitely generated groups, finite approximability and generic representations with finite orbits. We prove (Theorem 2.1) that for every sufficiently regular Fraïssé class $\mathcal{K}$ (to be more specific, for every Fraïssé class with amalgamation allowing for amalgamating partial automorphisms), with Fraïssé limit M, every action of a finitely generated group Γ on M is finitely approximable if and only if Γ has a generic representation in $\mathrm{Aut}(M)$ with finite orbits. In particular, this theorem, combined with our results on tournaments, implies that there is a comeager conjugacy class in the automorphism group $\mathrm{Aut}(\mathcal{T})$ of the random tournament $\mathcal{T}$.

We also study generic representations in the automorphism groups of several metric structures, namely the Urysohn space $\mathbb{U}$, the Urysohn sphere $\mathbb{U}_1$ and the Gurarij space $\mathbb{G}$. We show that no countably infinite discrete group has a generic representation in $\mathrm{Aut}(X)$, where $X \in \{\mathbb{U}, \mathbb{U}_1, \mathbb{G}\}$. After the first version of this paper was written, we were informed by Julien Melleray that he had already proved the last result for the Urysohn space and sphere in his habilation thesis, published in [19]. The result for the sphere is however stated without a proof there, so we decided to publish it here (as well as the result for the whole Urysohn space, which is simpler). This allows us to only sketch the proof for the Gurarij space as it is analogous to that one for the sphere. This answers a question of Melleray from [19]. We moreover show additionally that the conjugation action of $\mathrm{Iso}(\mathbb{U})$ on $\mathrm{Rep}(\Gamma, \mathbb{U})$ is generically turbulent, for any infinite Γ. The same ideas also work for the Urysohn sphere and the Gurarij space.

Terminology and basic facts

Let $\mathcal{K}$ be a countable, up to isomorphism, class of finite structures in a given language $\mathcal{L}$. We say that $\mathcal{K}$ is a *Fraïssé class* if it has the hereditary property HP (for every $A \in \mathcal{K}$, if B can be embedded in A, then $B \in \mathcal{K}$), the joint embedding property JEP (for any $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ which embeds both A and B), and the amalgamation property AP (for any $A, B_1, B_2 \in \mathcal{K}$ and any embeddings $\phi_i \colon A \to B_i$, $i = 1, 2$, there are $C \in \mathcal{K}$ and embeddings $\psi_i \colon B_i \to C$, $i = 1, 2$, such that $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$). By a theorem due to Fraïssé , for every Fraïssé family $\mathcal{K}$, there is a unique up to isomorphism countable ultrahomogeneous structure M (i.e., isomorphisms between finite substructures of M extend to automorphisms of M) which embeds every member of $\mathcal{K}$, and $\mathcal{K} = \mathrm{Age}(M)$, where $\mathrm{Age}(M)$ is the class of all finite structures that can be embedded in M. In that case, we call M the *Fraïssé limit* of $\mathcal{K}$, see [11, Section 7.1]. As a matter fact, M can be also characterized by its *extension property*: a locally finite countable structure X with $\mathrm{Age}(X) \subseteq \mathcal{K}$ is the Fraïssé limit of $\mathcal{K}$ if and only if for any $A, B \in \mathcal{K}$, and embeddings $\phi : A \to X$, $i : A \to B$ there exists an embedding $\psi : B \to X$ such that $\psi \circ i = \phi$.

A locally finite structure X such that $\mathrm{Age}(X) \subseteq \mathcal{K}$, for some class of finite structures $\mathcal{K}$, is called a *chain* from $\mathcal{K}$. Suppose that α is an action by automorphisms of a group Γ on a chain X from a class $\mathcal{K}$. We will say that α is *finitely approximable* (by structures from $\mathcal{K}$) if for every finite $F \subseteq \Gamma$, and $X_0 \subseteq X$ there exists $Y \in \mathcal{K}$, an action by automorphisms β of Γ on Y, and an injection $e : \alpha[F \times X_0] \to Y$ that embeds α restricted to F and X_0 into β, i.e.,

$$\beta(f, e(x)) = e(\alpha(f, x))$$

for $f \in F$, and $x \in X_0$. Note that when Γ is the free group on finitely many generators then this corresponds to the well-studied problem when a tuple of finite partial automorphisms extends to a tuple of finite automorphisms. This was first proved by Hrushovski in [12] for graphs and it is open till today for tournaments.

Denote by $\mathcal{A}$ the class of all chains from $\mathcal{K}$. We say that a Fraïssé class $\mathcal{K}$ has the Katětov functor if for any $X \in \mathcal{A}$ there are embeddings $\phi_X : X \hookrightarrow M$ and $\psi_X : \mathrm{Aut}(X) \hookrightarrow \mathrm{Aut}(M)$, where M is the Fraïssé limit of $\mathcal{K}$, such that for every $f \in \mathrm{Aut}(X)$ and $x \in X$ we have

$$\phi_X \circ f(x) = \psi_X(f) \circ \phi_X(x).$$

Note that a prototypical example is the metric Fraïssé class of metric spaces where it follows from the result of Uspenskij in [24] that this class has a Katětov functor. See [16] for a reference on this topic.

Let Γ be a countable discrete group, and let G be a Polish grpup. The space $\mathrm{Rep}(\Gamma, G)$ of all representations of Γ in G (i.e., homomorphisms of Γ into G) can be naturally endowed with a Polish topology by regarding it as a (closed) subspace of G^{Γ}. When G is the automorphism group of some structure X, we usually write $\mathrm{Rep}(\Gamma, X)$ instead of the more precise $\mathrm{Rep}(\Gamma, \mathrm{Aut}(X))$. We will say that $\alpha \in \mathrm{Rep}(\Gamma, G)$ is a *generic representation* if the orbit of α under the conjugation action of G on $\mathrm{Rep}(\Gamma, G)$ is comeager in $\mathrm{Rep}(\Gamma, G)$. Note that in most cases we encounter the topological 0-1 law for representations is valid, i.e. either there is a generic representation in $\mathrm{Rep}(\Gamma, G)$, or all the conjugacy classes are meager. This is the case e.g. when there is a dense conjugacy class in $\mathrm{Rep}(\Gamma, G)$ (see Theorem 8.46 in [13]).

1. Finite approximability

We recall that if Γ is a discrete group, the profinite topology on Γ is the group topology on Γ generated by the basic open sets gK, where $g \in \Gamma$, and K is a finite index normal subgroup of Γ. Thus, a subset $S \subseteq \Gamma$ is closed in the profinite topology on Γ if for any $g \in \Gamma \setminus S$, there is a finite index normal subgroup $K \leq \Gamma$ such that $g \notin SK$. Since this is a group topology, i.e., the group operations are continuous, Γ is Hausdorff if and only if $\{1\}$ is closed, i.e., if for any $g \neq 1$ there is a finite index normal subgroup K not containing g. In other words, Γ is Hausdorff if and only if it is residually finite. A stronger notion than residual finiteness is subgroup separability or being LERF (locally extended residually finite). Here a group Γ is subgroup separable, or LERF, if any finitely generated subgroup $H \leq \Gamma$ is closed in the profinite topology on Γ. An even stronger notion is what we shall call the n-Ribes-Zalesskii property, where $n \in \mathbb{N}$, or n-RZ property for brevity. Namely, for a fixed $n \in \mathbb{N}$, a group Γ is said to have the n-RZ property if any product $H_1 H_2 \dots H_n$ of finitely generated subgroups of Γ is closed in the profinite topology on Γ. Finally, Γ has the RZ property if it has the n-RZ property for every n. We refer to the paper [21] of Ribes and Zalesskii, where they prove that free groups have the RZ property.

In [22], Rosendal used the RZ property and its variants to characterize finite approximability of actions on metric spaces or graphs. In the following sections, we show that that this concept turns out to be useful also in studying K_n-free graphs (where K_n is a clique on n elements), and tournaments (recall that a tournament is a directed graph (X, R) such that for any $x, y \in X$ exactly one of the arrows (x, y), (y, x) is in R.)

1.1. **Tournaments.**

Theorem 1.1. *Let Γ be a finitely generated abelian group. Then every action of Γ on a tournament is finitely approximable.*

Proof. Fix an action α of Γ on a tournament (X, R). Fix a finite, symmetric $F \subseteq \Gamma$, and $A \subseteq X$. By possibly enlarging F, we can assume that no two elements of A are in the same orbit under α.

For $x \in A$, let $H_x \leq \Gamma$ be the stabilizer of x under α. Find subgroups $H'_x \leq \Gamma$, $x \in A$, such that, for $L_x = H_x + H'_x$ the following conditions are satisfied:

(1) the order of M_x is finite, and an odd number,
(2) the quotient mapping $F/H_x \mapsto (F/H_x)/H'_x \subseteq \Gamma/L_x$ is injective.

In order to see that it is always possible to find such H'_x, observe (or see Lemma 1.4 below) that, for no $x \in X$, there exists an element of even order in Γ/H_x. Indeed, if some $g \in \Gamma/H_x$ had even order, there would exist $h \in \Gamma$ such that $h^2.x = x$. But then, by invariance of the relation R under α, we would get that both $R(x, h.x)$ and $R(h.x, x)$ hold, which is impossible. Now, as Γ is finitely generated, each Γ/H_x is also finitely generated, and so it can be written as a direct sum $\bigoplus_i K_i$ of cyclic groups K_i. Moreover, by the above observation, each K_i with finite order has in fact order which is an odd number. Now it is straightforward to choose H'_x as required.

We define a binary relation S on $Y = \bigsqcup_{x \in A} \Gamma/L_x$ in the following way. Fix a linear ordering $\preceq$ on A. Fix $x \preceq y \in A$. Let $B_y, C_y \subseteq \Gamma/L_y$ be a partition of M_y such that

(a) $g + L_y \in B$ iff $-g + L_y \in C_y$,
(b) for $g \in F$, $g + L_y \in C_y$ iff $R(x, g.y)$.

Condition (a) can be met because of (1) (which implies there are no elements of order 2 in M_y), while (b) is satisfiable because of (2), and our assumption that F is symmetric. Then, for $f, g \in \Gamma$, we define

$$S(f + L_x, g + L_y) \text{ iff } (g - f) + L_y \in C_y,$$

$$S(g + L_y, f + L_x) \text{ iff } (g - f) + L_y \in B_y.$$

Observe that S defines a tournament on Y. Indeed, suppose that both $S(f + L_x, g + L_y)$ and $S(g + L_y, f + L_x)$ hold for some $f, g \in \Gamma$ and $x \preceq y \in A$. But then, by the definition of S, we have that $(g - f) + L_y \in C_y$, $(g - f) + L_y \in B_y$, which is impossible because $B_y \cap C_y = \emptyset$. Similarly, if $S(f + L_x, g + L_y)$ does not hold, then $(g - f) + L_y \notin C_y$. But then $(g - f) + L_y \in B_y$ because $B_y \cup C_y = \Gamma/L_y$, so $S(g + L_y, f + L_x)$ holds.

Clearly, (Y, S) is invariant under the action β of Γ on Y by left-translation. Moreover, (b) implies that the mapping $f.x \mapsto f + L_x$, $f \in F$, $x \in A$, is an embedding of α into β, when α is restricted to F on Γ, and A on X.

□

As a matter of fact, there exists a general algebraic condition on a group Γ, which implies finite approximability for tournaments. We start with the following definition.

Definition 1.2. Let Γ be a countable group and let $H \leq \Gamma$. We say H is *good* if there are no $g \in \Gamma \setminus H$ and $h_1, h_2 \in H$ such that $g \cdot h_1 \cdot g \cdot h_2 \in H$.

Remark 1.3. Clearly, if Γ is abelian, then $H \leq \Gamma$ is good if and only if Γ/H does not have elements of order 2.

The motivation for this definition comes from the following lemma.

Lemma 1.4. *Let Γ be a countable group and let $H \leq \Gamma$. Then H is good if and only if H is a point-stabilizer for some action of Γ on a tournament.*

Proof. Suppose Γ acts on a tournament (X, R) and H is the stabilizer of some $x \in X$. Suppose there are $g \in \Gamma \setminus H$ and $h_1, h_2 \in H$ such that $gh_1gh_2 \in H$. Without loss of generality, assume that we have $R(x, g.x)$. Since $gh_1gh_2 \in H$, and the action of Γ preserves the tournament relation R, we have

$$R(x, g.x) \Leftrightarrow R(gh_1gh_2.x, g.x) \Leftrightarrow$$

$$R(h_1gh_2.x, x) \Leftrightarrow R(g.x, x);$$

a contradiction.

Conversely, assume that H is good. We define a tournament structure on Γ/H, on which Γ will act canonically. Using Zorn's lemma find a maximal subset $F \subseteq \Gamma$ satisfying that for no $f, g \in F$ there are $h_1, h_2 \in H$ such that $fh_1gh_2 \in H$. Next we set $R(fH, gH)$ if and only if there are $h_1, h_2 \in H$ and $g' \in F$ such that $f^{-1}g = h_1g'h_2$. Clearly, the action of Γ preserves the relation R, so we must check that it is a tournament relation. Suppose there is $g \in \Gamma \setminus H$ such that both $R(H, gH)$ and $R(gH, H)$ hold true. Then also $R(H, g^{-1}H)$ holds, so there are $h_1, h_2, h_3, h_4 \in H$ such that $h_1gh_2, h_3g^{-1}h_4 \in F$. This clearly violates the condition imposed on F. So suppose now that there is $g \in \Gamma \setminus H$ such that neither $R(H, gH)$ nor $R(H, g^{-1}H)$ hold true. Then we claim we may add g into F contradicting the maximality of F. Indeed, suppose that by adding g into F we violate the condition imposed on F. Since H is good, we cannot have that $gh_1gh_2 \in H$ for some $h_1, h_2 \in \Gamma$. So there are $f \in F$ and $h_1, h_2 \in H$ such that $gh_1fh_2 \in H$ or $fh_1gh_2 \in H$. Assume the former case, the latter one is dealt with analogously. Then we have $gh_1fh_2 = h$ for some $h \in H$, so $g^{-1} = h_1fh_2h^{-1}$, so $R(H, g^{-1}H)$; a contradiction. ☐

We record some basic properties of good subgroups.

Lemma 1.5. *Let Γ be a countable group. We have*

(1) If $(H_i)_{i \in I}$ is a collection of good subgroups of Γ, then $\bigcap_{i \in I} H_i$ is also good.

(2) If $H \leq \Gamma$ is a good subgroup, then the maximal normal subgroup of Γ contained in H is also good. In particular, if H is a good subgroup of finite index, then there is a good normal subgroup of finite index.

Proof. (1) Suppose $(H_i)_{i \leq I} \leq \Gamma$ are good. Set $H = \bigcap_{i \in I} H_i$ and take some $g \in \Gamma \setminus H$ such that there are $h_1, h_2 \in H$ with $gh_1gh_2 \in H$. There exist $i \in I$ such that $g \notin H_i$. Then however $gh_1gh_2 \in H \leq H_i$ which violates that H_i is good.

(2): Suppose that $H \leq \Gamma$ is a good subgroup. The maximal normal subgroup of Γ contained in H is equal to $\bigcap_{g\in\Gamma} g^{-1}Hg$, so we are done by (1). □

Question 1.6. *We do not know whether for every countable group Γ if $H \leq \Gamma$ is a good subgroup and $N \leq \Gamma$ is a good normal subgroup, then the subgroup HN is good.*

The aim of the next definition is to isolated a Ribes-Zalesskii-like property of groups that would guarantee finite approximability of tournaments - in the same way as the 2-Ribes-Zalesskii property guarantees finite approximability for graphs. We shall call it tournament 2-Ribes-Zalesskii property as it is apparently very similar to the 2-Ribes-Zalesskii property, however not obviously equivalent, or weaker or stronger.

Definition 1.7. Γ has the *tournament 2-RZ property* if for any $g_i \in \Gamma$, $i \leq n$, finitely generated good subgroups $K_i, H_i \leq \Gamma$, $i \leq n$, such that $g_i \notin K_iH_i$, and any finitely generated good subgroups $M_j \leq \Gamma$, $j \leq m$, there exists a finite index normal subgroup $N \leq \Gamma$ such that

- $g_i \notin K_iH_iN$ for each $i \leq n$ (thus, in particular, Γ has the 2-RZ property),
- M_jN is good for each $j \leq m$.

Theorem 1.8. *Suppose that Γ is a finitely generated group with the tournament 2-RZ property. Then every action of Γ on a tournament is finitely approximable.*

Conversely, if every action of Γ on a tournament is finitely approximable, and, moreover, Question 1.6 has a positive answer for Γ, then Γ has the tournament 2-RZ property.

Proof. Suppose that Γ acts on a tournament (X, R). Take now some finite partial subaction of Γ on X. We may suppose it is given by the following data:

- finitely many orbits $O_1, \ldots, O_m$, each O_j with some base point x_j and a finitely generated stabilizer M_j of x_j;
- for each $j \leq m$ a finite subset $F_j \subseteq \Gamma$ such that for each $f \in F_j$ we have $R(x_j, f.x_j)$;
- for every $i \neq j \leq m$ we have a finite subset $F_{i,j} \subseteq \Gamma$ such that for every $f \in F_{i,j}$ we have $R(x_i, f.x_j)$.

Notice that for every $j \leq m$ and any two elements $f, g \in F_j$ we have

$$fM_jgM_j \cap M_j = \emptyset.$$

Indeed, suppose that $fh_1gh_2 = h_3$, for some $h_1, h_2, h_3 \in M_j$, i.e. $g = hf^{-1}h'$, for some $h, h' \in M_j$. We have $R(x_j, g.x_j)$ and $R(f^{-1}.x_j, x_j)$, thus, since h and h' fix x_j, $R(hf^{-1}h'.x_j, x_j)$ and $R(g.x_j, x_j)$, which is a contradiction.

We can now find a finite index subgroup N satisfying:

(1) for every $j \leq m$ and every $f, g \in F_j$ we have
$$fM_jNgM_jN \cap M_jN = \emptyset;$$
(2) for every $i \neq j \leq m$ and every $f \in F_{i,j}$ and $g \in F_{j,i}$ we have
$$M_ifM_jN \cap M_ig^{-1}M_jN = \emptyset$$
(note that this is equivalent to $gf \notin (gM_ig^{-1})M_jN$);
(3) for every $j \leq m$, M_jN is good.

Indeed, by Lemma 1.4 all the subgroups involved are good, so the tournament 2-RZ property applies.

Now we define a finite tournament extending this finite fragment. The underlying set is
$$Y = \Gamma/(M_1N) \sqcup \ldots \sqcup \Gamma/(M_mN).$$
For every $j \leq m$, let $F'_j \subseteq \Gamma$ be a finite subset satisfying

- $F_j \subseteq F'_j$ (extension);
- for every $f, g \in F'_j$ we have
$$fM_jNgM_jN \cap M_jN = \emptyset \text{ (consistency)};$$
- for every $f \in \Gamma \setminus (F'_jM_jN)$ there exist $g \in F'_j$ and $h_1, h_2 \in M_jN$ such that either $fh_1gh_2 \in M_jN$ or $gh_1fh_2 \in M_jN$ (maximality).

It is possible to find such a set by (1) and (3). Indeed, by (1) we have that that for every $f, g \in F_j$,
$$fM_jNgM_jN \cap M_jN = \emptyset.$$
We want to extend F_j as much as possible still satisfying this consistency property. Suppose that F'_j is a maximal subset containing F_j and satisfying that for every $f, g \in F'_j$ we have
$$fM_jNgM_jN \cap M_jN = \emptyset.$$
Suppose the third condition is not satisfied, i.e. there is $f \in \Gamma \setminus (F'_jM_jN)$ such that for every $g \in F'_j$ and $h_1, h_2 \in M_jN$ neither $fh_1gh_2 \in M_jN$, nor $gh_1fh_2 \in M_jN$. Then we claim we may extend F'_j by f contradicting its maximality. Suppose that not. Then necessarily $fh_1fh_2 \in M_jN$, for some $h_1, h_2 \in M_jN$, which however contradicts (3). Clearly, F'_j is finite.

Analogously, for each $i \neq j \leq m$, let $F'_{i,j} \subseteq \Gamma$ be a finite subset satisfying

- $F_{i,j} \subseteq F'_{i,j}$ (extension);
- for every $i \neq j \leq m$ and every $f \in F'_{i,j}$ and $g \in F'_{j,i}$ we have
$$M_ifM_jN \cap M_ig^{-1}M_jN = \emptyset \text{ (consistency)};$$
- for every $f \in \Gamma$ there are $h_i \in M_i$ and $h_j \in M_j$ such that either $h_ifh_j \in F_{i,j}$ or $h_jf^{-1}h_i \in F_{j,i}$ (maximality).

It is possible to find such sets $F'_{i,j}$ as follows. By (2) we have that $F_{i,j}$ satisfies the consistency condition above, so we again want to extend $F_{i,j}$ as much as possible. Suppose that $j > i$ then we set $F'_{j,i} = F_{j,i}$. So suppose that $i < j$. Then for each double coset $(M_iN\backslash\Gamma/M_jN) \setminus ((\bigcup_{f\in F_{i,j}} M_iNfM_jN) \cup (\bigcup_{g\in F_{j,i}} M_iNg^{-1}M_jN))$ we choose some representative and put it into $F'_{i,j}$.

Now we define the tournament relation S on Y. For any $j \leq m$ and any $f, g \in \Gamma$ we set $S(fM_jN, gM_jN)$ if and only if for some $h \in M_jN$ we have $f^{-1}gh \in F'_j$. We claim that exactly one of the options $S(fM_jN, gM_jN)$ and $S(gM_jN, fM_jN)$ happens. To simplify the notation, we show that for any $f \in \Gamma$ either $S(M_jN, fM_jN)$ or $S(fM_jN, M_jN)$ happens. First we show that at least one of the options happens, then that at most one of them happens.

If $f \in F'_jM_jN$, then $S(M_jN, fM_jN)$ by definition. So suppose that $f \in \Gamma \setminus (F'_jM_jN)$. Then by the maximality condition there exist $g \in F'_j$ and $h_1, h_2 \in M_jN$ such that either $fh_1gh_2 \in M_jN$ or $gh_1fh_2 \in M_jN$. Suppose the former. Then we have

$$S(M_jN, gM_jN) \Leftrightarrow S(M_jN, gh_2M_jN) \Leftrightarrow$$

$$S(fh_1M_jN, fh_1gh_2M_jN) \Leftrightarrow S(fM_jN, M_jN).$$

The latter condition is treated analogously.

Now we show that at most one of the conditions happens. Suppose on the contrary that both $S(M_jN, fM_jN)$ and $S(fM_jN, M_jN)$ hold. Then however $S(M_jN, f^{-1}M_jN)$ holds, therefore there are $h_1, h_2 \in M_jN$ such that $fh_1, f^{-1}h_2 \in F'_j$. This violates the consistency condition though as we have

$$fh_1M_jNf^{-1}h_2M_jN = M_jN.$$

Now for $i \neq j \leq m$ we set $S(fM_iN, gM_jN)$ if there are $h_i \in M_i$, $h_j \in M_j$ and $g' \in F'_{i,j}$ such that $f^{-1}g = h_ig'h_j$. We again claim that exactly one of the options $S(fM_iN, gM_jN)$ and $S(gM_jN, fM_iN)$ happens. Again it suffices to check that for $f \in \Gamma$ exactly one of the options $S(M_iN, fM_jN)$ and $S(fM_jN, M_iN)$ happens. First we show that if at least one of the options happens, then that at most one of them happens.

By the maximality condition there are $h_i \in M_i$ and $h_j \in M_j$ such that either $h_ifh_j \in F_{i,j}$ or $h_jf^{-1}h_i \in F_{j,i}$. In the first case we clearly have that $S(M_iN, fM_jN)$ holds true, while in the latter case we have $S(M_jN, f^{-1}M_iN)$, therefore $S(fM_jN, M_iN)$.

Suppose now that both $S(M_iN, fM_jN)$ and $S(fM_jN, M_iN)$ hold true. Then there are h_i, h_j, h'_i, h'_j such that $h_ifh_j \in F'_{i,j}$ and $h'_jf^{-1}h'_i \in F'_{j,i}$. Then it clearly violates the consistency condition above since

$$M_ih_ifh_jM_j \cap M_i(h')_i^{-1}f(h')_j^{-1}M_jN \neq \emptyset.$$

Now we suppose that we have the Γ-approximability for tournaments and that Question 1.6 has a positive answer for Γ. We show that Γ has the tournament 2-RZ property.

Take some finite number of triples $(g_1, K_1, H_1), \ldots, (g_n, K_n, H_n)$ where $g_i \notin K_i H_i$ and K_i and H_i are finitely generated good subgroups, for $i \leq n$, and some finite number of finitely generated good subgroups $M_1, \ldots, M_m$. We define an action of Γ on a tournament (X, R). The underlying set X will be the disjoint union (of orbits)

$$(\bigsqcup_{i \leq n} \Gamma/K_i \sqcup \Gamma/H_i) \sqcup \bigsqcup_{j \leq m} \Gamma/M_j,$$

and the action of Γ is the canonical one. On each orbit of X we define a tournament structure using Lemma 1.4. Now the arrows between different orbits are defined arbitrarily just to satisfy that for all $i \leq n$ we have $R(K_i, g_i H_i)$ and $R(H_i, K_i)$.

We want to finitely approximate this tournament action so that the stabilizers of the orbits are preserved and so that for each $i \leq n$ the relations $R(K_i, g_i H_i)$ and $R(H_i, K_i)$ are preserved. Therefore we get an action of Γ on a finite tournament (Y, S) with $2n + m$ orbits with stabilizers $K_i \leq K'_i$, $H_i \leq H'_i$, for $i \leq n$, and $M_j \leq M'_j$, for $j \leq m$, i.e. we may view Y as

$$Y = (\bigsqcup_{i \leq n} \Gamma/K'_i \sqcup \Gamma/H'_i) \sqcup \bigsqcup_{j \leq m} \Gamma/M'_j$$

with the natural action of Γ. Moreover, we have for all $i \leq n$, $S(K'_i, g_i H'_i)$ and $S(H'_i, K'_i)$, which implies that that for every $i \leq n$, $g_i \notin K'_i H'_i$. Since all the stabilizers are good subgroups of finite index, using Lemma 1.5 we can find good normal finite index subgroups $H''_i \leq H'_i$, $K''_i \leq K'_i$, for $i \leq n$, and $M''_j \leq M'_j$, for $j \leq n$. Again using Lemma 1.5 we get that the intersection N of all these good normal finite index subgroups is again a good normal finite index subgroup. Clearly, for every $i \leq n$ we have $g_i \notin K_i H_i N$. Now if the answer of Question 1.6 is positive for Γ, then $M_j N$ is good for every $j \leq m$, therefore N is as desired, and we are done. □

Question 1.9. *Do finitely generated free groups have the tournament 2-Ribes-Zalesskii property?*

1.2. **K_n-free graphs.** Now we turn to triangle-free graphs. Using techniques from [22], we prove that the 2-RZ and 3-RZ properties are the lower and the upper bounds for finite approximability of actions on triangle-free graphs (and on K_n-free graphs.)

Theorem 1.10. *Let Γ be a countable group satisfying the 3-RZ property. Then every action of Γ on a triangle-free graph is finitely approximable. More generally, every action of Γ on a K_n-free graph, for $n \geq 3$, is finitely approximable.*

Proof. Fix an action α of Γ on a triangle-free graph X identified with a metric space (X, d) with possible values of $d(x, y)$, $x \neq y \in X$, either 1, if there is an edge between x and y, or 2 otherwise. Fix finite $F \subseteq \Gamma$, and $A \subseteq X$. Without loss of generality, we can assume that $1 \in F$, and no two elements of A are in the same orbit under α.

For $x \in X$, let $H_x \leq \Gamma$ be the stabilizer of x. Fix $x, y, z \in X$, $f_1, f_2 \in \Gamma$. Observe that the assumption that X is triangle-free means that

$$d(x, f_1.y) = d(x, f_2.z) = d(y, f_1^{-1} f_2.z) = 1.$$

does not hold. Moreover, distances must be constant on appropriate double cosets, i.e., for every $g_1 \in H_x f_1 H_y$, $g_2 \in H_x f_2 H_z$,

$$d(x, g_1.y) = d(x, g_2.z) = d(y, g_1^{-1} g_2.z) = 1.$$

does not hold hold either. But this is equivalent to saying that for every $f_1, f_2, f_3 \in \Gamma$ such that

$$d(x, f_1.y) = d(x, f_2.z) = d(y, f_3.z) = 1$$

we have

$$H_y f_1 H_x f_2 H_z \cap H_y f_3 H_z = \emptyset,$$

or that

$$H_y f_1 H_x f_2 H_z \cap \{f_3\} = \emptyset.$$

Actually, the above formula can be also written as

$$f_1 f_2 (f_2^{-1} f_1^{-1} H_y f_1 f_2)(f_2^{-1} H_x f_2) H_z \cap \{f_3\} = \emptyset.$$

Because Γ has the 3-RZ property, we can find a finite-index normal subgroup $K \leq \Gamma$ such that for every $f_1, f_2, f_3 \in F$, and $x, y, z \in A$, with

$$d(x, f_1.y) = d(x, f_2.z) = d(y, f_3.z) = 1$$

we have

$$H_y f_1 H_x f_2 H_z \cap H_y f_3 H_z K = \emptyset,$$

and also, for every $f \in F$ and $x \in A$ with $f \notin H_x$,

$$H_x \cap fK = H_x K \cap \{f\} = \emptyset.$$

Let $L_x = H_x K$ for $x \in A$. We define a finite graph $Y = \coprod_{x \in A} \Gamma / L_x$ by specifying a metric ρ on Y so that

$$\rho(fL_x, gL_y) = 1$$

if and only if $f^{-1} g \in H_x g' H_y K$ for some $g' \in F$ with $d(x, g'.y) = 1$.

Note first that ρ is trivially a metric because the triangle inequality is satisfied for any mapping from $Y \times Y$ into $\{0, 1, 2\}$. Also, it is clearly invariant under the left-translation action β of Γ on Y. We need to verify that Y is triangle-free, and that α, when restricted to F and A, embeds into β via the mapping $f.x \mapsto fL_x$, for $f \in F$, $x \in A$.

In order to see that Y is triangle-free, suppose the contrary, and fix $x, y, z \in A$ and $g_1, g_2 \in \Gamma$ such that

$$\rho(L_x, g_1 L_y) = \rho(L_x, g_2 L_z) = \rho(g_1 L_y, g_2 L_z) = 1.$$

But this means that

$$g_1 \in H_x f_1 H_y K,\ g_2 \in H_x f_2 H_y K,\ g_1^{-1} g_2 \in H_y f_3 H_z K,$$

where $f_1, f_2, f_3 \in F$ are such that

$$d(x, f_1.y) = d(x, f_2.z) = d(y, f_3.z) = 1.$$

Since we have that

$$K H_y f_1 H x f_2 H_z K \cap H_y f_3 H_z K = H_y f_1 H x f_2 H_z \cap H_y f_3 H_z K = \emptyset,$$

for any such f_1, f_2, f_3, this is impossible.

In a similar fashion, we can show that $f.x \mapsto f L_x$ is a mapping embedding α restricted to F and A into β.

The general statement for K_n-free graphs, $n \geq 3$, can be proved in the same way. For example, for $n = 4$, the condition that X does not contain K_4 means that, for any fixed $X_0 = \{x, y, z, w\} \subseteq X$, and $f_1, f_2, f_3 \in \Gamma$

$$d(x, f_1.y) = d(x, f_2.z) = d(x, f_3.w) = d(y, f_1^{-1} f_2.z) = d(y, f_1^{-1} f_3.w) = d(z, f_2^{-1} f_3 w) = 1$$

does not hold. That can be rewritten that for any $f_{a,b} \in \Gamma$, $a \neq b \in X_0$ such that $d(a, f_{a,b}.b) = 1$, at least one of the intersections

$$H_b f_{a,b}^{-1} H_a f_{a,c} H_c \cap H_b f_{b,c} H_c,$$

is empty. Now, because Γ has the 3-RZ property, we can construct a K_4-free graph Y extending X exactly as above.

□

Theorem 1.11. *Let Γ be a countable group whose actions on triangle-free graphs are finitely approximable. Then Γ has the 2-RZ property.*

Proof. Fix finitely generated subgroups H_1, H_2 of Γ, and $g \in \Gamma \setminus H_1 H_2$. We need to show that there is a finite index subgroup $K \leq \Gamma$ such that $g \notin H_1 H_2 K$. Define a graph

$$X = \Gamma / H_1 \sqcup \Gamma / H_2$$

with metric satisfying $d(f H_i, g H_j) = 1$ iff $i = 1$, $j = 2$, and $f H_i \cap g H_j \neq \emptyset$. Clearly, this is a triangle-free graph, and Γ acts on X by left-translation. Note also, that $d(H_1, g H_2) = 2$. Let β be an action on a finite graph Y such that the left-translation action of Γ on X embeds into β when restricted to $A = \{H_1, H_2, g H_2\}$ and $F = \{\text{generators of } H_1, H_2\} \cup \{1, g\}$. Let $i : X \to Y$ be such an embedding, and let K_i be the stabilizer of $i(H_i)$, $i = 1, 2$.

Observe that $H_i \leq K_i$. Moreover, $g \notin K_1 K_2$ because for every $k_1 \in K_1$, $k_2 \in K_2$ we have that

$$d(H_1, k_1 k_2 H_2) = d(H_1, H_2) = 1.$$

Therefore $g \notin H_1 H_2 K$. □

2. Finite approximability and generic representations

In this section, we study connections between finite approximability and generic representations in the context of Fraïssé theory. Moreover, we present a simpler proof of genericity of permutation representations from [6].

We will say that a Fraïssé class $\mathcal{K}$ has *independent amalgamation* if for any $A_0, A_1, A_2 \in \mathcal{K}$ there exists an amalgam $B \in \mathcal{K}$ of A_1 and A_2 over A_0 such that for all automorphisms ϕ_1, ϕ_2 of A_1, A_2, respectively, that agree on A_0, $\phi_1 \cup \phi_2$ extends to an automorphism of B. Analogously, we define *independent joint embedding*. The simplest example of independent amalgamation is free amalgamation, present, e.g., in the class of finite graphs or K_n-free graphs.

Recall that a chain from a Fraïssé class $\mathcal{K}$ is a locally finite structure X such that $\mathrm{Age}(X) \subseteq \mathcal{K}$. It follows from [16] that if $\mathcal{K}$ is a Fraïssé class with a Katětov functor, then every chain X in $\mathcal{K}$ can be embedded in the Fraïssé limit M of $\mathcal{K}$ so that $\mathrm{Aut}(X)$ embeds in $\mathrm{Aut}(M)$. Thus, for Fraïssé classes with a Katětov functor (e.g., the class of finite graphs, by [16, Example 2.5], finite K_n-free graphs, by [16, Example 2.10], finite tournaments, by [16, Example 2.6], or finite metric spaces with rational distances), studying actions on chains (graphs, K_n-free graphs, tournaments or metric spaces with rational distances) reduces to studying actions on their Fraïssé limits (the random graph, the random K_n-free graph, the random tournament or the rational Urysohn space.)

Theorem 2.1. *Let Γ be a finitely generated discrete group, and let M be the Fraïssé limit of a relational Fraïssé class $\mathcal{K}$ with independent amalgamation and joint embedding, and with a Katětov functor. Then every action of Γ on a chain from $\mathcal{K}$ is finitely approximable if and only if Γ has a generic representation in $\mathrm{Aut}(M)$ with finite orbits .*

Proof. Clearly, if Γ has a generic representation in $\mathrm{Aut}(M)$ with finite orbits, then every action of Γ on M can be approximated by actions of Γ on finite $A \in \mathcal{K}$. Since $\mathcal{K}$ has a Katětov functor, this implies that every action of Γ on a chain from $\mathcal{K}$ is finitely approximable.

In order to prove the converse, we show, as in the proof of Theorem 2.4, that the class $\mathcal{K}_a$ of all actions of Γ on elements of $\mathcal{K}$, with equivariant injections as morphisms, is a Fraïssé class. The hereditary property is obvious. Amalgamation in $\mathcal{K}_a$ follows from independent amalgamation in $\mathcal{K}$: take actions $\alpha_0, \alpha_1, \alpha_2 \in \mathcal{K}_a$ on $A_0, A_1, A_2 \in \mathcal{K}$, respectively, such that $\alpha_1 \upharpoonright \Gamma \times A_0 = \alpha_2 \upharpoonright \Gamma \times A_0 = \alpha_0$. Fix an amalgam $B \in \mathcal{K}$ of A_1 and A_2 over A_0 with $A_1 \cup A_2$ as the underlying set, and chosen using independent amalgamation in $\mathcal{K}$. It is easy to verify that the mapping β defined on $\Gamma \times B$ by setting $\beta(\gamma, x) = \alpha_1(\gamma, x)$ if $x \in A_1$, and $\beta(\gamma, x) = \alpha_2(\gamma, x)$ if $x \in A_2 \setminus A_1$ is an action of Γ on B. For the same reasons, $\mathcal{K}_a$ has joint embedding. Because Γ is finitely generated, $\mathcal{K}_a$ is countable up to isomorphism, and so a Fraïssé class.

Observe that actually, we can assume that the Fraïssé limit of $\mathcal{K}_a$ is an action of Γ on M. This is because $\mathcal{K}$ has a Katětov functor, which implies, together with the assumption that actions of Γ on chains from $\mathcal{K}$ are finitely approximable, that for every $A \subseteq B \in \mathcal{K}$, and every action α of Γ on A, there exists $C \in \mathcal{K}$ such that $B \subseteq C$, and an action β on C extending α. Thus, in the standard construction of the Fraïssé limit of $\mathcal{K}_a$, where a cofinal increasing sequence of actions α_n on $A_n \in \mathcal{K}$, $n \in \mathbb{N}$, with the extension property is selected (see, e.g., [11, Theorem 7.1.2]), we can make sure that the sequence A_n, $n \in \mathbb{N}$, is cofinal in $\mathcal{K}$, and it has the extension property (in $\mathcal{K}$) as well. In other words, that $\bigcup_n A_n$ is the Fraïssé limit of $\mathcal{K}$.

Now, consider the family of all representations in $\mathrm{Rep}(\Gamma, M)$ with the extension property with regard to $\mathcal{K}_a$. It is straightforward to verify this is a dense G_δ condition. As any two representations with the extension property are conjugate, we are done. □

Corollary 2.2. *Let Γ be a finitely generated discrete group.*

1. *(1) If Γ has the 3-RZ property, then, for any $n \geq 3$, Γ has a generic representation in $\mathrm{Aut}(\mathcal{G}_n)$, where $\mathcal{G}_n$ is the random K_n-free graph,*
2. *(2) For any $n \geq n$, the group $\mathbb{Z}^n$ has a generic representation in $\mathrm{Aut}(\mathcal{T})$, where $\mathcal{T}$ is the random tournament; in particular, $\mathrm{Aut}(\mathcal{T})$ has a comeager conjugacy class,*
3. *(3) more generally, if Γ has the tournament 2-RZ property, then Γ has a generic representation in $\mathrm{Aut}(\mathcal{T})$,*
4. *(4) (Rosendal) Γ has the RZ property iff Γ has a generic representation in $\mathrm{Aut}(\mathbb{QU})$ finite orbits.*

Proof. As is has been already mentioned, the classes of triangle free graphs, tournaments, and finite metric spaces with rational distances have Katětov functors.

Now, the class of triangle-free graphs has free amalgamation and joint embedding. For finite tournaments, the amalgamation is not free but this class has independent amalgamation (as well as joint embedding) because, for given tournaments A_0, A_1, A_2 with $A_0 \subseteq A_0, A_1$, we can amalgamate A_1, A_2 over A_0 by adding an edge (a_1, a_2) iff $a_1 \in A_1$, $a_2 \in A_2$, for $a_1, a_2 \notin A_0$. Similarly, the class of finite metric spaces with rational distances has metric-free amalgamation, which is also a case of independent amalgamation. □

Remark 2.3. We note that Corollary 2.2 in particular implies that K_n-free graphs, for $n \geq 3$, have the Hrushovski property and the automorphism group of the random K_n-free graph has ample generics. This is a result originally proved by Herwig in [8].

2.1. **The theorem of Glasner, Kitroser and Melleray.** Finally, we give another proof of a characterization of groups with generic permutation representations that was proved by Glasner, Kitroser and Melleray in [6], which is in the spirit of our other proofs in this paper. Let us recall some terminology. For a countable group Γ, view the set $\mathrm{Sub}(\Gamma)$ of its subgroups as a closed subspace of the Cantor space 2^Γ. With this identification we give $\mathrm{Sub}(\Gamma)$ a compact Polish topology usually called the Chabauty topology (see [3] for the original reference). Glasner, Kitroser and Melleray call a countable group Γ *solitary* if the set of isolated points in $\mathrm{Sub}(\Gamma)$ is dense.

Theorem 2.4 (Glasner, Kitroser, Melleray)**.** *A countable group Γ has a generic permutation representation if and only if it is solitary.*

Remark 2.5. Notice the analogy of this theorem with the result that follows from [15] mentioned in the beginning of Section 3 which says that Γ has a generic unitary representation if and only if the set of isolated points in the unitary dual $\hat{\Gamma}$ equipped with the Fell topology is dense.

Proof. Let Γ be a countable solitary group. Let

$$\mathcal{G} = \{\Gamma_i : i \in \mathbb{N}\}$$

be the (at most) countable collection of isolated subgroups of Γ in $\mathrm{Sub}(\Gamma)$ which form a dense subset there. Let $\mathcal{C}$ be the set of all finite 'sums' of left quasi-regular actions

$$\Gamma \curvearrowright \Gamma/H_1 \sqcup \ldots \sqcup \Gamma/H_n,$$

where $H_i \in \mathcal{G}$, for $i \leq n$. It is immediate that it is a countable class with the amalgamation property, thus it has some Fraïssé limit $\alpha \in \mathrm{Rep}(\Gamma, S_\infty)$, which is uniquely, up to conjugation, characterized by the extension property. We show that the extension property can be expressed as a G_δ condition. Since $\mathcal{G}$ is dense, the extension property will thus define a dense G_δ set, therefore proving that α has a comeager conjugacy class. Consider the set

$$\{\beta \in \mathrm{Rep}(\Gamma, S_\infty) : \forall H \in \mathcal{G}\ \forall (x_i)_{i\leq n}\ \exists x \in \mathbb{N} \setminus \beta(\Gamma)[(x_i)_{i\leq n}]\ (\mathrm{Stab}^\beta_x = H)\}.$$

It is clearly dense. Moreover, it is G_δ. Indeed, the only non-trivial part is that '$\mathrm{Stab}^\beta_x = H$' is an open condition, which, however, follows since H is isolated, thus uniquely determined by finitely many group elements.

Suppose that Γ is not solitary. Then there exists a non-empty basic open set O without isolated points in $\mathrm{Sub}(\Gamma)$, consisting of subgroups of Γ containing some $g_1, \ldots, g_n \in \Gamma$ and not containing some $h_1, \ldots, h_m \in \Gamma$. For every $H \in O$, let

$$C(H) = \{\alpha \in \mathrm{Rep}(\Gamma, S_\infty) : \forall n \in \mathbb{N} \exists g \in \Gamma(\alpha(g).n = n \Leftrightarrow g \notin H)\}.$$

It is easy to check it is a G_δ set. Moreover it is dense. Indeed, fix a basic open neighborhood U of some $\beta \in \mathrm{Rep}(\Gamma, S_\infty)$ and some $n \in \mathbb{N}$. It suffices to show there is $\gamma \in U$ such that for some $g \in \Gamma$,

$$\gamma(g).n = n \Leftrightarrow g \notin H.$$

If β satisfies this condition we are done. Otherwise fix $n_1, \ldots, n_i \in \mathbb{N}$ and $g_1, \ldots, g_j \in \Gamma$ that determine U; we can suppose $n_1 = n$. Since H is not isolated, there exist $H' \leq \Gamma$ and $h \in \Gamma \setminus F$ such that for $k \leq j$, $g_k \in H$ iff $g_k \in H'$ and $h \in H$ iff $h \notin H'$. Therefore we can find some $\gamma \in U$ such that $\gamma(\Gamma).n \cong \Gamma/H'$, which means that γ satisfies the condition above.

Now suppose there is a comeager conjugacy class C. It must intersect the open set

$$D = \{\alpha \in \mathrm{Rep}(\Gamma, S_\infty) : \exists n \in \mathbb{N}\ \exists H \in O\ \alpha(\Gamma).n \cong \Gamma/H\}.$$

Then for some $H \in O$, and for every $\beta \in C$, there is $n \in \mathbb{N}$ such that $\beta(\Gamma).n \cong \Gamma/H$. This contradicts that C also intersects $C(H)$. □

3. Generic representations in metric structures

In this section, we investigate generic properties of representations of countable groups in automorphism groups of metric structures. Typically, there are no generic representations in this situation. However, perhaps the most interesting case, when the metric structure in question is the separable infinite-dimensional Hilbert space, is still open. It follows from Theorem 2.5 in [15] that when Γ is a group with the Kazhdan's property (T) whose finite-dimensional unitary representations form a dense set in the unitary dual $\hat{\Gamma}$, then Γ has a generic unitary representation. See also [4] for a more explicit statement of this theorem and more elementary proof. Nevertheless, the existence of such infinite Kazhdan groups is an open question, see e.g. Question 7.10 in [1].

Here, we prove that an at most countable group Γ has a generic representation in the isometry group of the Urysohn space and the Urysohn sphere if and only if Γ is finite. As mentioned in the introduction, Julien Melleray informed us he had proved it earlier in his habilitation theis, see Theorem 5.78 in [19]. As he did not publish the proof for the Urysohn sphere we use the opportunity to present our proof here (for the sake of completeness also with the proof for the Urysohn space which is a simpler version). We also show that every infinite countable group has meager conjugacy classes in the linear isometry group of the Gurarij space, which answers a question of Melleray from the same paper [19]. We also show that these methods can be used to prove that when one restricts to the space of free actions on the Random graph, the rational Urysohn space, etc., then every infinite group has meager classes.

Most importantly, we show that the conjugation action of the isometry group of the Urysohn space on the space of representations of a fixed infinite group Γ in the Urysohn space is generically turbulent.

3.1. **Urysohn space and Urysohn sphere.** Denote by $\mathbb{U}$ and by $\mathbb{U}_1$ the Urysohn universal metric space and the Urysohn sphere respectively. We refer the reader to Chapter 5 in [20] for information about the Urysohn space. We recall that the Fraïssé limit of finite rational-valued metric spaces is the so called rational Urysohn space, denoted here by $\mathbb{QU}$, and $\mathbb{U}$ is its completion. Analogously, the Fraïssé limit of finite rational-valued metric spaces bounded by one is the rational Urysohn sphere, denoted by $\mathbb{QU}_1$, and its completion is $\mathbb{U}_1$. Alternatively, one may obtain $\mathbb{U}_1$, as the name suggests, by picking any point from $\mathbb{U}$ and taking the subset of $\mathbb{U}$ of those points having distance one half from the chosen point.

For a fixed countable group Γ, we shall denote the Polish space of representations of Γ in the Polish groups $\mathrm{Iso}(\mathbb{U})$, resp. $\mathrm{Iso}(\mathbb{U}_1)$ by $\mathrm{Rep}(\Gamma, \mathbb{U})$, resp. $\mathrm{Rep}(\Gamma, \mathbb{U}_1)$.

We omit the proof of the following theorem which is straightforward, and probably known among the experts.

Theorem 3.1. *Let Γ be a finite group. Then the classes of all actions of Γ on finite rational metric spaces, resp. on finite rational metric spaces bounded by 1 are Fraïssé classes whose limit are an action of Γ on the $\mathbb{QU}$, resp. on $\mathbb{QU}_1$. Their completions are generic actions of Γ on $\mathbb{U}$, resp. on $\mathbb{U}_1$.*

Recall that a *pseudonorm* (or *length function*) on a group Γ is a function $\lambda : \Gamma \to \mathbb{R}$ satisfying $\lambda(1_\Gamma) = 0$, $\lambda(g) = \lambda(g^{-1})$, for $g \in \Gamma$, and $\lambda(g \cdot h) \leq \lambda(g) + \lambda(h)$, for $g, h \in \Gamma$. We shall generalize this notion below.

Definition 3.2. Let Γ be a group and I some index set. A *generalized pseudonorm* on the pair (Γ, I) is a function $N : \Gamma \times I^2 \to [0, \infty)$ satisfying

- $N(g, i, j) = N(g^{-1}, j, i)$, for all $g \in \Gamma$ and $i, j \in I$;
- $N(1_\Gamma, i, i) = 0$, for every $i \in I$, and $N(g, i, j) > 0$ for all $g \in \Gamma$ (including 1_Γ) whenever $i \neq j$;
- $N(gh, i, j) \leq N(g, i, k) + N(h, k, j)$, for all $g, h \in \Gamma$ and $i, j, k \in I$.

For any $i \in I$, we shall denote by N_i the function defined by $N_i(g) = N(g, i, i)$. Clearly, it is a pseudonorm on Γ.

Generalized pseudonorms correspond to actions of Γ on metric spaces by isometries together with representatives of each orbit. Indeed, let I be some index set, and let N be a generalized pseudonorm on (Γ, I). For each $i \in I$, let $H_i \leq \Gamma$ be the subgroup defined as the kernel of N_i. We define a metric d on $M = \bigcup_{i \in I} \Gamma / H_i$ as follows: for $g, h \in \Gamma$ and $i, j \in I$ we set $d(gH_i, hH_j) = N(g^{-1}h, i, j)$. It is straightforward to check that it is a metric, and, moreover, that the natural action of Γ on M (defined as $g.(hH_i) = (gh)H_i$) is an action by isometries.

Conversely, let (M, d) be a metric space on which Γ acts by isometries. Let I be an index set for all the orbits in M of this action, and for each $i \in I$ select a representative $x_i \in M$ from this orbit. Now the function $N : \Gamma \times I^2 \to \mathbb{R}$ defined by $N(g, i, j) = d(x_i, gx_j)$ is readily checked to be a generalized pseudonorm.

A function $P : \Gamma \times I^2 \to [0, \infty)$ satisfying all the axioms of the generalized pseudonorm except the triangle inequality is called *generalized pre-pseudonorm*.

Fact 3.3. *If $P : \Gamma \times I^2 \to \mathbb{R}$ is a generalized pre-pseudonorm, then there exists a maximal generalized pseudonorm N satisfying $N(g,i,j) \leq P(g,i,j)$ for all $g \in \Gamma$ and $i,j \in I$.*

Proof. Consider a complete graph with the set of vertices $\Gamma \times I$, i.e. $|I|$ disjoint copies of Γ and consider a real valuation of its edges P', where the value $P'((g,i),(h,j))$ is defined to be $P(h^{-1}g,i,j)$, for $g,h \in \Gamma$ and $i,j \in I$. Now consider the corresponding graph metric N' on $\Gamma \times I$. It is clear that the function $N(g,i,j) = N'((1_\Gamma,i),(g,j))$ is the desired generalized pseudonorm. Note that another equivalent way how to define N is to set

$$N(g,i,j) = \inf\{\sum_{i=1}^{m} P(g_i,k_i,l_i) : g = g_1 \dots g_m, l_i = k_{i+1} \forall i < m\}.$$

□

An analogous notion is that of a partial generalized pseudonorm. A function $P : \Gamma \times I^2 \to \mathbb{R}$ is a *partial generalized pseudonorm* if it satisfies all the axioms of the generalized pseudonorm except that it is defined partially, i.e. its domain is a proper subset of $\Gamma \times I^2$. Let now $P : A \subseteq \Gamma \times I^2 \to \mathbb{R}$ be some partial generalized pseudonorm. Take again a graph V with the set of vertices $\Gamma \times I$ and connect two vertices (g,i) and (h,j) by an edge if and only if $(h^{-1}g,i,j) \in A$. If V is connected, then we say that the partial generalized pseudonorm is *sufficient*. Note that this is the case when for example $A = F \times I^2$, where F is some symmetric generating subset of Γ. Also, in this case we can extend the partial generalized pseudonorm onto a genuine generalized pseudonorm.

Fact 3.4. *Let $P : A \subseteq \Gamma \times I^2 \to \mathbb{R}$ be a sufficient partial generalized pseudonorm. Then there exists a maximal generalized pseudonorm N on $\Gamma \times I^2$ which extends P.*

Proof. The proof proceeds as the proof of Fact 3.3. We take graph V with the set of vertices $\Gamma \times I$ and we connect two vertices (g,i) and (h,j) by an edge if and only if $(h^{-1}g,i,j) \in A$, as above. For every edge $((g,i),(h,j))$ in V we define a value $P'((g,i),(h,j))$ of this edge to be $P(h^{-1}g,i,j)$. Then we take the graph metric which gives the generalized pseudonorm N as in the proof of Fact 3.3. The fact that N extends P follows from the property that P satisfies all the axioms of the generalized pseudonorm on its domain. □

We are now prepared to prove the meagerness of conjugacy classes for the Urysohn space and the Urysohn sphere.

Theorem 3.5 (see also Theorem 5.78 in [19])**.** *Let Γ be an infinite group. Then every $\alpha \in \mathrm{Rep}(\Gamma,\mathbb{U})$ has all conjugacy classes meager. Analogously, every $\alpha \in \mathrm{Rep}(\Gamma,\mathbb{U}_1)$ has all conjugacy classes meager.*

Proof. Fix an infinite group Γ. First, we work with the full Urysohn space. Let λ be an arbitrary pseudonorm on Γ. It suffices to show that the set

$C(\lambda)$ of all $\alpha \in \mathrm{Rep}(\Gamma, \mathbb{U})$ such that for every $x \in \mathbb{QU}$ there exists $g \in \Gamma$ with

$$|\lambda(g) - d_{\mathbb{U}}(\alpha(g)x, x)| > 1/4$$

is comeager. Indeed, if the conjugacy class of any $\alpha' \in \mathrm{Rep}(\Gamma, \mathbb{U})$ was non-meager, it would have a non-empty intersection with $C(\lambda_0)$, where $\lambda_0(g) = d_{\mathbb{U}}(\alpha'(g)x, x)$ for some fixed $x \in \mathbb{QU}$; this is clearly a contradiction.

To see that $C(\lambda)$, for any pseudonorm λ on Γ, is comeager, it suffices to show that for a fixed $x \in \mathbb{QU}$ the open set of all $\alpha \in \mathrm{Rep}(\Gamma, \mathbb{U})$ such that there exists $g \in \Gamma$ with

$$|\lambda(g) - d_{\mathbb{U}}(\alpha(g)x, x)| > 1/4,$$

is dense in $\mathrm{Rep}(\Gamma, \mathbb{U})$.

Let U be an open neighborhood of some $\beta \in \mathrm{Rep}(\Gamma, \mathbb{U})$ given by some finite set $x_1, \ldots, x_n$, where $x = x_1$ and each x_i, x_j lie in different orbits of β, some finite symmetric set $F \subseteq \Gamma$ containing the unit and some $\varepsilon > 0$. Set $F' = \{g^{-1}h : g, h \in F\}$, $I = \{1, \ldots, n\}$ and let $N_0 : F' \times I^2 \to \mathbb{R}$ be a function defined as $N_0(g, i, j) = d_{\mathbb{U}}(x_i, \beta(g)x_j)$, for every $g \in F'$, $i, j \in I$. Let $N_0' : F' \times I^2 \to \mathbb{R}$ be a function defined as

$$N_0'(g, i, j) = \begin{cases} 0 & g = 1_\Gamma, i = j; \\ N(g, i, j) + \varepsilon/2 & \text{otherwise.} \end{cases}$$

Note that N' is a partial generalized pseudonorm. Let

$$M' = \max\{N_0'(g, i, j) : g \in F', i, j \in I\}.$$

If the pseudonorm λ is bounded by some K, then we set $M = \max\{M', K + 1/4\}$; if it is unbounded, we set $M = M'$. and finally, let $N_0'' : \Gamma \times I^2 \to \mathbb{R}$ be a generalized pre-pseudonorm extending N_0' which coincides with N_0' on its domain, and everywhere else it is constantly M. Now we use Fact 3.3 to find a maximal generalized pseudonorm N bounded by N_0''.

Note that

- $N(g, i, j) = 0$ if and only if $g = 1_\Gamma$ and $i = j$;
- $N(g, i, j) = N_0'(g, i, j)$ if $g \in F'$;
- for every $i \in I$ for all but finitely many $g \in \Gamma$ we have $N(g, i, i) = M$.

N gives an action γ' by isometries on a metric space $Y = \bigcup_{i \leq n} \Gamma.y_i$, where $d_Y(\gamma'(g)y_i, \gamma'(h)y_j) = N(h^{-1}g, i, j)$, for $g, h \in \Gamma$ and $i, j \in I$. Using the Katětov functor for metric spaces we can extend the action γ' on Y to an action γ on $\mathbb{U}$, and then, using the finite extension property of the Urysohn space, we can find an isometry ϕ of $\mathbb{U}$ such that $\phi(y_1) = x_1$, and for every $i \leq n$ and $g \in F$ we have $d_{\mathbb{U}}(\phi(\gamma(g)y_i), \beta(g)x_i) < \varepsilon$. Without loss of generality, we can assume that ϕ is the identity map, therefore γ is in the neighborhood U of β. By the construction, there exists $g \in \Gamma$ such that $d_{\mathbb{U}}(x, \gamma(g)x) = M$, while $\lambda(g) \leq M - 1/4$ in case λ was bounded, or $\lambda(g) > M + 1/4$ in case λ was unbounded. This finishes the proof for the

Urysohn space.

Now we work with the Urysohn sphere. Fix some pseudonorm λ on Γ which is now bounded by 1. Analogously as above, it suffices to show that for a fixed $x \in \mathbb{QU}_1$ the open set of all $\alpha \in \mathrm{Rep}(\Gamma, \mathbb{U})$ such that there exists $g \in \Gamma$ with

$$|\lambda(g) - d_{\mathbb{U}_1}(\alpha(g)x, x)| \geq 1/4,$$

is dense in $\mathrm{Rep}(\Gamma, \mathbb{U}_1)$.

The proof proceeds similarly in the beginning, however we cannot take an advantage of the fact that λ may be unbounded. Let U be an open neighborhood of some $\beta \in \mathrm{Rep}(\Gamma, \mathbb{U}_1)$ again given by some finite set $x_1, \ldots, x_n$, where $x = x_1$, and each x_i, x_j lies in a different orbit of β, some finite symmetric set $F \subseteq \Gamma$ containing the unit, and some $\varepsilon > 0$. Set $F' = \{g^{-1}h : g, h \in F\}$, $I = \{1, \ldots, n\}$, and let $N_0 : F' \times I^2 \to \mathbb{R}$ be a function defined as $N_0(g, i, j) = d_{\mathbb{U}_1}(x_i, \beta(g)x_j)$, for every $g \in F'$, $i, j \in I$. Let $N_0' : F' \times I^2 \to \mathbb{R}$ be a function defined as

$$N_0'(g, i, j) = \begin{cases} 0 & g = 1_\Gamma, i = j; \\ \max\{N(g, i, j) + \varepsilon/2, 1\} & \text{otherwise.} \end{cases}$$

Set $m = \min\{N_0'(g, i, j) : g \neq 1_\Gamma \text{ or } i \neq j\}$. Note that $m > 0$. Take now $M = \lceil 1/m \rceil + 1$.

Suppose first that there are infinitely many $g \in \Gamma$ such that $\lambda(g) \leq 3/4$. Then we set $N_0'' : \Gamma \times I^2 \to \mathbb{R}$ to be a generalized pre-pseudonorm extending N_0' which coincides with N_0' on its domain, and everywhere else it is constantly 1. Now we use Fact 3.3 to find a maximal generalized pseudonorm N bounded by N_0''. It is clear that N coincides with N_0' on its domain and it is equal to 1 at co-finitely many elements.

If, on the other hand, for all but finitely many g we have $\lambda(g) > 3/4$, then we may and will find some $g \in \Gamma$ satisfying $\lambda(g) > 3/4$ and such that g cannot be obtained as a product of less than M-many elements of F'. Then we set $N_0'' : \Gamma \times I^2 \to \mathbb{R}$ to be a generalized pre-pseudonorm extending N_0' which coincides with N_0' on its domain, it is equal to $1/2$ for at $(g, 1, 1)$, and everywhere else it is constantly 1. We again use Fact 3.3 to find a maximal generalized pseudonorm N bounded by N_0''. Clearly $N(g, 1, 1) \leq 1/2$. We now check that N coincides with N_0' on its domain. Suppose that there is some (f, i, j) from the domain of N_0' such that $N(f, i, j) < N_0'(f, i, j)$. By the construction of N, it means there are $g_1, \ldots, g_n \in F'$ and $k_1, l_1, \ldots, k_n, l_n \in I$ such that $f = g_1 \ldots g_n$ and $l_i = k_{i+1}$ for $i < n$ and $N(f, i, j) = \sum_{i=1}^n N_0''(g_i, k_i, l_i) < N_0'(f, i, j)$. We claim that $n < M - 1$ since otherwise $N(f, i, j) = \sum_{i=1}^n N_0''(g_i, k_i, l_i) \geq \sum_{i=1}^n m \geq 1$ (note that for each $i \leq n$ we have $N_0''(g_i, k_i, l_i) \geq m$). It is clear that for at least one $i \leq n$ we must have $(g_i, k_i, l_i) = (g, 1, 1)$ since otherwise all (g_i, k_i, l_i) are from the domain of N_0' and N_0' is a restriction of a genuine generalized pseudonorm, so the triangle inequalities are satisfied. there. On the other hand, there is at most one $i \leq n$ such that $(g_i, k_i, l_i) = (g, 1, 1)$ since $N_0''(g, 1, 1) = 1/2$. Let $i \leq n$ be such that $g_i = g$. We have $f = g_1 \ldots g_{i-1} g g_{i+1} \ldots g_n$, so $g = g_{i-1}^{-1} \ldots g_1^{-1} f g_n^{-1} \ldots g_{i+1}^{-1}$, but that is in contradiction with the assumption that g cannot be obtain as a product of less than M-many elements of F'.

Now we finish the proof as in the Urysohn space case. N gives an action γ' by isometries on a metric space $Y = \bigcup_{i \leq n} \Gamma.y_i$ bounded by 1, where $d_Y(\gamma'(g)y_i, \gamma'(h)y_j) = N(h^{-1}g, i, j)$, for $g, h \in \Gamma$ and $i, j \in I$. Using the Katětov functor for metric spaces bounded by 1, we can extend the action γ' on Y to an action γ on $\mathbb{U}_1$, and then, using the finite extension property of the Urysohn sphere, we can find an isometry ϕ of $\mathbb{U}$ such that $\phi(y_1) = x_1$, and for every $i \leq n$ and $g \in F$ we have $d_{\mathbb{U}_1}(\phi(\gamma(g)y_i), \beta(g)x_i) < \varepsilon$. Without loss of generality, we can assume that ϕ is the identity map, therefore γ is in the neighborhood U of β.

Now if λ was such that for infinitely many g we have $\lambda(g) \leq 3/4$, then we have guaranteed that there are co-finitely many $g \in \Gamma$ such that $d_{UrS}(x, \gamma(g)x = 1)$. If, on the other hand, for all but finitely many g we have $\lambda(g) > 3/4$, then we have guaranteed an existence of $g \in \Gamma$ such that $\lambda(g) > 3/4$, while $d_{UrS}(x, \gamma(g)x \leq 1/2)$. This finishes the proof. □

An object analogous with its properties to the Urysohn space in the category of Banach spaces is the Gurarij space ([7]), denoted here by $\mathbb{G}$. We refer the reader to the paper [17] for information about this Banach space. We can use similar methods as in the proof of Theorem 3.5 to show that also every representation of every infinite group has meager conjugacy classes in the linear isometry group of the Gurarij space. Since the arguments are repetitive we provide only a sketch of the proof. The existence of the Katětov functor in the category of Banach spaces is shown in [2]. We mention that this theorem answers Question 5.7 from [19].

Theorem 3.6. *Let Γ be a countably infinite group. Then every conjugacy class in* $\mathrm{Rep}(\Gamma, \mathbb{G})$ *is meager.*

Sketch of the proof. Fix an infinite group Γ and some countable dense subset D of the sphere in $\mathbb{G}$. Notice that for any $\alpha \in \mathrm{Rep}(\Gamma, \mathbb{G})$ and any $x \in \mathbb{G}$ of norm one, the function on Γ defined as $g \to \|\alpha(g)x - x\|$ is again a pseudonorm (bounded by 2). Therefore, as in the case of the Urysohn sphere, it suffices to show that for any pseudonorm λ on Γ, bounded by 2, we have that the set of those $\alpha \in \mathrm{Rep}(\Gamma, \mathbb{G})$ such that for every $x \in D$ there exists $g \in \Gamma$ with

$$|\lambda(g) - \|\alpha(g)x - x\|| > 1/4$$

is comeager. That again reduces to showing that for a fixed $x \in D$ the open set of all $\alpha \in \mathrm{Rep}(\Gamma, \mathbb{G})$ such that there exists $g \in \Gamma$ with

$$|\lambda(g) - \|\alpha(g)x - x\|| > 1/4,$$

is dense in $\mathrm{Rep}(\Gamma, \mathbb{G})$.

Take U to be an open neighborhood of some $\beta \in \mathrm{Rep}(\Gamma, \mathbb{G})$ given by some finite set $x_1, \dots, x_n \in D$ of unit linearly independent vectors, where $x = x_1$ and each x_i, x_j lie in different orbits of β, some finite symmetric set $F \subseteq \Gamma$ containing the unit and some $\varepsilon > 0$. Perturbing the representation β by less than ε if necessary, we may assume that the set $S = \{\beta(f)x_i : f \in F, i \leq n\}$ consists of linearly independent elements. As in the Urysohn sphere case, we get that $N' : F \times I^2 \to \mathbb{R}$ defined as $N'(f, i, j) = \|x_i - \beta(f)x_j\|$ is a partial generalized pseudonorm, where $I = \{1, \dots, n\}$. By the same arguments as for the Urysohn sphere we get that N' extends to a generalized pseudonorm $N : \Gamma \times I^2 \to \mathbb{R}$ that 'avoids' λ, i.e. there exists $g \in \Gamma$ such that $|\lambda(g) - N(g, 1, 1)| > 1/4$.

Now let X be the vector space spanned by the set $\{g.x_i : g \in \Gamma, i \leq n\}$. Note that Γ acts canonically on X, so for any $g \in \Gamma$ and $x \in X$ the element $g.x$ is defined. By Y we denote the finite-dimensional subspace spanned by $\{g.x_i : g \in F, i \leq n\}$. We define a norm on X. First we define 'a partial norm' $\|\cdot\|'$ on X and then show how it naturally extends to a norm on X so that the canonical action of Γ on X is by linear isometries. Since we may identify Y with the finite-dimensional subspace of $\mathbb{G}$ spanned by S we set for any $x \in Y$,

$$\|x\|' = \|x\|_{\mathbb{G}}.$$

Next, for any $g, h \in \Gamma$ and $i, j \leq n$ we set

$$\|g.x_i - h.x_j\|' = \|h.x_j - g.x_i\|' = N(g^{-1}h, i, j).$$

Note that in case that $g.x_i - h.x_j \in Y$ we have $\|g.x_i - h.x_j\|_{\mathbb{G}} = N(g^{-1}h, i, j)$, so our definition is consistent. For any $g \in \Gamma$ and $i \leq n$ we set

$$\|g.x_i\|' = 1,$$

this is again consistent, and finally we make $\|\cdot\|'$ invariant under the canonical action of Γ, i.e. for any $g \in \Gamma$ and $x \in X$ such that $\|x\|'$ has been defined above we set

$$\|g.x\|' = \|x\|'.$$

This is again readily checked to be consistent.

Now we set $\|\cdot\|$ to be the greatest norm on X that extends $\|\cdot\|'$. That can be formally defined as follows. For any $x \in X$ we set

$$\|x\| = \inf\{\sum_{i=1}^{m} |\alpha_i| \|x_i\|' : (\alpha_i)_{i \leq m} \subseteq \mathbb{R}, (x_i)_{i \leq m} \subseteq \mathrm{dom}(\|\cdot\|'), x = \sum_{i=1}^{m} \alpha_i x_i\}.$$

Finally, using the Katětov functor for Banach spaces, we may extend the action Γ on X to an action γ of Γ on $\mathbb{G}$, and moreover in such a way that $\gamma \in U$ and $|\|\gamma(g)x - x\| - \lambda(g)| > 1/4$ which is what we were supposed to show. □

Remark 3.7. One may ask what happens when Γ is a finite group. Is there a generic representation of Γ in $\mathbb{G}$? Indeed, this is the case and it is again possible to construct such generic representation by Fraïssé theory. It is perhaps less clear though what is the right Fraïssé class. We refer the reader to Section 4 in [5] where similar Fraïssé classes of representations of groups in Banach spaces were considered.

3.2. **Free actions on countable metric spaces.** Let Γ be a countable discrete group and X a countable structure. Notice that the set $\mathrm{Rep}_F(\Gamma, X)$ of all free actions of Γ on X is a G_δ set invariant under the conjugacy action of Γ and therefore a Polish space itself. One may thus also study conjugacy classes in these spaces. It is obvious from the proofs of Theorems 3.5 and 3.6 that the main difference between spaces $\mathrm{Rep}(\Gamma, \mathrm{Aut}(M))$ and $\mathrm{Rep}(\Gamma, \mathrm{Iso}(N))$, where M is a countable metric space viewed as a countable discrete structure and N is a Polish metric space, that in the latter 'locally free' actions are dense. Here by 'locally free actions are dense' we mean that for any finite set $\{x_1, \ldots, x_n\}$ the set of those actions whose restriction on the orbit of x_i, for $i \leq n$, is regular is dense. Since this is also satisfied, for obvious reasons, in the spaces of free actions we can prove by the same means as in the proof of Theorem 3.5 the following.

Theorem 3.8. *Let Γ be a countably infinite group. Then all conjugacy classes are meager in the spaces $\mathrm{Rep}_F(\Gamma, \mathrm{Aut}(\mathbb{QU}))$, $\mathrm{Rep}_F(\Gamma, \mathrm{Aut}(\mathbb{QU}_1))$ and $\mathrm{Rep}_F(\Gamma, \mathrm{Aut}(\mathcal{R}))$, where $\mathcal{R}$ is the random graph.*

This stands in stark contrast with Rosendal's results from [23] which say that all finitely generated groups with the RZ property have a generic representation in $\mathrm{Iso}(\mathbb{QU})$, and all finitely generated groups with the 2-RZ property have a generic representation in $\mathrm{Aut}(\mathcal{R})$.

3.3. **Generic turbulence.** The notion of turbulence was introduced by Hjorth in [10] in order to develop methods for proving non-classifiability by countable structures. Suppose that G is a Polish group acting continuously on a Polish space X. Fix a point $x \in X$, some open neighborhood U of x in X and some open neighborhood V of the unit in G. The *local orbit* $O(U, V)$ of x is the set $\{y \in U : \exists g_1, \ldots, g_n \in V \ (y = g_1 \ldots g_n.x \wedge \forall i \leq n \ (g_1 \ldots g_i.x \in U))\}$.

A point $x \in X$ is *turbulent* if for every open neighborhood U of x in X and every open neighborhood V of $1_G \in G$, the local orbit $O(U, V)$ of x is somewhere dense (in U). If there is a G-invariant comeager subset $Y \subseteq X$ such that every G-orbit in Y is dense and meager and every point in Y is turbulent, then we say that the G-action on X is generically turbulent. It is shown in [10] that as a consequence the corresponding orbit equivalence relation on X is not classifiable by countable structures.

Our aim is now to show that the orbit equivalence given by the action of $\mathrm{Iso}(\mathbb{U})$ on $\mathrm{Rep}(\Gamma, \mathbb{U})$ by conjugation is generically turbulent whenever Γ is infinite. By Theorem 3.21 in [10], to show that it is sufficient to prove that all equivalences classes are meager, which we have proved already, and that there exists a turbulent element whose equivalence class is dense. That will be the content of the following theorem.

We note that Kerr, Li and Pichot in [15] prove analogous statements for unitary representation. Namely, in Theorem 3.3 they prove that for any countable group Γ the action of $U(\mathcal{H})$ on the space $\mathrm{Rep}_\lambda(\Gamma, \mathcal{H})$ of all unitary representations weakly contained in the regular representation is generically turbulent. It follows from Theorem 2.5 there that for any countable group without property (T) the action of $U(\mathcal{H})$ on $\mathrm{Rep}(\Gamma, \mathcal{H})$ is generically turbulent.

Theorem 3.9. *Let Γ be a countably infinite group. Then there exists $\alpha \in \mathrm{Rep}(\Gamma, \mathbb{U})$ whose conjugacy class is dense and which is turbulent.*

Proof. First we work with finitely generated groups. Fix some finitely generated group Γ. Let I be some finite index set and call a generalized pseudonorm N on $\Gamma \times I^2$ *finitely generated* if there exists a sufficient partial generalized pseudonorm with finite domain such that N is its maximal extension (guaranteed by Fact 3.4). An *embedding* between two generalized pseudonorms N_1 on $\Gamma \times I_1^2$ and N_2 on $\Gamma \times I_2^2$ is an injection $\iota : I_1 \hookrightarrow I_2$ such that for every $g \in \Gamma$ and $i, j \in I_1$ we have $N_1(g, i, j) = N_2(g, \iota(i), \iota(j))$. Typically, ι will be just the inclusion.

Lemma 3.10. *The class of all finitely generated rational valued generalized pseudonorms is a Fraïssé class.*

Note that being rational valued means that it is the maximal extension of some partial generalized pseudonorm with finite domain which is rational valued.

Proof. This is straightforward, we only check the amalgamation property. Suppose we are given such generalized pseudonorms N_1, N_2 and N_3 defined on $\Gamma \times I_1$, $\Gamma \times I_2$ and $\Gamma \times I_3$ respectively, and $I_1 \subseteq I_2 \cap I_3$. That is, there are embeddings from N_1 into N_2 and N_3. Suppose that N_i is the maximal extension of a partial generalized pseudonorm P_i defined on $A_i \subseteq \Gamma \times I_i^2$, where $i \in \{2, 3\}$. Set $I_4 = I_2 \cup I_3$ and $A_4 = A_2 \cup A_3$. Define a partial generalized pseudonorm P_4 on $A_4 \subseteq \Gamma \times I_4^2$ so that it extends P_2, resp. P_3. Note that this is well-defined as P_2 and P_3 agree on $A_2 \cap A_3$. Set N_4 to be the maximal extension of P_4. This is readily checked to be the desired amalgam. □

The Fraïssé limit is some rational valued generalized pseudonorm N_F defined on $\Gamma \times I_F^2$, for some infinite I_F. N_F corresponds to an action of Γ on some rational valued countable metric space with orbits indexed by I_F and for each $i \in I_F$ there is a distinguished base point x_i of that orbit. It is straightforward to check that this metric space is isometric to $\mathbb{QU}$ and that $D = \{x_i : i \in I_F\}$ is dense. Denote by α this action on $\mathbb{QU}$ and use the same letter also for its extension on $\mathbb{U}$, the completion of $\mathbb{QU}$. We aim to show that α is the desired element from the statement of Theorem 3.9.

First we observe that the conjugacy class of α is dense. That is easy. Fix an open neighborhood O of some $\beta \in \mathrm{Rep}(\Gamma, \mathbb{U})$ given by some $x_1, \ldots, x_n \in \mathbb{U}$, some finite symmetric $F \subseteq \Gamma$ containing the unit and some $\varepsilon > 0$. Set $F' = \{g^{-1}h : g, h \in F\}$. Set $I = \{1, \ldots, n\}$. Define a partial generalized pseudonorm P on $F' \times I^2 \subseteq \Gamma \times I^2$ as follows: set $P(g, i, j) = d_{\mathbb{U}}(x_i, g.x_j)$ for $(g, i, j) \in F' \times I^2$. By ε-perturbing P a bit if necessary, we may assume that P is rational valued. Now using the property of the Fraïssé limits we see that that there are $i_1, \ldots, i_n \in I_F$ such that $N_F(g, i_k, i_l) = P(g, k, l)$ for every $g \in F'$ and $k, l \in I$. Clearly we can then find an isometry $\phi \in \mathrm{Iso}(\mathbb{U})$ such that $\phi^{-1}\alpha\phi \in O$.

We now check that α is turbulent. Suppose, without loss of generality, that $I_F = \mathbb{N}$. Fix some open neighborhood U of α which we may assume is given by $x_1, \ldots, x_n \in D \subseteq \mathbb{QU}$, finite symmetric $F \subseteq \Gamma$ containing the unit and some $\varepsilon > 0$, and some open neighborhood V of id in $\mathrm{Iso}(\mathbb{U})$ which we may assume is given also by the same $x_1, \ldots, x_n \in D \subseteq \mathbb{QU}$ and $\varepsilon > 0$. Recall that by the construction, for every $f, g \in F$, $i, j \leq n$ we have $d_{\mathbb{U}}(\alpha(f).x_i, \alpha(g).x_j) = N_F(g^{-1}f, i, j)$. Set $F' = \{g^{-1}f : g, f \in F\}$ and $I = \{1, \ldots, n\}$. The restriction of N_F onto $\Gamma \times I^2$ is a finitely generated generalized pseudonorm. Without loss of generality we may assume that it is generated by the values on $A = F' \times I^2$.

Now we check that the local orbit $O(U, V)$ of α is dense in U. Take some $\beta \in U$ and some its open neighborhood $W \subseteq U$ given by $x_1, \ldots, x_n, \ldots, x_m \in D \subseteq \mathbb{QU}$, some finite symmetric $F \subseteq H \subseteq \Gamma$ containing the unit and some $\varepsilon > \varepsilon' > 0$. Set again $H' = \{g^{-1}f : g, f \in H\}$, $J = \{1, \ldots, m\}$ and let P_β be the partial generalized pseudonorm on $H' \times J^2$ defined as $P_\beta(g, i, j) = d_{\mathbb{U}}(x_i, \beta(g).x_j)$. Again, by perturbing P_β a bit if necessary, we may assume that P_β is rational valued. Analogously, set P_α to be the restriction of N_F onto $H' \times J^2$. We may suppose, without loss of generality, that the finitely generated generalized pseudonorm $N_F \restriction \Gamma \times J^2$ is generated by its values on $H' \times J^2$, i.e. it is generated by P_α.

Before we proceed further we need a few notions and basic lemmas. First, suppose that N_1 and N_2 are two partial generalized pseudonorms defined on the same set of the form $A \times L^2$ for some $A \subseteq \Gamma$. Then we define the distance $D(N_1, N_2)$ between them as the supremum distance, i.e. $\sup_{a \in A \times L^2} |N_1(a) - N_2(a)|$. For any $n \in \mathbb{N}$ denote by $L \times n$ the set $\{(i, j) : i \in L, j \leq n\}$ and for any $j \leq n$ denote by $L(j) \subseteq L \times n$ the subset $\{(i, j) : i \in L\}$. Finally, denote by $(L \times n)' \subseteq (L \times n)^2$ the set $\{((i, j), (i, j+1)), ((i, j+1), (i, j)) : i \in L, j < n\}$.

Lemma 3.11. *There exists a partial generalized pseudonorm N defined on $B = (A \times L(1)^2) \cup (A \times L(2)^2) \cup (\{1_\Gamma\} \times (L \times n)')$ such that*

- $N(a, (i, k), (j, k)) = N_k(a, i, j)$*, for* $k \in \{1, 2\}$*,* $a \in A$ *and* $i, j \in L$*;*
- $N(1_\Gamma, (i, j), (i, j+1)) = D(N_1, N_2)$*.*

Proof. Define N as in the statement of the lemma. We must check that it is a partial generalized pseudonorm. The only non-trivial thing to check is the triangle inequality. Suppose that the triangle inequality does not hold. That is, there are $g, g_1, \ldots, g_j \in A$, $(i_1, i'_1), \ldots, (i_{j+1}, i'_{j+1}) \in L \times n$ such that

- $g = g_1 \ldots g_j$;
- $(g_l, (i_l, i'_l), (i_{l+1}, i'_{l+1})) \in B$ for $l \leq j$ and $(g, (i_1, i'_1), (i_{j+1}, i'_{j+1})) \in B$;
- $N(g, (i_1, i'_1), (i_{j+1}, i'_{j+1})) > \sum_{l=1}^{j} N(g_l, (i_l, i'_l), (i_{l+1}, i'_{l+1}))$.

The case when $g = 1_\Gamma$ and $i'_1 \neq i'_{j+1}$, i.e. $(g, (i_1, i'_1), (i_{j+1}, i'_{j+1})) \in \{1_\Gamma\} \times (L \times n)'$ is straightforward. So let us assume that $i'_1 = i'_{l+1} = 1$, the case when it is equal to 2 is symmetric.

Suppose also that j above is the least possible. We claim that for no $l < j$ we have $i'_l = i'_{l+1} = i'_{l+2}$. Indeed, by the triangle inequality we have

$$N(g_l, (i_l, i'_l), (i_{l+1}, i'_{l+1})) + N(g_{l+1}, (i_{l+1}, i'_{l+1}), (i_{l+2}, i'_{l+2})) =$$
$$N_{i_l}(g_l, (i_l, i'_l), (i_{l+1}, i'_{l+1})) + N_{i_l}(g_{l+1}, (i_{l+1}, i'_{l+1}), (i_{l+2}, i'_{l+2})) \geq$$
$$N_{i_l}(g_l g_{l+1}, (i_l, i'_l), (i_{l+2}, i'_{l+2})) = N(g_l g_{l+1}, (i_l, i'_l), (i_{l+2}, i'_{l+2})).$$

So we may shorten the decomposition of g which contradicts that j was the least possible.

It follows that without loss of generality we may assume that for $l \leq j$ odd we have that $g_l \in A$ and $i'_l = i'_{l+1}$, while for $l \leq j$ even we have $g_l = 1_\Gamma$ and $i'_l \neq i'_{l+1}$. Suppose that for some $l \leq j$ we have that $i'_l = 2$. It follows that $l < j$ therefore by the paragraph above we have

$$N(g_{l-1}, (i_{l-1}, i'_{l-1}), (i_l, i'_l)) + N(g_l, (i_l, i'_l), (i_{l+1}, i'_{l+1})) +$$
$$N(g_{l+1}, (i_{l+1}, i'_{l+1}), (i_{l+2}, i'_{l+2})) = D(N_1, N_2) + N_2(g_l, i_l, i_{l+1}) + D(N_1, N_2) \geq$$
$$N_1(g_l, i_l, i_{l+1}).$$

Therefore we may again shorten the decomposition which again contradicts that j was the least possible. It follows that for no $l \leq j$ we have that $i'_l = 2$. However, then we are clearly done. □

The proof of the following lemma is very easy and left to the reader. Note in particular that if N is a generalized pseudonorm and $r > 0$ is a positive real, then rN is a generalized pseudonorm, and if N_1, N_2 are two generalized pseudonorms, then $N_1 + N_2$ is as well.

Lemma 3.12. *Suppose that we have generalized pseudonorms N_1 and N_2 as above. Take any $0 < t < 1$. Then for the convex combination $N_3 = tN_1 + (1-t)N_2$ we have $D(N_1, N_3) = (1-t)D(N_1, N_2)$ and $D(N_2, N_3) = tD(N_1, N_2)$.*

Now set $M = D(P_\alpha, P_\beta)$. Let $\delta > 0$ be an arbitrary rational number such that $\delta < \varepsilon$ and $k = M/\delta$ is in $\mathbb{N}$. For every $1 < j < k$ let P_j be the convex combination $\frac{j}{k} P_\alpha + \frac{k-j}{k} P_\beta$ defined on $H' \times J^2$. Notice that since for every $i, i' \in J$ we have $P_\alpha(1_\Gamma, i, i') = P_\beta(1_\Gamma, i, i')$, we also have for every $1 < j < k$, $P_\alpha(1_\Gamma, i, i') = P_j(1_\Gamma, i, i')$. Moreover, since for every $i, i' \in I$ and every $g \in F'$ we have $|P_\alpha(g, i, i') - P_\beta(g, i, i')| < \varepsilon$, we also have for every

$1 < j < k$, $|P_\alpha(g,i,i') - P_j(g,i,i')| < \varepsilon$.

Now we define a rational valued sufficient partial generalized pseudonorm N on $C = (\bigcup_{j\leq k} H' \times J(j)) \cup (\{1_\Gamma\} \times (J \times k)')$ as follows:

- For any $b \in \{1_\Gamma\} \times (J \times k)'$ we set $N(b) = \delta$.
- For any $g \in H'$ and $i, i' \leq m$ we set $N(g, (i,1), (i',1)) = P_\alpha(g,i,i')$ and $N(g,(i,k),(i',k)) = P_\beta(g,i,i')$.
- For any $g \in H'$, $i, i' \leq m$ and $1 < j < k$ we set $N(g,(i,j),(i',j)) = P_j(g,i,i')$.

It follows from Lemma 3.12 and repeated use of Lemma 3.11 that N is indeed a rational valued sufficient generalized pseudonorm.

By the extension property of the Fraïssé limit N_F we may realize N as the extension of $N_F \restriction \Gamma \times I^2$ and as a subfunction of N_F. To simplify the notation, we view C as a subset of $\Gamma \times I_F^2$ and assume that $N_F \restriction C = N \restriction C$.

Now we may find $\phi_1 \in V \subseteq \mathrm{Iso}(\mathbb{U})$ such that for every $i \leq m$ and every $1 < j \leq k$, $\phi_1(\alpha(f).x_{(i,j)} = \alpha(f).x_{(i,j-1)}$.

Indeed, that follows from the fact that for every $j < k$, the subspaces $\{x_{(i,j)} : i \leq m\}$ and $\{x_{(i,j+1)} : i \leq m\}$ are isometric.

Analogously, for every $1 < l < k$ we may find $\phi_l \in V$ such that for every $i \leq m$ and every $l < j < k$, $\phi_l \circ \ldots \circ \phi_1(x_{(i,j)} = x_{(i,j-l)}$.

That shows that for every $l < k-1$ we have

$$\phi_l \circ \ldots \circ \phi_1 \circ \alpha \circ \phi_1^{-1} \circ \ldots \circ \phi_l^{-1} \in U$$

and

$$\phi_{k-1} \circ \ldots \circ \phi_1 \circ \alpha \circ \phi_1^{-1} \circ \ldots \circ \phi_{k-1}^{-1} \in W$$

and we are done.

If Γ is infinitely generated, then write Γ as an increasing union $\Gamma_1 \leq \Gamma_2 \leq \ldots$ of finitely generated subgroups. Denote by α_n, for $n \in \mathbb{N}$, the action of Γ_n on $\mathbb{QU}$ obtained as a Fraïssé limit as above. Since the Fraïssé limits are uniquely characterized by the extension property, it is easy to check that for every $n \in \mathbb{N}$ the restriction of α_{n+1} onto Γ_n is isomorphic to α_n. Denote by U_n the copy of $\mathbb{QU}$ on which α_n acts. By the identification of α_n with the restriction of α_{n+1} we may view U_n as a subspace of U_{n+1}. Moreover, it is easy to see that U_n is dense in U_{n+1}. Therefore we get the direct limit of the actions α_n to be an action α on the union $U = \bigcup_n U_n$ which is also isometric to $\mathbb{QU}$ and each U_n is dense in U. Therefore α is also naturally a direct limit of the actions α_n on the completion $\mathbb{U}$ and it follows from the argument above, for finitely generated groups, that α is a turbulent element in $\mathrm{Rep}(\Gamma, \mathbb{U})$. □

Let us mention that analogous proof can be also used to show that the conjugacy action of $\mathrm{Iso}(\mathbb{U}_1)$ on $\mathrm{Rep}(\Gamma, \mathbb{U}_1)$, for any infinite Γ, is generically turbulent. The turbulent element there is the completion of the Fraïssé limit of finitely generated *bounded* rational generalized pseudonorms on Γ.

Proposition 3.13. *The conjugacy action of* $Iso(\mathbb{U}_1)$ *on* $\mathrm{Rep}(\Gamma, \mathbb{U}_1)$ *is generically turbulent for any countably infinite group* Γ.

To proof of the same result for the Gurarij space is more technical. The turbulent element can be again constructed as an appropriate Fraïssé limit of certain 'finitely generated actions' of Γ on finite-dimensional Banach spaces. We refer the reader the reader again to Section 4 of [5] where similar Fraïssé classes of finitely generated actions on Banach spaces were considered.

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