Long-time behaviour of solutions to hyperbolic equations with hysteresis

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Abstract

For a quasilinear hyperbolic system with Dirichlet boundary conditions and with hysteretic constitutive law describing waves in elastoplastic solids, we give an overview of results on existence, uniqueness, and asymptotic stability of solutions if either initial data or the time-periodicity condition are prescribed. Convexity in the hysteresis diagrams implies the existence of a second order dissipation term which in turn prevents the system from formation of shocks.

Contents

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Introduction

The main problem in quasilinear hyperbolic equations consists in the fact that shock-free solutions exist in general only for small times even if the data are smooth, and weak solutions are difficult to handle. A recent overview can be found e. g. in [4]. The situation is substantially different if the nonlinearity in the constitutive law has a hysteresis character. In many cases, shocks do not occur and the solution remains regular. The monograph [14] was an attempt to give an explanation of this fact. It was shown that in the p-system, trajectories of entropy solutions to the Riemann problem tend to follow the convex hull of the constitutive graph. If now the constitutive graph consists of *convex hysteresis loops*, it is natural to expect that the solution trajectories will follow the hysteresis branches and, in terms of gas dynamics, only rarefaction takes place. There is a substantial difference between viscous (rate-dependent) and plastic (rate-independent) dissipation in hyperbolic equations. While the former transforms the problem into essentially a parabolic one with unbounded speed of propagation, the latter preserves the hyperbolic character of the balance equations and the propagation speed is bounded independently of the hysteresis term, see Proposition 1.0.2 below.

Existence, uniqueness, and qualitative properties of solutions to the wave equation with elastoplastic hysteresis were studied in [14] for a fairly general class of hysteresis operators and under mixed boundary conditions, where displacement is prescribed on one end and stress on the other end of the space interval. The material presented here is mostly new. Besides the interaction with additional lower order nonlinearities, we consider the case of Dirichlet boundary conditions, where the displacement (or velocity, depending on the setting) is prescribed on both ends. This is more difficult, since the a priori estimate for the space derivative of the stress depends on the sup-norm of the stress for which no boundary condition is available. The estimation technique thus has to use more refined arguments based on special properties of the hysteresis memory. In order to reduce the complexity, we restrict ourselves to the so-called *Prandtl-Ishlinskii hysteresis operator*, although much of the results remain valid for a larger class of convex hysteresis operators, and an example is presented in Subsection 1.4.

If convexity of the hysteresis loops is lost, like in Maxwell's equations for large amplitude electromagnetic waves in ferromagnetic materials, then the regular behaviour cannot be expected and shocks are again likely to occur. The only publication on this subject seems to be [26], where existence of weak solutions on a bounded time interval has been established for a very general class of hysteresis operators.

The text is organized as follows. In Section 1 we state and solve the problem of existence, uniqueness, and asymptotic stability of regular solutions to an initial-boundary value problem for a wave equation with a Prandtl-Ishlinskii stress-strain law. We also show that the regularity is preserved under nonlinear perturbations of the constitutive law if convexity is not violated. The time-periodic problem is investigated in Section 2 and we again prove results of existence, uniqueness, and asymptotic stability for this case. The last Section 3 is a collection of known results on the Prandtl-Ishlinskii model. This part has an auxiliary character and has only be included in order to keep the exposition as self-contained as possible.

1 An initial-boundary value problem

We consider the model problem

$$\left\{\begin{array}{rcl} \partial_t v &=& \partial_x \sigma + f(\sigma, v, x, t)\,, \\ \partial_t \varepsilon &=& \partial_x v\,, \\ \varepsilon &=& \mathcal{F}[\lambda, \sigma] \end{array}\right. \tag{1.0.1}$$

describing e. g. longitudinal oscillations of an elastoplastic beam, where $(x,t) \in Q_T := \,]0,1[\, \times \,]0,T[$. We assume that $\mathcal{F}$ is a hysteresis operator and f is a function, both satisfying Hypothesis 1.0.1 below. For System (1.0.1) we prescribe initial and boundary conditions

$$v(x,0) \;=\; v^0(x),\;\; \varepsilon(x,0) \;=\; \varepsilon^0(x) \qquad \text{for} \quad x \in \,]0,1[\,, \tag{1.0.2}$$

$$v(0,t) \;=\; v(1,t) \;=\; 0 \qquad \text{for} \quad t \in \,]0,T[\,. \tag{1.0.3}$$

Hypothesis 1.0.1.

(i) $\mathcal{F}$ *is a Prandtl-Ishlinskii operator of the form (3.5.2) with non-decreasing generating function* $h : [0,\infty[\, \to \,]0,\infty[$ *, and with a given initial configuration* $\lambda \in C([0,1];\Lambda_K)$ *for some* $K > 0$*;*

(ii) $f : \mathbb{R}^2 \times Q_T \to \mathbb{R}$ *is a given function such that* $f(\sigma, v, \cdot, \cdot), \partial_t f(\sigma, v, \cdot, \cdot) : Q_T \to \mathbb{R}$ *are measurable for all* $(\sigma, v) \in \mathbb{R}^2$*,* $f(\cdot, \cdot, x, t), \partial_t f(\cdot, \cdot, x, t) : \mathbb{R}^2 \to \mathbb{R}$ *are continuous for a. e.* $(x,t) \in Q_T$*,* $f^0 := f(0,0,\cdot,\cdot) \in C([0,T];L^2(0,1))$*, and there exist functions* $\alpha_f \in L^1(0,T)$ *and* $\beta_f \in L^1(0,T;L^2(0,1))$ *such that the inequalities*

$$\left\{\begin{array}{rcl} |f(\sigma_1, v_1, x, t) - f(\sigma_2, v_2, x, t)| &\leq& \alpha_f(t)\,(|\sigma_1 - \sigma_2| + |v_1 - v_2|)\,, \\ |\partial_t f(\sigma, v, x, t)| &\leq& \beta_f(x,t)\,, \end{array}\right. \tag{1.0.4}$$

hold for a. e. $(x,t) \in Q_T$ *and every* $\sigma, \sigma_1, \sigma_2, v, v_1, v_2 \in \mathbb{R}$*.*

(iii) $v^0, \varepsilon^0 \in W^{1,2}(0,1)$*,* $v^0(0) = v^0(1) = 0$*.*

In the case without hysteresis, i. e. $h(r) = h(0)$ for all $r \geq 0$, System (1.0.1) is semilinear hyperbolic with wave propagation speed

$$c_0 \;=\; \frac{1}{\sqrt{h(0)}}\,. \tag{1.0.5}$$

We now show that even if hysteresis is present, the speed of propagation is bounded by the same constant c_0 *independently of the initial data and of the hysteresis dissipation*, so that the hyperbolic character of the problem is not violated, at variance with the case where viscosity is included into the model.

Proposition 1.0.2. *Let Hypothesis 1.0.1 be fulfilled, and let* c_0 *be the constant in (1.0.5). Let there exist an interval* $[x_1, x_2] \subset]0,1[$ *such that the data* $\sigma^0, v^0, \lambda, f$ *satisfy* $\sigma^0(x) = v^0(x) = \lambda(x,\cdot) \equiv 0$ *for* $x \in [x_1, x_2]$, $f(0,0,x,t) = 0$ *for a. e.* $(x,t) \in \Omega := \{(x,t) \in Q_T;\, x_1 + c_0 t < x < x_2 - c_0 t\}$. *Then every solution* (v, σ) *of (1.0.1) vanishes in* $\bar{\Omega}$.

Proof. We apply the classical energy method proposed by Courant & Hilbert, see [5]. Put $U(x,t) = \mathcal{U}[\lambda, \sigma](x,t)$, where $\mathcal{U}$ is the potential energy operator associated with $\mathcal{F}$ according to (3.4.1). We have the pointwise inequality

$$U(x,t) \;\geq\; \frac{1}{2c_0^2}\sigma^2(x,t) \qquad \text{a. e.} \tag{1.0.6}$$

For $t \in [0,T]$ set

$$A(t) \;=\; R\int_0^t \alpha_f(t')\,dt'\,, \quad \text{where } \; R = 1 + \sqrt{1+c_0^2}\,. \tag{1.0.7}$$

For a. e. $(x,t) \in \Omega$ we have by (3.4.2), (3.4.3), and (1.0.4) that

$$\begin{aligned}
&\partial_t\left(\mathrm{e}^{-A(t)}\left(\frac{1}{2}v^2 + U\right)\right) - \partial_x\left(\mathrm{e}^{-A(t)} v\sigma\right)\\
&\quad= \mathrm{e}^{-A(t)}\left(-R\alpha_f(t)\left(\frac{1}{2}v^2 + U\right) + v\left(\partial_t v - \partial_x \sigma\right) + \partial_t U - \sigma\partial_x v\right)\\
&\quad\leq \mathrm{e}^{-A(t)}\left(-R\alpha_f(t)\left(\frac{1}{2}v^2 + U\right) + v\, f(\sigma, v, x, t)\right)\\
&\quad\leq \alpha_f(t)\,\mathrm{e}^{-A(t)}\left(-R\left(\frac{1}{2}v^2 + \frac{1}{2c_0^2}\sigma^2\right) + |v|\left(|\sigma| + |v|\right)\right) \;\leq\; 0\,.
\end{aligned}$$

For an arbitrary $\tau \in [0, (x_2 - x_1)/(2c_0)] \cap [0,T]$ we denote $\Omega_\tau = \{(x,t) \in \Omega;\, t < \tau\}$. The Green Theorem yields

$$\begin{aligned}
0 \;&\geq\; \iint_{\Omega_\tau}\left(\partial_t\left(\mathrm{e}^{-A(t)}\left(\frac{1}{2}v^2 + U\right)\right) - \partial_x\left(\mathrm{e}^{-A(t)} v\sigma\right)\right) dx\, dt\\
&= \int_{x_1 + c_0\tau}^{x_2 - c_0\tau} \mathrm{e}^{-A(\tau)}\left(\frac{1}{2}v^2 + U\right)\right)(x,\tau)\, dx\\
&\quad + \int_{x_1}^{x_1 + c_0\tau} \mathrm{e}^{-A((x-x_1)/c_0)}\left(\frac{1}{2}v^2 + U + \frac{1}{c_0}v\sigma\right)\left(x, \frac{x - x_1}{c_0}\right) dx\\
&\quad + \int_{x_2 - c_0\tau}^{x_2} \mathrm{e}^{-A((x_2-x)/c_0)}\left(\frac{1}{2}v^2 + U - \frac{1}{c_0}v\sigma\right)\left(x, \frac{x_2 - x}{c_0}\right) dx\,.
\end{aligned}$$

All three integrals on the right-hand side of the above inequality are non-negative by (1.0.6). This is only possible if both v and σ vanish in $\bar{\Omega}$, which completes the proof. ∎

1.1 Existence and uniqueness of solutions

Theorem 1.1.1. *Let Hypothesis 1.0.1 be fulfilled. Then there exists a unique solution* $(v,\sigma,\varepsilon) \in C(\bar{Q}_T;\mathbb{R}^3)$ *of System (1.0.1)–(1.0.3) such that* $\partial_t v, \partial_x v, \partial_t \sigma, \partial_x \sigma, \partial_t \varepsilon$ *belong to the space* $L^\infty(0,T;\, L^2(0,1))$, *and (1.0.1) holds almost everywhere in* Q_T.

Proof. The uniqueness argument is straightforward. We consider two solutions $(v_1,\sigma_1,\varepsilon_1), (v_2,\sigma_2,\varepsilon_2)$, and put $\bar{v} = v_1 - v_2$, $\bar{\sigma} = \sigma_1 - \sigma_2$, $\bar{\varepsilon} = \varepsilon_1 - \varepsilon_2$. The hypothesis yields that $\bar{v}(x,0) = \bar{\varepsilon}(x,0) = 0$ for all $x \in [0,1]$. From Proposition 3.2.1 it follows that $\sigma_i = \hat{\mathcal{F}}[\mu,\varepsilon_i]$ for $i = 1,2$, hence also $\bar{\sigma}(x,0) = 0$. For a. e. $(x,t) \in Q_T$ we have

$$\left\{ \begin{array}{rcl} \partial_t \bar{v} & = & \partial_x \bar{\sigma} + f(\sigma_1, v_1, x, t) - f(\sigma_2, v_2, x, t)\,, \\ \partial_t \bar{\varepsilon} & = & \partial_x \bar{v}\,. \end{array} \right. \tag{1.1.1}$$

Testing the first equation in (1.1.1) by $\bar{v}$, the second by $\bar{\sigma}$, and using (1.0.4), we obtain

$$\int_0^1 (\bar{v}\,\partial_t \bar{v} + \bar{\sigma}\,\partial_t \bar{\varepsilon})\,dx \;\le\; \alpha_f(t) \int_0^1 (|\bar{v}| + |\bar{\sigma}|)\,|\bar{v}|\,dx\,. \tag{1.1.2}$$

Let c_0 be as in (1.0.5). From (3.3.2) and the elementary inequality $2pq \le \delta p^2 + q^2/\delta$ for $p,q,\delta > 0$ it follows for $t \ge 0$ that

$$\int_0^1 \left(|\bar{v}|^2 + \frac{1}{c_0^2}|\bar{\sigma}|^2 \right)(t)\,dx \;\le\; \left(1 + \sqrt{1 + c_0^2}\right) \int_0^t \alpha_f(\tau) \int_0^1 \left(|\bar{v}|^2 + \frac{1}{c_0^2}|\bar{\sigma}|^2 \right)(\tau)\,dx\,d\tau\,, \tag{1.1.3}$$

and the Gronwall argument yields that $\bar{v} = \bar{\sigma} = 0$ a. e.

The existence statement will be proved by space semi-discretization. For $n \in \mathbb{N}$ we consider the system of equations

$$\dot{v}_j \;=\; n(\sigma_{j+1} - \sigma_j) + f_j(\sigma_j, v_j, t)\,, \quad j = 1, \dots n-1\,, \tag{1.1.4}$$

$$\dot{\varepsilon}_j \;=\; n(v_j - v_{j-1})\,, \quad j = 1, \dots n\,, \tag{1.1.5}$$

$$\varepsilon_j \;=\; \mathcal{F}[\lambda_j, \sigma_j]\,, \quad j = 1, \dots n\,, \tag{1.1.6}$$

coupled with “boundary conditions”

$$v_0(t) = v_n(t) = 0\,, \tag{1.1.7}$$

and initial conditions

$$v_j(0) = v_j^0\,, \quad \varepsilon_j(0) = \varepsilon_j^0\,, \tag{1.1.8}$$

where we set

$$f_j(\sigma, v, t) = n \int_{(j-1)/n}^{j/n} f(\sigma, v, x, t)\,dx\,, \quad \lambda_j(r) = n \int_{(j-1)/n}^{j/n} \lambda(x, r)\,dx\,, \tag{1.1.9}$$

$$v_j^0 = n \int_{(j-1)/n}^{j/n} v^0(x)\,dx\,, \quad \varepsilon_j^0 = n \int_{(j-1)/n}^{j/n} \varepsilon^0(x)\,dx \tag{1.1.10}$$

for all admissible values of arguments and indices, except for the compatibility condition $v_0^0 = v_n^0 = 0$. It follows from Propositions 3.2.1 and 3.2.2 that Eq. (1.1.6) can be written in the form

$$\sigma_j \;=\; \hat{\mathcal{F}}[\mu_j, \varepsilon_j] \tag{1.1.11}$$

with Lipschitz continuous operators $\hat{\mathcal{F}}[\mu_j, \cdot] : C[0,T] \to C[0,T]$. System (1.1.4)–(1.1.5) is of the type

$$\dot{z}(t) \;=\; \Phi[z, \cdot](t)\,, \quad z(0) = z^0 \tag{1.1.12}$$

for an unknown function $z : [0,T] \to \mathbb{R}^{2n-1}$, $z = (v_1, \dots, v_{n-1}, \varepsilon_1, \dots, \varepsilon_n)$, with a mapping $\Phi : C([0,T];\mathbb{R}^{2n-1}) \times [0,T] \to C([0,T];\mathbb{R}^{2n-1})$ given by the right-hand side of (1.1.4)–(1.1.5). We solve (1.1.12) as a fixed point problem in $C([0,T];\mathbb{R}^{2n-1})$ for the operator

$$S[z](t) \;=\; z^0 + \int_0^t \Phi[z, \cdot](\tau)\, d\tau\,. \tag{1.1.13}$$

By Hypothesis 1.0.1 and Proposition 3.2.2 there exists $a \in L^1(0,T)$ such that for all $z_1, z_2 \in C([0,T];\mathbb{R}^{2n-1})$ and $t \in [0,T]$ we have

$$|\Phi[z_1, \cdot](t) - \Phi[z_2, \cdot](t)| \;\le\; a(t)\, \|z_1 - z_2\|_{[0,t]}\,, \tag{1.1.14}$$

hence

$$|S[z_1] - S[z_2]|(t) \;\le\; \int_0^t a(\tau)\, \|z_1 - z_2\|_{[0,\tau]}\, d\tau\,. \tag{1.1.15}$$

For $t \in [0,T]$ set $A(t) = \exp(\int_0^t a(\tau)\, d\tau)$ and

$$\|z\|_{A,[0,t]} \;=\; \max_{\tau\in[0,t]} \left(\frac{1}{A(\tau)}\, \|z\|_{[0,\tau]}\right) \qquad \text{for } z \in C([0,T];\mathbb{R}^{2n-1})\,. \tag{1.1.16}$$

In particular, $\|\cdot\|_{A,[0,T]}$ is a norm in $C([0,T];\mathbb{R}^{2n-1})$ which is equivalent to $\|\cdot\|_{[0,T]}$. By (1.1.15), we have for all $t \in [0,T]$ that

$$\begin{aligned} \frac{1}{A(t)}\, \|S[z_1] - S[z_2]\|_{[0,t]} \;&\le\; \frac{1}{A(t)} \int_0^t a(\tau) A(\tau) \frac{1}{A(\tau)}\, \|z_1 - z_2\|_{[0,\tau]}\, d\tau \\ &\le\; \frac{A(t)-1}{A(t)}\, \|z_1 - z_2\|_{A,[0,t]} \;\le\; \frac{A(T)-1}{A(T)}\, \|z_1 - z_2\|_{A,[0,T]}\,. \end{aligned} \tag{1.1.17}$$

We see that S is a contraction on $C([0,T];\mathbb{R}^{2n-1})$ endowed with norm $\|\cdot\|_{A,[0,T]}$, and its unique fixed point is a solution to (1.1.4)–(1.1.8).

We now derive estimates which enable us to pass to the limit as $n \to \infty$. To this end, we differentiate Eqs. (1.1.4)–(1.1.5) and test by $\dot{v}_j$ and $\dot{\sigma}_j$, respectively. From Hypothesis 1.0.1 we obtain

$$\ddot{v}_j \dot{v}_j \;\le\; n(\dot{\sigma}_{j+1} - \dot{\sigma}_j)\dot{v}_j + \big(\beta_j(t) + \alpha_f(t)(|\dot{\sigma}_j| + |\dot{v}_j|)\big)|\dot{v}_j|\,, \tag{1.1.18}$$

$$\ddot{\varepsilon}_j \dot{\sigma}_j \;=\; n(\dot{v}_j - \dot{v}_{j-1})\dot{\sigma}_j\,, \tag{1.1.19}$$

where $\beta_j(t) = n\int_{(j-1)/n}^{j/n} \beta_f(x,t)\,dx$. The boundary conditions (1.1.7) yield

$$\sum_{j=1}^{n-1}(\dot\sigma_{j+1} - \dot\sigma_j)\dot v_j + \sum_{j=1}^{n}(\dot v_j - \dot v_{j-1})\dot\sigma_j \;=\; 0\,, \tag{1.1.20}$$

and by virtue of Theorem 3.4.1 we have for all $t \in\,]0,T]$ that

$$\int_0^t \ddot\varepsilon_j\dot\sigma_j\,d\tau \;\ge\; \frac{1}{2}(\dot\varepsilon_j\dot\sigma_j)(t-) - \frac{1}{2}(\dot\varepsilon_j\dot\sigma_j)(0+)\,. \tag{1.1.21}$$

From (1.1.5) and (1.1.8) it follows that $\dot\varepsilon_j(0+) = \dot\varepsilon_j(0) = n(v_j^0 - v_{j-1}^0)$. Using (1.1.11) and (3.2.11) we obtain

$$|\dot\sigma_j(0+)| = \lim_{t\to 0+}\frac{1}{t}|\sigma_j(t) - \sigma_j(0)| \le \lim_{t\to 0+}\frac{2}{h(0)t}\,\|\varepsilon_j - \varepsilon_j(0)\|_{[0,t]} = \frac{2}{h(0)}|\dot\varepsilon_j(0)|\,, \tag{1.1.22}$$

hence

$$\frac{1}{n}\sum_{j=1}^{n}(\dot\sigma_j\dot\varepsilon_j)(0+) \;\le\; \frac{2n}{h(0)}\sum_{j=1}^{n}|v_j^0 - v_{j-1}^0|^2\,. \tag{1.1.23}$$

The right-hand side of (1.1.23) can be estimated independently of n. To do so, we decompose the sum as

$$n\sum_{j=1}^{n}|v_j^0 - v_{j-1}^0|^2 \;=\; n\left(|v_1^0|^2 + |v_{n-1}^0|^2 + \sum_{j=2}^{n-1}|v_j^0 - v_{j-1}^0|^2\right)\,. \tag{1.1.24}$$

Formula (1.1.10) yields

$$n|v_1^0|^2 = n^3\left|\int_0^{1/n}\int_0^{(1/n)-x}\partial_x v^0(x')\,dx'\,dx\right|^2 \le n\int_0^{1/n}\int_0^{(1/n)-x}|\partial_x v^0(x')|^2\,dx'\,dx\,,$$

and similarly

$$n|v_{n-1}^0|^2 \le n\int_0^{1/n}\int_{((n-1)/n)-x}^{1}|\partial_x v^0(x')|^2\,dx'\,dx\,.$$

For $j = 2,\dots,n-1$ we have

$$\begin{aligned} n|v_j^0 - v_{j-1}^0|^2 \;&=\; n^3\left|\int_0^{1/n}\int_{((j-1)/n)-x}^{(j/n)-x}\partial_x v^0(x')\,dx'\,dx\right|^2 \\ &\le\; n\int_0^{1/n}\int_{((j-1)/n)-x}^{(j/n)-x}|\partial_x v^0(x')|^2\,dx'\,dx\,. \end{aligned}$$

From the above computations it follows that

$$n\sum_{j=1}^{n}|v_j^0 - v_{j-1}^0|^2 \;\le\; \int_0^1|\partial_x v^0(x')|^2\,dx'\,. \tag{1.1.25}$$

We similarly have

$$\frac{1}{n}\sum_{j=1}^{n-1}|\dot{v}_j(0)|^2 \ \le\ \frac{1}{n}\sum_{j=1}^{n-1}\big(n|\sigma_{j+1}(0)-\sigma_j(0)|+|f_j(\sigma_j(0),v_j(0),0)|\big)^2 . \tag{1.1.26}$$

From (1.1.11) and Propositions 3.2.1–3.2.2 it follows that

$$|\sigma_{j+1}(0)-\sigma_j(0)| \ \le\ \frac{2}{h(0)}|\varepsilon^0_{j+1}-\varepsilon^0_j|$$

and arguing as in (1.1.25) we obtain

$$n\sum_{j=1}^{n-1}|\sigma_{j+1}(0)-\sigma_j(0)|^2 \ \le\ \left(\frac{2}{h(0)}\right)^2\int_0^1|\partial_x\varepsilon^0(x)|^2\,dx . \tag{1.1.27}$$

We further have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1}|f_j(\sigma_j(0),v_j(0),0)|^2 \ \le\ \int_0^1|f(\sigma(x,0),v^0(x),x,0)|^2\,dx \tag{1.1.28}$$

and

$$\frac{1}{n}\sum_{j=1}^{n-1}\beta_j^2(t) \ \le\ \int_0^1\beta_f^2(x,t)\,dx . \tag{1.1.29}$$

Let us introduce auxiliary functions

$$W_n(t) \ =\ \frac{1}{n}\sum_{j=1}^{n-1}\dot{v}_j^2(t), \quad S_n(t) \ =\ \frac{1}{n}\sum_{j=1}^{n}\dot{\sigma}_j^2(t) . \tag{1.1.30}$$

In the following series of estimates, C denotes any positive constant independent of n and T, and C_T is a constant independent of n and possibly dependent on T.

We integrate (1.1.18)–(1.1.19) from 0 to t. Combining the inequality $\dot{\varepsilon}_j(t)\dot{\sigma}_j(t) \ge h(0)\dot{\sigma}_j^2(t)$ (which follows from (3.2.17)) with (1.1.20)–(1.1.29), we obtain

$$S_n(t)+W_n(t) \ \le\ C\left(1+\int_0^t\sqrt{W_n(\tau)\int_0^1\beta_f^2(x,\tau)\,dx}+\alpha(\tau)(S_n(\tau)+W_n(\tau))\,d\tau\right) . \tag{1.1.31}$$

Using the inequality $W_n^{1/2}\le(1+W_n)/2$ and the hypothesis that the function $\tilde{\alpha}(t) = (\int_0^1\beta_f^2(x,t)\,dx)^{1/2}+\alpha_f(t)$ belongs to $L^1(0,T)$, we obtain from (1.1.31) that

$$S_n(t)+W_n(t) \ \le\ C_T\left(1+\int_0^t(\tilde{\alpha}(\tau)(S_n(\tau)+W_n(\tau)))\,d\tau\right) \tag{1.1.32}$$

for almost all $t\in\,]0,T[$, and Gronwall's Lemma yields that

$$S_n(t)+W_n(t) \ \le\ C_T \qquad \text{a. e.} \tag{1.1.33}$$

We now define piecewise linear and piecewise constant approximations of (v, σ) by the formula

$$
\begin{aligned}
\hat{v}^{(n)}(x,t) &= v_{j-1}(t) + n\left(x - \frac{j-1}{n}\right)(v_j(t) - v_{j-1}(t))\,, && (1.1.34)\\
\hat{\sigma}^{(n)}(x,t) &= \sigma_{j-1}(t) + n\left(x - \frac{j-1}{n}\right)(\sigma_j(t) - \sigma_{j-1}(t))\,, && (1.1.35)\\
\bar{v}^{(n)}(x,t) &= v_{j-1}(t)\,, && (1.1.36)\\
\bar{\sigma}^{(n)}(x,t) &= \sigma_j(t)\,, && (1.1.37)\\
\underline{\sigma}^{(n)}(x,t) &= \sigma_{j-1}(t)\,, && (1.1.38)\\
\bar{\lambda}^{(n)}(x,t) &= \lambda_j(t)\,, && (1.1.39)\\
\bar{f}(\sigma, v, x, t) &= f_{j-1}(\sigma, v, t)\,, && (1.1.40)\\
\bar{\varepsilon}^{(n)} &= \mathcal{F}[\bar{\lambda}^{(n)}, \bar{\sigma}^{(n)}] && (1.1.41)
\end{aligned}
$$

for $x \in [(j-1)/n, j/n[\ ,\ j = 1, \ldots, n$, continuously extended to $x = 1$, where we set $f_0(\sigma, v, t) = 0$. Equations (1.1.4)–(1.1.5) then have the form

$$
\begin{aligned}
\partial_t \bar{v}^{(n)} &= \partial_x \hat{\sigma}^{(n)} + \bar{f}(\underline{\sigma}^{(n)}, \bar{v}^{(n)}, x, t)\,, && (1.1.42)\\
\partial_t \bar{\varepsilon}^{(n)} &= \partial_x \hat{v}^{(n)}\,. && (1.1.43)
\end{aligned}
$$

From (1.1.34) it follows that the functions $\partial_t \hat{v}^{(n)}, \partial_t \hat{\sigma}^{(n)}, \partial_t \bar{v}^{(n)}, \partial_t \bar{\sigma}^{(n)}$ are bounded in $L^\infty(0,T\,;\,L^2(0,1))$ independently of n. We further have

$$
\begin{aligned}
\int_0^1 \left|\bar{f}(\underline{\sigma}^{(n)}, \bar{v}^{(n)}, x, t)\right|^2 dx &= \frac{1}{n}\sum_{j=1}^{n-1} f_j^2(\sigma_j, v_j, t) && (1.1.44)\\
&\leq \frac{1}{n}\sum_{j=1}^{n-1}\left(f_j(\sigma_j(0), v_j(0), 0) + \int_0^t \frac{\partial}{\partial \tau}(f_j(\sigma_j, v_j, \tau))\,d\tau\right)^2\\
&\leq C_T\,,
\end{aligned}
$$

hence also $\partial_x \hat{v}^{(n)}, \partial_x \hat{\sigma}^{(n)}$ are bounded in $L^\infty(0,T\,;\,L^2(0,1))$ independently of n. Denote $H^{\infty,2}(Q_T) = \{u \in L^2(Q_T)\,;\, \partial_t u, \partial_x u \in L^\infty(0,T\,;\,L^2(0,1))\}$. By compact embedding $H^{\infty,2}(Q_T) \hookrightarrow\hookrightarrow C(\bar{Q}_T)$ we find functions $v, \sigma \in H^{\infty,2}(Q_T)$ and a subsequence of $\{(\hat{v}^{(n)}, \hat{\sigma}^{(n)})\}$ (still indexed by n) such that

$$
\left.\begin{aligned}
\partial_t \hat{\sigma}^{(n)} &\to \partial_t \sigma\\
\partial_x \hat{\sigma}^{(n)} &\to \partial_x \sigma\\
\partial_t \hat{v}^{(n)} &\to \partial_t v\\
\partial_x \hat{v}^{(n)} &\to \partial_x v
\end{aligned}\right\}\quad \text{weakly-star in } L^\infty(0,T\,;\,L^2(0,1))\,, \qquad (1.1.45)
$$

$$
\left.\begin{aligned}
\hat{\sigma}^{(n)} &\to \sigma\\
\hat{v}^{(n)} &\to v
\end{aligned}\right\}\quad \text{uniformly in } C(Q_T)\,. \qquad (1.1.46)
$$

We furthermore have for $x \in [(j-1)/n, j/n[$ that

$$\begin{aligned}\left|\hat{\sigma}^{(n)}(x,t) - \underline{\sigma}(x,t)\right|^2 &\le |\sigma_j(t) - \sigma_{j-1}(t)|^2 \le \sum_{j=1}^{n} |\sigma_j(t) - \sigma_{j-1}(t)|^2 \\ &\le \frac{1}{n}\|\partial_x \hat{\sigma}^{(n)}\|^2_{L^\infty(0,T\,;\,L^2(0,1))}\end{aligned}$$

and similarly

$$\begin{aligned}\left|\hat{\sigma}^{(n)}(x,t) - \bar{\sigma}(x,t)\right|^2 &\le \frac{1}{n}\|\partial_x \hat{\sigma}^{(n)}\|^2_{L^\infty(0,T\,;\,L^2(0,1))}\,, \\ \left|\hat{v}^{(n)}(x,t) - \bar{v}(x,t)\right|^2 &\le \frac{1}{n}\|\partial_x \hat{v}^{(n)}\|^2_{L^\infty(0,T\,;\,L^2(0,1))}\,,\end{aligned}$$

hence

$$\left.\begin{array}{rcl}\bar{\sigma}^{(n)} & \to & \sigma \\ \underline{\sigma}^{(n)} & \to & \sigma \\ \bar{v}^{(n)} & \to & v\end{array}\right\} \quad \text{uniformly in } \ L^\infty(0,1;C[0,T])\,. \tag{1.1.47}$$

The results of Subsection 3.5 enable us to pass to the limit in (1.1.41)–(1.1.43) and check that (v,σ) is a solution of (1.0.1)–(1.0.3). The fact that ε is continuous with respect to x follows directly from Proposition 3.2.2 with $\lambda_i(r) = \lambda(x_i,r)$ and $w_i(t) = \sigma(x_i,t)$, $i = 1,2$ for any $x_1, x_2 \in [0,1]$. ■

1.2 Global boundedness

In order to investigate the asymptotic behaviour of solutions as $t \to \infty$, we first establish conditions under which the solutions remain globally bounded. In particular, we assume that the right-hand side of (1.0.1) is independent of σ. In other words, we consider the system

$$\left\{\begin{array}{rcl}\partial_t v &=& \partial_x \sigma + f(v,x,t)\,, \\ \partial_t \varepsilon &=& \partial_x v\,, \\ \varepsilon &=& \mathcal{F}[\lambda,\sigma]\end{array}\right. \tag{1.2.1}$$

under the following hypotheses.

Hypothesis 1.2.1. *The right-hand side* $f : \mathbb{R}\times\,]0,1[\,\times\,]0,\infty[\,\to \mathbb{R}$ *of (1.2.1) is such that the functions* $f(v,\cdot,\cdot), \partial_t f(v,\cdot,\cdot) :]0,1[\,\times\,]0,\infty[\,\to \mathbb{R}$ *are measurable for all* $v \in \mathbb{R}$, $f(\cdot,x,t), \partial_t f(\cdot,x,t) : \mathbb{R} \to \mathbb{R}$ *are continuous for a. e.* $(x,t) \in\,]0,1[\,\times\,]0,\infty[$, $f^0 := f(0,\cdot,\cdot) \in L^\infty(0,\infty;L^2(0,1))$, *and there exist a constant* $\gamma_f > 0$ *and a function* $\beta_f \in L^\infty(0,\infty;L^2(0,1))$ *such that for almost all arguments we have*

$$-\gamma_f \le \partial_v f(v,x,t) \le 0\,, \tag{1.2.2}$$

$$|\partial_t f(v,x,t)| \le \beta_f(x,t)\,. \tag{1.2.3}$$

Hypothesis 1.2.2. *Let h and κ be the functions associated with the operator $\mathcal{F}$ according to (3.2.1) and (3.4.8). For $p > 0$ set*

$$\mu(p) \;=\; \max\left\{h(p), \frac{h^{3/4}(p)}{\kappa^{1/2}(p)}\right\}, \tag{1.2.4}$$

and assume that

$$\lim_{p\to\infty} \frac{\mu^2(p)\,h(p)}{p^2} \;=\; 0\,. \tag{1.2.5}$$

Hypothesis 1.2.3. *The function H defined by (3.2.3) satisfies the implication*

$$\exists L, m, R_0 > 0 \;\; \forall r > 0 : \; R \ge \max\{R_0, Lr\} \;\Rightarrow\; 2H(R-r) - H(R) \;\ge\; mR\,. \tag{1.2.6}$$

Remark 1.2.4. Condition (1.2.2) says that "negative friction" is excluded. The hysteresis dissipation is not strong enough to keep the solution away from resonance if the energy supply becomes dominant. On the other hand, the results remain valid even if no friction ($\partial_v f = 0$) is present.

The functions $h(p)$ and $\kappa(p)$ characterize the *slope* and the *curvature* of the hysteresis branches, respectively. Condition (1.2.5) is fulfilled if, for example, there exist $0 < h_* \le h^*$, $r_0 > 0$, and $-1 < \alpha < 2/3$, such that

$$h_* \max\{r_0, r\}^{\alpha-1} \;\le\; h'(r) \;\le\; h^* \max\{r_0, r\}^{\alpha-1} \qquad \text{a. e.} \tag{1.2.7}$$

A sufficient condition for (1.2.6) reads for instance

$$\exists \delta_0 > 0 \; : \; \limsup_{r\to\infty} \frac{H((1+\delta_0)r)}{H(r)} \;<\; 2\,. \tag{1.2.8}$$

Indeed, (1.2.8) can be rewritten in the form

$$\exists \delta_0, m, q_0 > 0 \;\; \forall \delta \in \,]0, \delta_0] : \; q \ge q_0 \;\Rightarrow\; \frac{H((1+\delta)q)}{H(q)} \;\le\; 2 - \frac{(1+\delta_0)m}{h(0)}\,. \tag{1.2.9}$$

Then (1.2.6) follows from (1.2.9) with $L = 1 + 1/\delta_0$, $r = \delta q$, $R_0 = (1+\delta_0)q_0$, $R = (1+\delta)q$.

A variant of l'Hôpital's rule implies that condition (1.2.8) is in turn satisfied if, for example, $h(r)$ is concave for large r.

The existence and uniqueness of a global solution to Problem (1.2.1) coupled with (1.0.2)–(1.0.3) under Hypotheses 1.0.1 and 1.2.1 follows from Theorem 1.1.1. The aim of this subsection is to prove the following global boundedness result.

Theorem 1.2.5. *Let Hypotheses 1.0.1 and 1.2.1–1.2.3 hold. Then there exists a constant $C > 0$ independent of t such that the solution (v, σ) to (1.2.1), (1.0.2)–(1.0.3) satisfies a. e. the conditions*

$$\int_0^1 \left((\partial_t v)^2 + (\partial_x v)^2 + (\partial_t \sigma)^2 + (\partial_x \sigma)^2\right)(x,t)\,dx \;\le\; C\,, \tag{1.2.10}$$

$$|v(x,t)| + |\sigma(x,t)| \;\le\; C\,. \tag{1.2.11}$$

The proof of Theorem 1.2.5 is split into two parts. We first prove Lemma 1.2.6 below. Since no boundary conditions for σ are prescribed, the transition from Lemma 1.2.6 to (1.2.10)–(1.2.11) is not straightforward and a deeper result on hysteresis memory structure will have to be established in the subsequent Lemma 1.2.7.

In accordance with the previous notation, we define for $t > 0$ the sets $Q_t = [0,1] \times [0,t]$ and for $u \in L^\infty(Q_t)$ put

$$\|u\|_{Q_t} = \sup \mathrm{ess} \{|u(x,\tau)| \,;\, (x,\tau) \in Q_t\} . \tag{1.2.12}$$

Lemma 1.2.6. *Let Hypotheses 1.0.1 and 1.2.1–1.2.2 hold, and let (v, σ) be the solution to (1.2.1), (1.0.2)–(1.0.3). Then for every $\delta > 0$ there exists $p_0 > 0$ such that for all $p \geq p_0$ the following implication holds true for a. e. $t \geq 0$.*

$$\|\sigma\|_{Q_t} \leq p \implies \frac{1}{p^2} \int_0^1 \left((\partial_t \sigma)^2 + (\partial_x \sigma)^2\right)(x,t)\, dx \leq \delta . \tag{1.2.13}$$

Proof of Lemma 1.2.6. Taking into account the convergences (1.1.45)–(1.1.47), it suffices to consider the discrete system

$$\dot v_j = n(\sigma_{j+1} - \sigma_j) + f_j(v_j, t) , \quad j = 1, \dots n-1 , \tag{1.2.14}$$

$$\dot\varepsilon_j = n(v_j - v_{j-1}) , \quad j = 1, \dots n , \tag{1.2.15}$$

$$\varepsilon_j = \mathcal{F}[\lambda_j, \sigma_j] , \quad j = 1, \dots n \tag{1.2.16}$$

analogous to (1.1.4)–(1.1.6) with boundary and initial conditions (1.1.7)–(1.1.8), and prove that for every $\delta > 0$ there exists $p_0 > 0$ independent of n and t such that for all $p \geq p_0$ the following implication holds.

$$\max_{j=1,\dots,n} \|\sigma_j\|_{[0,t]} \leq p \implies \frac{1}{p^2} \left(\frac{1}{n} \sum_{j=1}^{n} \dot\sigma_j^2(t) + n \sum_{j=1}^{n-1} (\sigma_{j+1} - \sigma_j)^2(t) \right) \leq \delta . \tag{1.2.17}$$

To do so, we define for $j = 1, \dots, n-1$ auxiliary functions

$$G_j(v,t) = v\, f_j(v,t) - \int_0^v f_j(v', t)\, dv' \qquad \text{for } (v,t) \in \mathbb{R} \times [0, \infty[, \tag{1.2.18}$$

and set

$$E(t) = \frac{1}{2n} \left(\sum_{j=1}^{n-1} \dot v_j^2(t) + \sum_{j=1}^{n} \dot\varepsilon_j(t) \dot\sigma_j(t) \right) , \tag{1.2.19}$$

$$S(t) = \frac{1}{n} \sum_{j=1}^{n} |\dot\sigma_j(t)|^3 , \tag{1.2.20}$$

$$W(t) = \frac{1}{n} \sum_{j=1}^{n-1} \dot v_j^2(t) , \tag{1.2.21}$$

$$Z(t) = \frac{1}{n} \sum_{j=1}^{n-1} \left(G_j(v_j(t), t) - v_j(t) \dot v_j(t) \right) . \tag{1.2.22}$$

We have by definition

$$-\gamma_f\, v_j^2(t) \;\le\; G_j(v_j(t),t) \;\le\; 0\,, \tag{1.2.23}$$

The boundary condition (1.1.7) and Eq. (1.1.5) yield

$$\frac{1}{n}\sum_{j=1}^{n-1} v_j^2(t) \;\le\; n\sum_{j=1}^{n}(v_j - v_{j-1})^2(t) \;=\; \frac{1}{n}\sum_{j=1}^{n}\varepsilon_j^2(t)\,. \tag{1.2.24}$$

Assume now that $\max_{j=1,\dots,n}\|\sigma_j\|_{[0,T]} \le p$ for some $T>0$ and $p \ge K$. From (3.2.16) we obtain for $t\in[0,T]$ that

$$|\dot\varepsilon_j(t)| \le h(p)|\dot\sigma_j(t)| \qquad \text{for } j=1,\dots,n\,. \tag{1.2.25}$$

This, together with (3.2.17), implies that

$$\frac{1}{n}\sum_{j=1}^{n-1} v_j^2(t) \;\le\; h(p)\frac{1}{n}\sum_{j=1}^{n}\dot\varepsilon_j(t)\dot\sigma_j(t)\,. \tag{1.2.26}$$

We now fix a constant $c^* > 0$ such that

$$|Z(t)| \;\le\; c^*\, h(p)\, E(t) \tag{1.2.27}$$

for all $t\in[0,T]$. Using (1.2.14)–(1.2.15) yields for a. e. $t\in\,]0,T[$ that

$$\begin{aligned}\dot Z(t) + W(t) \;&=\; \frac{1}{n}\sum_{j=1}^{n}\dot\varepsilon_j(t)\dot\sigma_j(t) + \frac{1}{n}\sum_{j=1}^{n-1}(\partial_t G_j - v_j\,\partial_t f_j)(t) \\ &\le\; \frac{1}{n}\sum_{j=1}^{n}\dot\varepsilon_j(t)\dot\sigma_j(t) + \frac{1}{n}\sum_{j=1}^{n-1}|v_j(t)|\,|\beta_j(t)|\,,\end{aligned} \tag{1.2.28}$$

where β_j is as in (1.1.18), hence, by (1.2.25)–(1.2.26),

$$\begin{aligned}\dot Z(t) + \frac{1}{2}W(t) + E(t) \;&\le\; \frac{3}{2n}\sum_{j=1}^{n}\dot\varepsilon_j(t)\dot\sigma_j(t) + C\left(\frac{1}{n}\sum_{j=1}^{n-1}v_j^2(t)\right)^{1/2} \\ &\le\; C\,h(p)\left(1+S^{2/3}(t)\right)\,,\end{aligned} \tag{1.2.29}$$

where C denotes as before any positive constant independent of n, T, and p. The counterpart of (1.1.18)–(1.1.19) reads

$$\ddot v_j \dot v_j \;\le\; n(\dot\sigma_{j+1} - \dot\sigma_j)\dot v_j + |\beta_j(t)|\,|\dot v_j(t)|\,, \tag{1.2.30}$$

$$\ddot\varepsilon_j\dot\sigma_j \;=\; n(\dot v_j - \dot v_{j-1})\dot\sigma_j\,, \tag{1.2.31}$$

hence

$$\frac{1}{n}\sum_{j=1}^{n-1}\ddot v_j(t)\dot v_j(t) + \frac{1}{n}\sum_{j=1}^{n}\ddot\varepsilon_j(t)\dot\sigma_j(t) \;\le\; C\,W^{1/2}(t) \qquad \text{a. e.} \tag{1.2.32}$$

From Theorem 3.4.1 it follows that for all $0 \le s < t \le T$ we have

$$E(t-) - E(s+) + \frac{1}{4}\kappa(p) \int_s^t S(\tau)\, d\tau \ \le\ C \int_s^t W^{1/2}(\tau)\, d\tau\,. \tag{1.2.33}$$

Let c^* be as in (1.2.27). Inequalities (1.2.29) and (1.2.33) yield for all $0 \le s < t$ that

$$\begin{aligned} &(Z + 2c^*\, \mu(p)\, E)(t-) - (Z + 2c^*\, \mu(p)\, E)(s+) \qquad (1.2.34)\\ &\qquad + \frac{1}{2}c^*\, \mu(p)\kappa(p) \int_s^t S(\tau)\, d\tau + \int_s^t \left(\frac{1}{2}W(\tau) + E(\tau)\right)\, d\tau \\ &\quad \le\ C \int_s^t \left(\mu(p)\, W^{1/2}(\tau) + h(p)\left(1 + S^{2/3}(\tau)\right)\right)\, d\tau\,, \end{aligned}$$

hence

$$\begin{aligned} (Z + 2c^*\, \mu(p)\, E)(t-) - (Z + 2c^*\, \mu(p)\, E)(s+) + \int_s^t E(\tau)\, d\tau \qquad (1.2.35)\\ \le\ C\left(\mu^2(p) + h(p) + \frac{h^3(p)}{\kappa^2(p)\mu^2(p)}\right)(t-s) \\ \le\ C^*\left(1 + \mu^2(p)\right)(t-s)\,, \end{aligned}$$

where C^* is some frozen value of C, and set

$$E_1(t)\ =\ Z(t) + 2c^*\, \mu(p)\, E(t)\,, \quad E_2(t)\ =\ \int_0^t E(\tau)\, d\tau - C^*(1 + \mu^2(p))\, t\,. \tag{1.2.36}$$

By (1.2.35), the function $E_1 + E_2$ is non-increasing, hence for every non-negative absolutely continuous test function $\eta(t)$ and every $t \in [0, T]$ we have

$$\int_0^t (E_1 + E_2)(\tau)\, \dot\eta(\tau)\, d\tau\ \ge\ (E_1(t-) + E_2(t))\, \eta(t) - E_1(0+)\, \eta(0)\,, \tag{1.2.37}$$

or equivalently,

$$E_1(t-)\, \eta(t)\ \le\ E_1(0+)\, \eta(0) + \int_0^t E_1(\tau)\, \dot\eta(\tau)\, d\tau - \int_0^t \dot E_2(\tau)\, \eta(\tau)\, d\tau\,. \tag{1.2.38}$$

We choose

$$\eta(t)\ =\ \mathrm{e}^{qt}\,, \quad q\ =\ \frac{1}{3c^*\, \mu(p)}\,. \tag{1.2.39}$$

Using (1.1.23)–(1.1.28), we estimate $E_1(0+)$ by $C(1 + \mu(p))$, so that (1.2.38) yields

$$E_1(t-)\, \mathrm{e}^{qt}\ \le\ C(1 + \mu(p)) + \int_0^t (qE_1 - E)\,(\tau)\, \mathrm{e}^{q\tau}\, d\tau + \frac{1}{q}C^*(1 + \mu^2(p))\left(\mathrm{e}^{qt} - 1\right)\,. \tag{1.2.40}$$

We have $qE_1(\tau) - E(\tau) = qZ(\tau) - \frac{1}{3}E(\tau) \le 0$, hence

$$E_1(t-)\ \le\ C(1 + \mu(p))\, \mathrm{e}^{-qt} + \frac{1}{q}\, C^*(1 + \mu^2(p))\,, \tag{1.2.41}$$

that is,

$$E(t) \ \leq \ \frac{1}{c^* \, \mu(p)} E_1(t) \ \leq \ C(1+\mu^2(p)) \qquad \text{for} \quad \text{a.e.} \ \ t \in [0,T]\,. \tag{1.2.42}$$

Using (1.2.26) we have

$$\frac{1}{n} \sum_{j=1}^{n-1} v_j^2(t) \ \leq \ Ch(p)(1+\mu^2(p))\,, \tag{1.2.43}$$

and (1.2.14) yields

$$n \sum_{j=1}^{n} (\sigma_{j+1} - \sigma_j)^2(t) \ \leq \ Ch(p)(1+\mu^2(p)) \qquad \text{for} \ \ t \in [0,T]\,, \tag{1.2.44}$$

and (1.2.17) follows from (1.2.5) and (3.2.17). This completes the proof of Lemma 1.2.6. ■

Lemma 1.2.7. *Let Hypothesis 1.2.3 hold, and let* $\lambda : [0,1] \to \Lambda_K$ *,* $\varepsilon, \sigma : [0,1] \times [0,T] \to \mathbb{R}$ *be continuous mappings,* $\varepsilon = \mathcal{F}[\lambda, \sigma]$ *. Assume that there exist constants* c_1, c_2, c_3 *independent of* t *such that*

$$\begin{aligned} \left| \int_0^1 \varepsilon(x,t)\, dx \right| \ &\leq \ c_1 \quad \forall t \in [0,T]\,, & (1.2.45) \\ |\sigma(x,0)| \ &\leq \ c_2 \quad \forall x \in [0,1]\,, & (1.2.46) \\ \int_0^1 |\partial_x \sigma(x,t)|\, dx \ &\leq \ 2c_3 \quad \forall t \in [0,T]\,. & (1.2.47) \end{aligned}$$

Put

$$R \ = \ \max\left\{ Lc_3, \frac{c_1+1}{m}, c_2+1, K, R_0 \right\}\,. \tag{1.2.48}$$

Then $|\sigma(x,t)| < R$ *for all* $(x,t) \in [0,1] \times [0,T]$ *.*

Proof of Lemma 1.2.7. Assume that the statement is false. Then there exist x_0, t_0 such that one of the following two alternatives occurs.

(i) $\sigma(x_0,t_0) = R,\ \sigma(x,t) > -R$ for $(x,t) \in [0,1] \times [0,t_0[$;

(ii) $\sigma(x_0,t_0) = -R,\ \sigma(x,t) < R$ for $(x,t) \in [0,1] \times [0,t_0[$.

The two cases are similar, we therefore consider only (i).

For all $x \in [0,1]$ we have

$$\sigma(x,t_0) \geq \sigma(x_0,t_0) - \int_0^1 |\partial_x \sigma(x,\tau)|\, d\tau \geq R - 2c_3\,, \tag{1.2.49}$$

hence, by definition (3.1.13) of the play,

$$\mathfrak{p}_r[\lambda,\sigma](x,t_0) \;\ge\; \sigma(x,t_0) - r \;\ge\; R - 2c_3 - r \tag{1.2.50}$$

for all $r > 0$. By Lemma 3.1.2, we have

$$\mathfrak{p}_r[\lambda,\sigma](x,t) \ge \min\{\lambda(x,r), -R+r\} \tag{1.2.51}$$

for all $r > 0$ and $(x,t) \in [0,1]\times[0,t_0]$. We have

$$\lambda(x,r) \;=\; \begin{cases} 0 & \text{for } r \ge K\,, \\ \lambda(x,r) - \lambda(x,K) \ge -K + r & \text{for } 0 < r < K\,, \end{cases} \tag{1.2.52}$$

hence

$$\mathfrak{p}_r[\lambda,\sigma](x,t) \;\ge\; \min\{0, -R+r\}\,. \tag{1.2.53}$$

Consequently, combining (1.2.50) with (1.2.53), we obtain

$$\mathfrak{p}_r[\lambda,\sigma](x,t_0) \ge \max\{R - 2c_3 - r,\, \min\{0, -R+r\}\}\,, \tag{1.2.54}$$

and formula (3.2.1) with w replaced by σ yields for every $x \in [0,1]$ that

$$\begin{aligned} \varepsilon(x,t_0) \;&\ge\; h(0)(R-2c_3) + \int_0^{R-c_3} (R - 2c_3 - r)\,dh(r) + \int_{R-c_3}^{R} (-R+r)\,dh(r) \quad (1.2.55)\\ &=\; 2H(R-c_3) - H(R) \;\ge\; 1 + c_1\,, \end{aligned}$$

which is a contradiction with the assumption (1.2.45). ∎

We are now ready to pass to the proof of Theorem 1.2.5 and thus conclude this subsection.

Proof of Theorem 1.2.5. By virtue of the boundary conditions for v, the solution to (1.2.1) has the property

$$\int_0^1 \varepsilon(x,t)\,dx \;=\; \int_0^1 \varepsilon^0(x)\,dx$$

for every $t \ge 0$. We therefore can apply Lemma 1.2.7 and find a constant $\bar{C} > 0$ independent of t such that for every $t \ge 0$ we have

$$\|\sigma\|_{Q_t} \;\le\; \bar{C}\left(1 + \int_0^1 (\partial_x\sigma)^2(x,t)\,dx\right)\,. \tag{1.2.56}$$

By Lemma 1.2.6 and (1.2.56), we have the implication

$$\forall \delta > 0\; \exists p_0 > 0\; \forall p \ge p_0\; \forall t \ge 0 \,:\; \|\sigma\|_{Q_t} \;\le\; p \;\Rightarrow\; \|\sigma\|_{Q_t} \;\le\; \bar{C}\,(1 + \delta p)\,. \tag{1.2.57}$$

Choosing for instance $\delta = 1/(2\bar{C})$, we see that $|\sigma(x,t)|$ cannot exceed the value $2(p_0 + \bar{C})$. Using again Lemma 1.2.6 we obtain uniform $L^2(0,1)$-bounds for $\partial_t\sigma(\cdot,t)$ and $\partial_x\sigma(\cdot,t)$, which in turn (as a consequence of (3.2.16)) imply a uniform $L^2(0,1)$-bound for $\partial_t\varepsilon(\cdot,t)$. The bounds for $\partial_t v$ and $\partial_x v$ follow directly from the equations (1.2.1). ∎

1.3 Asymptotic stabilization

Solutions to hyperbolic equations with linear viscous terms and without forcing asymptotically vanish with exponential rate. This is not the case if hysteresis is the only source of dissipation. In our situation, the decay rate is of the order $1/t$, and Example 1.3.3 confirms that this estimate is optimal. Keeping initial and boundary conditions (1.0.2)–(1.0.3), we consider the system with time-independent external forcing

$$\left\{\begin{array}{rcl} \partial_t v &=& \partial_x \sigma + f(v,x)\,, \\ \partial_t \varepsilon &=& \partial_x v\,, \\ \varepsilon &=& \mathcal{F}[\lambda,\sigma] \end{array}\right. \tag{1.3.1}$$

under the following hypothesis.

Hypothesis 1.3.1. *The function* $f : \mathbb{R}\times\,]0,1[\,\to \mathbb{R}$ *is such that the functions* $f(v,\cdot) :]0,1[\,\to \mathbb{R}$ *is measurable for all* $v \in \mathbb{R}$*,* $f(\cdot,x) : \mathbb{R} \to \mathbb{R}$ *is continuous for a. e.* $x \in]0,1[$ *,* $f^0 := f(0,\cdot) \in L^2(0,1)$*, and there exists a constant* $\gamma_f > 0$ *such that for almost all arguments we have*

$$-\gamma_f \;\le\; \partial_v f(v,x) \;\le\; 0\,. \tag{1.3.2}$$

The main result of this subsection reads as follows.

Theorem 1.3.2. *Let Hypotheses 1.0.1 and 1.3.1 hold, and let*

$$\lim_{p\to\infty} \frac{h(p)}{p^2} \;=\; 0\,. \tag{1.3.3}$$

Then there exists a constant $C > 0$ *independent of* t *such that the solution* (v,σ) *to (1.3.1), (1.0.2)–(1.0.3) satisfies a. e. the conditions*

$$\int_0^1 \left((\partial_t v)^2 + (\partial_x v)^2 + (\partial_t \sigma)^2 + (\partial_x \sigma)^2\right)(x,t)\,dx \;\le\; C\,, \tag{1.3.4}$$

$$|v(x,t)| + |\sigma(x,t)| \;\le\; C\,. \tag{1.3.5}$$

If moreover the function κ *from (3.4.8) does not vanish on* $[0,\infty[$ *, then there exist constants* $\sigma_\infty \in \mathbb{R}$ *and* $C > 0$ *independent of* t *such that*

$$\int_0^1 \left((\partial_t v)^2 + (\partial_x v)^2 + (\partial_t \sigma)^2 + (\partial_x \sigma + f^0)^2\right)(x,t)\,dx \;\le\; \frac{C}{(1+t)^2}\,, \tag{1.3.6}$$

$$|v(x,t)| \;\le\; \frac{C}{1+t}\,, \tag{1.3.7}$$

$$\lim_{t\to\infty} \left|\sigma(x,t) + \int_0^x f^0(x')\,dx' - \sigma_\infty\right| \;=\; 0\,, \tag{1.3.8}$$

and the limit in (1.3.8) is uniform with respect to x*.*

Proof. We proceed as in the proof of Theorem 1.2.5. Relations (1.2.30)–(1.2.33) remain valid with $\beta_j \equiv 0$ and $C = 0$, hence $E(t)$ is non-increasing in $[0,\infty[$. The value of $E(0+)$ is bounded by a constant according to (1.1.23)–(1.1.28), hence so is $E(t)$. In order to simplify the notation, we argue formally using Eqs. (1.3.1), having however in mind the discrete system of the form (1.2.14)–(1.2.16). With this convention, we have

$$\int_0^1 \left((\partial_t v)^2 + \partial_t\sigma\, \partial_t\varepsilon\right)(x,t)\, dx \ \leq\ C\,, \tag{1.3.9}$$

where C is some constant independent of $t > 0$. We now fix $T > 0$ and $p > 0$ such that

$$\|\sigma\|_{Q_T} \ \leq\ p\,. \tag{1.3.10}$$

By (1.3.1) and (3.2.16)–(3.2.17) we have

$$\int_0^1 (\partial_x v)^2(x,t)\, dx \ =\ \int_0^1 (\partial_t\varepsilon)^2(x,t)\, dx \ \leq\ h(p)\int_0^1 (\partial_t\sigma\, \partial_t\varepsilon)(x,t)\, dx \ \leq\ C\, h(p) \tag{1.3.11}$$

with a constant C (here and in the sequel) independent of $t \in [0,T]$ and $p > 0$, hence

$$\int_0^1 v^2(x,t)\, dx \ \leq\ C\, h(p) \quad \forall t \in [0,T]\,, \tag{1.3.12}$$

and

$$\int_0^1 (\partial_x\sigma)^2(x,t)\, dx \ \leq\ C + \int_0^1 \left((\partial_t v)^2 + \gamma_f v^2\right)(x,t)\, dx \ \leq\ C(1+h(p))\,. \tag{1.3.13}$$

From Lemma 1.2.7 it follows that $\|\sigma\|_{Q_T} \leq C\sqrt{1+h(p)}$. Choosing p sufficiently large we thus obtain from (1.3.3) the global bounds (1.3.4)–(1.3.5).

To prove (1.3.6)–(1.3.7), we pass again to the space-discrete approximations. Note that a uniform upper bound for $|\sigma(x,t)|$ is already available by virtue of (1.3.5). We therefore do not have to consider the dependence on p in (1.2.25)–(1.2.34). Using again the fact that $\beta_j \equiv 0$ in (1.2.28), (1.2.30), and that κ is positive, we obtain the counterpart of (1.2.29) and (1.2.33) in the form

$$\dot Z(t) + E(t) \ \leq\ C\, S^{2/3}(t) \qquad \text{a.e.}\ , \tag{1.3.14}$$

$$E(t-) - E(s+) + c\int_s^t S(\tau)\, d\tau \ \leq\ 0 \quad \forall 0 \leq s < t \tag{1.3.15}$$

with some constants $c, C > 0$. In agreement with (1.2.27), we now fix some $m > 0$ such that

$$|Z(t)| \ \leq\ \frac{1}{8m} E(t) \qquad \text{a.e.} \tag{1.3.16}$$

and set

$$E_m(t) \ :=\ E(t) + \frac{4m}{1+mt}\, Z(t) \ \geq\ \frac{1}{2} E(t)\,. \tag{1.3.17}$$

We have for all $0 \le s < t$ that

$$E_m(t-) - E_m(s+) \ + \ \int_s^t \left(c\,S(\tau) + \frac{4m}{1+m\tau} E(\tau) + \frac{4m^2}{(1+m\tau)^2}\, Z(\tau)\right) d\tau \qquad (1.3.18)$$
$$\le \ \int_s^t \frac{4mC}{1+m\tau} S^{2/3}(\tau)\, d\tau\,,$$

hence, by Hölder's inequality, there exists another constant $C > 0$ such that

$$E_m(t-) - E_m(s+) + \int_s^t \left(\frac{4m}{1+m\tau} E(\tau) + \frac{4m^2}{(1+m\tau)^2}\, Z(\tau)\right) d\tau \ \le \ \int_s^t \frac{C}{(1+m\tau)^3}\, d\tau\,. \qquad (1.3.19)$$

In view of (1.3.16), we have

$$4E(\tau) + \frac{4m}{(1+m\tau)}\, Z(\tau) \ \ge \ 3E_m(\tau) \qquad \text{a. e.}\ \ , \qquad (1.3.20)$$

hence

$$E_m(t-) - E_m(s+) + \int_s^t \frac{3m}{1+m\tau} E_m(\tau)\, d\tau \ \le \ \int_s^t \frac{C}{(1+m\tau)^3}\, d\tau \qquad (1.3.21)$$

for all $0 \le s < t$. We argue similarly as in (1.2.37). The function

$$t \ \mapsto \ E_m(t) + \int_0^t \left(\frac{3m}{1+m\tau} E_m(\tau) - \frac{C}{(1+m\tau)^3}\right) d\tau$$

is non-increasing, hence for every non-negative absolutely continuous test function $\eta(t)$ we have

$$\int_0^t \left(E_m(\tau)\,\dot\eta(\tau) - \left(\frac{3m}{1+m\tau} E_m(\tau) - \frac{C}{(1+m\tau)^3}\right) \eta(\tau)\right) d\tau \qquad (1.3.22)$$
$$\ge \ E_m(t-)\,\eta(t) - E_m(0+)\,\eta(0)\,.$$

For $\eta(t) = (1+mt)^3$ this yields

$$E_m(t-)\,(1+mt)^3 \ \le \ E_m(0+) + Ct\,. \qquad (1.3.23)$$

From (1.3.23) and (1.3.17) we obtain $E(t) \ \le \ C\,(1+t)^{-2}$ a. e., and inequalities (1.3.6)–(1.3.7) easily follow. It remains to prove the convergence (1.3.8) of σ.

We fix constants $R > K$, $F_0 > 0$ such that

$$\int_0^1 |f^0(x')|\, dx' \ \le \ F_0\,, \quad |\sigma(x,t)| \ \le \ R \quad \forall (x,t) \in [0,1] \times [0,\infty[\,, \qquad (1.3.24)$$

and define auxiliary functions

$$\tilde\sigma(x,t) \ = \ \sigma(x,t) + \int_0^x f^0(x')\, dx'\,, \qquad (1.3.25)$$

$$\tilde\lambda(x,r) \ = \ \begin{cases} \lambda(x,r) & \text{for } 0 \le r \le R\,, \\ P[0,\tilde\lambda(x,R)](r-R) & \text{for } r \ge R\,, \end{cases} \qquad (1.3.26)$$

$$\tilde\varepsilon(x,t) \ = \ \mathcal{F}[\tilde\lambda,\tilde\sigma](x,t)\,, \qquad (1.3.27)$$

where the mapping P is defined by (3.1.16). Note that $|\tilde\lambda(x,R)| \le F_0$, hence $\tilde\lambda(x,r) = 0$ for $r \ge R+F_0$. We now claim that

$$\partial_t \tilde\varepsilon(x,t) \;=\; \partial_t \varepsilon(x,t) \qquad \text{a. e.} \tag{1.3.28}$$

To check this conjecture, we denote

$$\xi_r(x,t) \;=\; \mathfrak{p}_r[\lambda,\sigma](x,t)\,, \quad \tilde\xi_r(x,t) \;=\; \mathfrak{p}_r[\tilde\lambda,\tilde\sigma](x,t) \tag{1.3.29}$$

for $r>0$ and $(x,t) \in [0,1]\times[0,\infty[$. By (3.3.1), we have

$$\frac{\partial}{\partial t}\left(\tilde\xi_r(x,t) - \xi_r(x,t) - \int_0^x f^0(x')\,dx'\right)^2 \;\le\; 0\,, \tag{1.3.30}$$

hence

$$\left|\tilde\xi_r(x,t) - \xi_r(x,t) - \int_0^x f^0(x')\,dx'\right| \;\le\; \left|\tilde\xi_r(x,0) - \xi_r(x,0) - \int_0^x f^0(x')\,dx'\right| \tag{1.3.31}$$

for all admissible values of r, x, and t. We have by (3.1.10), (3.1.16) that

$$\xi_r(x,0) \;=\; P[\lambda(x,\cdot),\sigma(x,0)](r)\,, \quad \tilde\xi_r(x,0) \;=\; P[\tilde\lambda(x,\cdot),\tilde\sigma(x,0)](r)\,,$$

hence

$$\tilde\xi_r(x,0) = \xi_r(x,0) + \int_0^x f^0(x')\,dx' \qquad \text{for } 0<r\le R\,, \tag{1.3.32}$$

and (1.3.31) implies that

$$\frac{\partial}{\partial t}\tilde\xi_r(x,t) \;=\; \frac{\partial}{\partial t}\xi_r(x,t) \qquad \text{a. e.} \quad \text{for } 0<r\le R\,. \tag{1.3.33}$$

On the other hand, we have $\lambda(x,r)=0$ for $r\ge R$ and $|\tilde\lambda(x,R)-\tilde\sigma(x,t)|\le R$, hence $\left\| m_{\tilde\lambda(x,\cdot)}(\tilde\sigma(x,\cdot))\right\|_{[0,t]} \le R$ for all x and t. From Lemma 3.1.2 we conclude that

$$\frac{\partial}{\partial t}\tilde\xi_r(x,t) \;=\; \frac{\partial}{\partial t}\xi_r(x,t) \qquad \text{a. e.} \quad \text{for } r>R\,. \tag{1.3.34}$$

Combining (1.3.33) with (1.3.34), we obtain (1.3.28) from the definition (3.2.1) of the operator $\mathcal{F}$. This enables us to rewrite the system (1.3.1) in the form

$$\left\{\begin{array}{rcl} \partial_t v &=& \partial_x\tilde\sigma + f(v,x) - f(0,x)\,, \\ \partial_t\tilde\varepsilon &=& \partial_x v \end{array}\right. \tag{1.3.35}$$

together with the identity (1.3.27). In particular, we have for all $t\ge 0$ that

$$\int_0^1 \tilde\varepsilon(x,t)\,dx \;=\; \int_0^1 \tilde\varepsilon(x,0)\,dx \;=\; const. \tag{1.3.36}$$

Put $s(t) = \int_0^1 \tilde\sigma(x,t)\,dx$. The estimate

$$|\dot s(t)| \;\le\; \frac{C}{1+t}$$

which follows from (1.3.6) is not sufficient for concluding that $s(t)$ converges as $t \to \infty$. To prove that this convergence indeed takes place, we have to use again special properties of the operator $\mathcal{F}$, more precisely Lemma 3.1.2. Set

$$\bar{s} \; = \; \limsup_{t\to\infty} s(t)\,, \quad \underline{s} \; = \; \liminf_{t\to\infty} s(t)\,, \tag{1.3.37}$$

and assume that $\bar{s} > \underline{s}$. We fix some $\alpha > 0$ sufficiently small (it will be specified below in (1.3.47)), and using (1.3.6) we find $0 < t_0 < t_1 < t_2$ such that

$$\int_0^1 |\partial_x \tilde{\sigma}(x,t)|\,dx \; \le \; \alpha\,, \quad \underline{s} - \alpha \le s(t) \le \bar{s} + \alpha \qquad \text{for } t \ge t_0\,, \tag{1.3.38}$$

$$s(t_1) \; \le \; \underline{s} + \alpha\,, \quad s(t_2) \; \ge \; \bar{s} - \alpha\,. \tag{1.3.39}$$

For all $x \in [0,1]$ and $t \ge t_0$ we have

$$|\tilde{\sigma}(x,t) - s(t)| \; \le \; \int_0^1 |\partial_x \tilde{\sigma}(x,t)|\,dx \; \le \; \alpha\,,$$

hence

$$\tilde{\sigma}(x,t_1) \; \le \; \underline{s} + 2\alpha\,, \quad \tilde{\sigma}(x,t_2) \; \ge \; \bar{s} - 2\alpha\,. \tag{1.3.40}$$

For $r > 0$ set $\lambda_i(x,r) = \mathfrak{p}_r[\tilde{\lambda}, \tilde{\sigma}](x,t_i)$, $i = 1,2$. On the one hand, we have by definition of the play that

$$\lambda_1(x,r) \; \le \; \tilde{\sigma}(x,t_1) + r \; \le \; \underline{s} + 2\alpha + r\,, \quad \lambda_2(x,r) \; \ge \; \tilde{\sigma}(x,t_2) - r \; \ge \; \bar{s} - 2\alpha - r\,, \tag{1.3.41}$$

on the other hand, Lemma 3.1.2 and the semigroup property (3.1.21) yield that

$$\lambda_2(x,r) \; \ge \; \min\{\lambda_1(x,r), \bar{s} - 2\alpha + r\}\,, \tag{1.3.42}$$

hence

$$\lambda_2(x,r) \; \ge \; \min\{\lambda_1(x,r), \lambda_1(x,r) - 4\alpha\} \; = \; \lambda_1(x,r) - 4\alpha\,. \tag{1.3.43}$$

Combining (1.3.43) with (1.3.41), we obtain

$$\lambda_2(x,r) \; \ge \; \max\{\lambda_1(x,r), \bar{s} + 2\alpha - r\} - 4\alpha\,, \tag{1.3.44}$$

consequently

$$\lambda_2(x,r) - \lambda_1(x,r) \; \ge \; \max\{0, \bar{s} + 2\alpha - r - \lambda_1(x,r)\} - 4\alpha \; \ge \; \max\{0, \bar{s} - \underline{s} - 2r\} - 4\alpha\,. \tag{1.3.45}$$

Inserting the inequality (1.3.45) into the integral in the definition (3.2.1) of $\mathcal{F}$ (note that h is non-decreasing), we obtain

$$\begin{aligned} \tilde{\varepsilon}(x,t_2) - \tilde{\varepsilon}(x,t_1) \; &= \; h(0)(\tilde{\sigma}(x,t_2) - \tilde{\sigma}(x,t_1)) \\ &\quad + \int_0^{R+F_0} (\lambda_2(x,r) - \lambda_1(x,r))\,dh(r) \\ &\ge \; 2H\left(\frac{\bar{s} - \underline{s}}{2}\right) - 4\alpha\, h(R+F_0)\,. \end{aligned} \tag{1.3.46}$$

Choosing $\alpha > 0$ such that

$$2H\left(\frac{\bar{s} - \underline{s}}{2}\right) - 4\alpha\, h(R + F_0) \;\geq\; \alpha\,, \tag{1.3.47}$$

we obtain

$$\int_0^1 (\tilde{\varepsilon}(x, t_2) - \tilde{\varepsilon}(x, t_1))\, dx \;\geq\; \alpha \tag{1.3.48}$$

in contradiction with (1.3.36). We conclude that

$$\bar{s} \;=\; \underline{s} \;=:\; \sigma_\infty\,, \tag{1.3.49}$$

and the proof of Theorem 1.3.2 is complete. ■

Note that $\varepsilon(x, t)$ also converges uniformly to some function $\varepsilon_\infty \in C[0, 1]$. Indeed, by (3.2.11) we have for $0 \leq s < t$ that

$$|\varepsilon(x, t) - \varepsilon(x, s)| \;\leq\; C\, \|\sigma(x, \cdot) - \sigma(x, s)\|_{[s,t]}$$

with a constant $C > 0$ independent of x, t, and s. The convergence $\tilde{\sigma}(\cdot, t) \to \sigma_\infty$ is uniform with respect to x, hence also $\varepsilon(\cdot, t)$ converge uniformly. Since all $\varepsilon(\cdot, t)$ are continuous by Theorem 1.1.1, we conclude that ε_∞ is continuous.

Example 1.3.3. In order to illustrate the optimality of the $1/t$ decay rate as $t \to \infty$, we consider the following ODE system describing an elastoplastic spring-mass oscillator

$$\dot{v} \;=\; -\sigma\,, \tag{1.3.50}$$

$$\dot{\varepsilon} \;=\; v\,, \tag{1.3.51}$$

$$\varepsilon \;=\; \mathcal{F}[0, \sigma]\,, \tag{1.3.52}$$

with initial conditions

$$v(0) = 0\,, \quad \varepsilon(0) = \varepsilon_0 > 0\,, \tag{1.3.53}$$

where $\mathcal{F}[0, \cdot]$ is the Prandtl-Ishlinskii operator (3.2.1) with $\lambda \equiv 0$, and ε_0 is given. In fact, (1.3.50)–(1.3.52) is related to the space-discrete system (1.1.4)–(1.1.6) for $n = 2$ which is of the form

$$\dot{v}_1 \;=\; 2(\sigma_2 - \sigma_1)\,, \tag{1.3.54}$$

$$\dot{\varepsilon}_1 \;=\; 2v_1\,, \tag{1.3.55}$$

$$\dot{\varepsilon}_2 \;=\; -2v_1\,, \tag{1.3.56}$$

$$\varepsilon_j \;=\; \mathcal{F}[0, \sigma_j]\,, \quad j = 1, 2\,. \tag{1.3.57}$$

Indeed, assuming $\varepsilon_1(0) = -\varepsilon_2(0)$ and using the fact that both $\mathcal{F}$ and ist inverse $\hat{\mathcal{F}}$ (see Proposition 3.2.1) are odd, we obtain $\varepsilon_1 = -\varepsilon_2, \sigma_1 = -\sigma_2$, so that, after suitable rescaling, System (1.3.54)–(1.3.57) is equivalent to (1.3.50)–(1.3.52). We thus may use the argument of the proof of Theorem 1.3.2 and conclude that there exists a constant $C > 0$ such that

$$|v(t) + |\sigma(t)| \;\leq\; \frac{C}{1 + t} \quad \forall t \geq 0\,. \tag{1.3.58}$$

It will immediately follow from Proposition 1.3.4 below that this decay rate is optimal. On the other hand, we show in Remark 1.3.6 that $\varepsilon(t)$ does not asymptotically vanish, but converges as $t \to \infty$ to some positive limit $\varepsilon_\infty > 0$. In mechanical interpretation, this means in agreement with practical experience that the initial deformation is not completely recovered during free elastoplastic oscillations.

Proposition 1.3.4. *Let the generating function h of the Prandtl-Ishlinskii operator $\mathcal{F}$ be locally Lipschitz continuous in $[0,\infty[$, $h(0) > 0$, $h'(r) > 0$ a. e. Then there exist sequences $0 = t_0 < t_1 < t_2 < \dots$ and $\sigma_0 > \sigma_1 > \sigma_2 > \dots > 0$ such that the solution (v,σ) to (1.3.50)–(1.3.53) has the properties*

$$\lim_{k\to\infty} (t_k - t_{k-1}) \;=\; \inf_{k=1,2,\dots} (t_k - t_{k-1}) \;=\; \pi\sqrt{h(0)}\,, \tag{1.3.59}$$

$$(-1)^k\sigma \quad \textit{is increasing in} \quad [t_{k-1}, t_k]\,, \quad (-1)^k\sigma(t_k) \;=\; \sigma_k\,, \tag{1.3.60}$$

$$\exists c > 0 : \quad \sigma_k \;\geq\; \frac{c}{1+k} \quad \forall k \in \mathbb{N}\,. \tag{1.3.61}$$

The proof will be carried out by induction. The following Lemma constitutes a basis for the induction step.

Lemma 1.3.5. *Under the hypotheses of Proposition 1.3.4, let $t_* \geq 0$ be such that*

$$\sigma_* \;:=\; \sigma(t_*) \;>\; 0\,, \quad \dot\sigma(t_*) \;=\; 0 \tag{1.3.62}$$

$$\exists r_* \geq \sigma_* : \quad \mathfrak{p}_r[0,\sigma](t_*) \;=\; \sigma_* - r \quad \textit{for } r \in [0, r_*]\,. \tag{1.3.63}$$

Then there exists $t^* > t_*$ such that $\dot\sigma(t) < 0$ in $]t_*, t^*[$, $\dot\sigma(t^*) = 0$, and $\sigma^* := \sigma(t^*) \in\,]-\sigma_*, 0[$.

Proof of Lemma 1.3.5. By virtue of (3.2.16)–(3.2.17) we have $v(t_*) = 0$, $\dot v(t_*) < 0$, hence there exists $\bar t > t_*$ such that $\dot\sigma(t) < 0$ in $]t_*, \bar t[$. We set

$$t^* \;=\; \sup\{\bar t > t_* \,;\, \dot\sigma(t) < 0 \;\text{ in }\;]0, \bar t[\;,\; \sigma(\bar t) > -\sigma_*\}\,. \tag{1.3.64}$$

As in the proof of Proposition 1.0.2, we compute the balance of the total energy $\frac12 v^2 + \mathcal{U}[0,\sigma]$, but in a slightly refined form. Let us first evaluate explicitly $\mathfrak{p}_r[0,\sigma](t)$ for $t \in [t_*, t^*]$. Set $\lambda_*(r) = \mathfrak{p}_r[0,\sigma](t_*)$. The identity (3.1.20) and assumption (1.3.63) yield

$$\mathfrak{p}_r[0,\sigma](t) \;=\; \begin{cases} \sigma(t) + r & \text{for } r < \frac12(\sigma_* - \sigma(t))\,, \\ \lambda_*(r) & \text{for } r \geq \frac12(\sigma_* - \sigma(t))\,, \end{cases} \tag{1.3.65}$$

and formulæ (3.4.1)–(3.4.3) yield

$$\begin{aligned} \frac{d}{dt}\left(\frac12 v^2 + \mathcal{U}[0,\sigma]\right)(t) \;&=\; -\int_0^\infty \left|\frac{d}{dt}\mathfrak{p}_r[0,\sigma](t)\right| r\, dh(r) \\ &=\; \dot\sigma(t)\int_0^{(1/2)(\sigma_*-\sigma(t))} r\, dh(r) \;=\; \dot\sigma(t)\,\Gamma\left(\frac12(\sigma_* - \sigma(t))\right)\,, \end{aligned} \tag{1.3.66}$$

where we set $\Gamma(s) = sh(s) - H(s)$ for $s \geq 0$, H being given by (3.2.3). We denote by $\mathcal{H}$ the function

$$\mathcal{H}(r) \;=\; \int_0^r H(s)\, ds \qquad \text{for } r \geq 0\,. \tag{1.3.67}$$

With this notation, we can integrate (1.3.66) from t_* to t and obtain for $t \in [t_*, t^*[$ that

$$\begin{aligned}\frac{1}{2}v^2(t) - (\sigma_* + \sigma(t))\, H\left(\frac{1}{2}(\sigma_* - \sigma(t))\right) \;&=\; -(\sigma_* - \sigma(t))\, H\left(\frac{1}{2}(\sigma_* - \sigma(t))\right) \\ &\quad + 4\mathcal{H}\left(\frac{1}{2}(\sigma_* - \sigma(t))\right)\,.\end{aligned} \tag{1.3.68}$$

The function H is strictly convex, hence the right-hand side of (1.3.68) is negative for $t \in]t_*, t^*[$ and $\sigma_* + \sigma(t)$ thus remains negative even for $t \to t^*$. By definition (1.3.64) of t^* only one of the following two cases can occur.

(a) $t^* = \infty$, $\sigma^* := \lim_{t\to\infty} \sigma(t) > -\sigma_*$;

(b) $t^* < \infty$, $\dot\sigma(t^*) = 0$, $\sigma^* := \sigma(t^*) > -\sigma_*$.

Case (a) can easily be excluded. Indeed, we then would have

$$\lim_{t\to\infty} \dot\sigma(t) = \lim_{t\to\infty} \dot\varepsilon(t) = \lim_{t\to\infty} v(t) \;=\; 0\,. \tag{1.3.69}$$

We rewrite (1.3.68) in the form

$$\frac{1}{2}v^2(t) - 2\sigma(t)\, H\left(\frac{1}{2}(\sigma_* - \sigma(t))\right) \;=\; 4\mathcal{H}\left(\frac{1}{2}(\sigma_* - \sigma(t))\right)\,, \tag{1.3.70}$$

and passing to the limit as $t \to \infty$ we obtain

$$-2\sigma^*\, H\left(\frac{1}{2}(\sigma_* - \sigma^*)\right) \;=\; 4\mathcal{H}\left(\frac{1}{2}(\sigma_* - \sigma^*)\right)\,. \tag{1.3.71}$$

This implies that $\sigma^* < 0$, and from (1.3.50) we obtain $\lim_{t\to\infty} \dot v(t) = -\sigma^* > 0$, which contradicts (1.3.69). Consequently, the case (b) takes place together with (1.3.71), and the assertion of Lemma 1.3.5 follows. ∎

Proof of Proposition 1.3.4. We first apply Lemma 1.3.5 at $t_* = 0$. We have by (3.2.4) that $\sigma_0 := \sigma(0) = H(\varepsilon_0) > 0$, hence the conditions are fulfilled with $r_* = \sigma_0$. We conclude that there exists $t_1 > 0$ such that $\dot\sigma(t_1) = 0$, $\sigma_1 := -\sigma(t_1) \in]0, \sigma_0[$, and setting $r_0 := \sigma_0$, $r_1 := \frac{1}{2}(\sigma_0 + \sigma_1) \in]\sigma_1, r_0[$, we obtain from (1.3.65) that

$$\mathfrak{p}_r[0,\sigma](t_1) \;=\; \begin{cases} -\sigma_1 + r & \text{for } r < r_1\,, \\ \sigma_0 - r & \text{for } r \in [r_1, r_0[\,, \\ 0 & \text{for } r \geq r_0\,. \end{cases} \tag{1.3.72}$$

Recall that the play operator $\mathfrak{p}_r[0,\cdot]$ is odd. We therefore may use again Lemma 1.3.5 for $-\sigma$ instead of σ with $t_* = t_1$, $r_* = r_1$, $\sigma_* = \sigma_1$, and find $t_2 > t_1$ such that $\dot\sigma(t_2) = 0$, $\sigma_2 := \sigma(t_2) \in]0, \sigma_1[$, $\sigma > 0$ in $]t_1, t_2[$, and

$$\mathfrak{p}_r[0,\sigma](t_2) \;=\; \begin{cases} \sigma_2 - r & \text{for } r < r_2\,, \\ \mathfrak{p}_r[0,\sigma](t_1) & \text{for } r \ge r_2\,. \end{cases} \tag{1.3.73}$$

By induction we now construct a decreasing sequence $\{\sigma_k\}_{k=1}^{\infty}$ of positive numbers, and an increasing sequence $0 = t_0 < t_1 < t_2 < \dots$ such that (1.3.60) holds, and

$$\mathfrak{p}_r[0,\sigma](t_k) \;=\; \begin{cases} (-1)^k(\sigma_k - r) & \text{for } r < r_k := \frac{1}{2}(\sigma_{k-1} + \sigma_k)\,, \\ \mathfrak{p}_r[0,\sigma](t_{k-1}) & \text{for } r \ge r_k\,. \end{cases} \tag{1.3.74}$$

for all $r > 0$ and $k \in \mathbb{N}$. By Eq. (1.3.71), we furthermore have

$$\sigma_k \;=\; \frac{2\mathcal{H}(r_k)}{H(r_k)} \;=:\; A(r_k) \quad \forall k \in \mathbb{N}\,, \tag{1.3.75}$$

hence

$$\sigma_{k-1} - \sigma_k \;=\; 2(r_k - A(r_k)) \;=:\; B(r_k)\,. \tag{1.3.76}$$

We can differentiate the function B defined in (1.3.76) and obtain for all $x > 0$ the identity

$$B'(x) \;=\; \frac{4}{H^2(x)} \int_0^x H(s)(h(x) - h(s))\, ds\,. \tag{1.3.77}$$

By hypothesis, h is positive, increasing, and Lipschitz continuous on $[0, \sigma_0]$, hence B is increasing, $B(0+) = 0$, and

$$0 \;<\; B(x) \;\le\; Cx^2 \quad \forall x > 0\,, \tag{1.3.78}$$

with the convention that, similarly as in previous subsections, C denotes any positive constant independent of k. Using (1.3.75)–(1.3.78) we see, on the one hand, that the limit $\sigma_\infty = \lim_{k\to\infty} \sigma_k$ fulfils $B(\sigma_\infty) = 0$, hence $\sigma_\infty = 0$. On the other hand, we have

$$\frac{1}{\sigma_k} - \frac{1}{\sigma_{k-1}} \;=\; \frac{\sigma_{k-1} - \sigma_k}{\sigma_k\, \sigma_{k-1}} \;\le\; C\,\frac{r_k}{\sigma_k} \;\le\; C\,\frac{r_k H(r_k)}{\mathcal{H}(r_k)}\,. \tag{1.3.79}$$

The function $x \mapsto (xH(x)/\mathcal{H}(x)$ is bounded in $]0, \sigma_0]$, since its limit at $x \to 0+$ is $1/2$. Hence,

$$\frac{1}{\sigma_k} - \frac{1}{\sigma_{k-1}} \;\le\; C \qquad \text{for } k \in \mathbb{N}\,, \tag{1.3.80}$$

and (1.3.61) follows.

It remains to check that (1.3.59) holds. Assume for definiteness that k is even; the other case is similar. For $t \in [t_{k-1}, t_k]$ we have as in (1.3.65) and (1.3.74) that

$$\mathfrak{p}_r[0,\sigma](t) \;=\; \begin{cases} \sigma(t) - r & \text{for } r < r(t) := \frac{1}{2}(\sigma(t) - \sigma(t_{k-1}))\,, \\ \mathfrak{p}_r[0,\sigma](t_{k-1}) & \text{for } r \ge r(t)\,, \end{cases} \tag{1.3.81}$$

hence, by (3.2.1),

$$\varepsilon(t) - \varepsilon(t_{k-1}) \;=\; 2\,H(r(t))\,, \tag{1.3.82}$$

or equivalently

$$\sigma(t) - \sigma(t_{k-1}) \;=\; 2\,H^{-1}\left(\frac{1}{2}(\varepsilon(t) - \varepsilon(t_{k-1}))\right)\,. \tag{1.3.83}$$

As a consequence of (1.3.50)–(1.3.51) and (1.3.83), the function ε solves the differential equation (according to our notation we have $\sigma(t_{k-1}) = -\sigma_{k-1}$)

$$\ddot\varepsilon(t) - \sigma_{k-1} + 2\,H^{-1}\left(\frac{1}{2}(\varepsilon(t) - \varepsilon(t_{k-1}))\right) \;=\; 0\,. \tag{1.3.84}$$

Testing (1.3.84) by $\dot\varepsilon(t)$, integrating from t_{k-1} to t, and using the fact that $\dot\varepsilon(t_{k-1}) = 0$, $\dot\varepsilon > 0$ otherwise, we obtain

$$\dot\varepsilon(t) \;=\; \sqrt{2\sigma_{k-1}(\varepsilon(t) - \varepsilon(t_{k-1})) - 8\,\hat{\mathcal{H}}\left(\frac{1}{2}(\varepsilon(t) - \varepsilon(t_{k-1}))\right)}\,, \tag{1.3.85}$$

where

$$\hat{\mathcal{H}}(r) \;=\; \int_0^r H^{-1}(s)\,ds\,. \tag{1.3.86}$$

For $t = t_k$ we obtain from (1.3.85) in particular that

$$\sigma_{k-1}(\varepsilon(t_k) - \varepsilon(t_{k-1})) = 4\,\hat{\mathcal{H}}\left(\frac{1}{2}(\varepsilon(t_k) - \varepsilon(t_{k-1}))\right)\,. \tag{1.3.87}$$

Set $p_k = \frac{1}{2}(\varepsilon(t_k) - \varepsilon(t_{k-1}))$. Then (1.3.85) can be rewritten as

$$\dot\varepsilon(t) \;=\; \frac{2\sqrt{2}}{\sqrt{p_k}}\sqrt{\frac{1}{2}(\varepsilon(t) - \varepsilon(t_{k-1}))\,\hat{\mathcal{H}}\,(p_k) - p_k\,\hat{\mathcal{H}}\left(\frac{1}{2}(\varepsilon(t) - \varepsilon(t_{k-1}))\right)}\,. \tag{1.3.88}$$

The substitution

$$s(t) \;=\; \frac{1}{2p_k}(\varepsilon(t) - \varepsilon(t_{k-1}))$$

yields

$$\dot s(t) \;=\; \frac{\sqrt{2}}{p_k}\sqrt{s(t)\,\hat{\mathcal{H}}(p_k) - p_k\,\hat{\mathcal{H}}(s(t))}\,, \tag{1.3.89}$$

hence

$$\frac{\sqrt{2}}{p_k}\,(t_k - t_{k-1}) \;=\; \int_0^1 \frac{ds}{\sqrt{s\,\hat{\mathcal{H}}(p_k) - p_k\,\hat{\mathcal{H}}(s)}}\,. \tag{1.3.90}$$

The function $\hat h = (H^{-1})'$ is decreasing and

$$s\,\hat{\mathcal{H}}(p_k) - p_k\,\hat{\mathcal{H}}(s) \;=\; s\int_0^{p_k}\int_{sr}^{r} \hat h(z)\,dz\,dr$$

for $s \in [0,1]$, hence

$$\frac{\sqrt{2}}{p_k\sqrt{\hat{h}(p_k)}}\int_0^1 \frac{ds}{\sqrt{s\,(1-s)}} \ \geq\ \frac{\sqrt{2}}{p_k}\,(t_k - t_{k-1})\ \geq\ \frac{\sqrt{2}}{p_k\sqrt{\hat{h}(0)}}\int_0^1 \frac{ds}{\sqrt{s\,(1-s)}}\,. \quad (1.3.91)$$

We have by (1.3.82) that $p_k = H(r_k)$, hence $\hat{h}(p_k) = 1/h(r_k)$, and we conclude that

$$\pi\sqrt{h(r_k)}\ \geq\ t_k - t_{k-1}\ \geq\ \pi\sqrt{h(0)}\,. \quad (1.3.92)$$

We already know that $\lim_{k\to\infty} r_k = 0$, and the proof of Proposition 1.3.4 is complete. ■

Remark 1.3.6. We have seen that $\{\varepsilon(t_k)\}$ is an alternating sequence of decreasing local maxima and increasing local minima of ε whose differences p_k tend to 0, hence $\varepsilon_\infty = \lim_{t\to\infty}\varepsilon(t)$ exists. It cannot be expected, however, that $\varepsilon_\infty = 0$. This follows from the identity

$$\varepsilon_\infty \ =\ \varepsilon_0 + 2\sum_{k=1}^{\infty}(-1)^k p_k \ =\ \sum_{k=1}^{\infty}(p_{2k-2} - 2p_{2k-1} + p_{2k})\,, \quad (1.3.93)$$

provided we put $p_0 := \varepsilon_0$. We still have $p_k = H(r_k)$ for all $k \geq 0$, and the relation

$$r_{2k-2} - 2r_{2k-1} + r_{2k} \ =\ \frac{1}{2}(B(r_{2k-2}) - B(r_{2k-1}))\ >\ 0 \quad (1.3.94)$$

holds for every $k \in \mathbb{N}$ by virtue of (1.3.76)–(1.3.77). The function H is increasing and convex, hence

$$p_{2k-1} \ =\ H(r_{2k-1})\ <\ H\left(\frac{1}{2}(r_{2k-2} + r_{2k})\right)\ \leq\ \frac{1}{2}(p_{2k-2} + p_{2k})\,, \quad (1.3.95)$$

and we see that ε_∞ in (1.3.93) is positive. Similarly, the total energy $E(t) = \frac{1}{2}v^2(t) + \mathcal{U}[0,\sigma](t)$ does not asymptotically vanish. Putting $E_k = E(t_k) = \mathcal{U}[0,\sigma](t_k)$ for $k \geq 0$, we obtain by a computation similar as in the proof of Lemma 1.3.5 that

$$E_0 = \mathcal{H}(\sigma_0)\,, \quad E_{k-1} - E_k \ =\ (\sigma_{k-1} - \sigma_k)\,H(r_k) \qquad \text{for } k \in \mathbb{N}\,. \quad (1.3.96)$$

Since H is strictly convex, we have

$$(\sigma_{k-1} - \sigma_k)\,H(r_k)\ <\ \int_{\sigma_k}^{\sigma_{k-1}} H(s)\,ds \ =\ \mathcal{H}(\sigma_{k-1}) - \mathcal{H}(\sigma_k)\,,$$

hence, using the fact that $E(t)$ is non-increasing, we have

$$\lim_{t\to\infty} E(t) \ =\ E_0 - \sum_{k=1}^{\infty}(E_{k-1} - E_k)\ >\ 0\,. \quad (1.3.97)$$

This fact is not surprising either. A non-zero part of the initial energy is stored in the remanent deformation of the spring, the rest is dissipated into heat.

To conclude this section, let us note that if we allow $h'(0+)$ to be infinite, then the decay rate may be faster than $1/t$. The computation in [14, Example III.2.6] shows, however, that it is never exponential. The case where $\mathcal{F}$ in (1.3.52) is replaced by the Preisach operator (3.2.18) is investigated in [15].

1.4 Quasilinear perturbations

The results listed in the previous subsections are related to the variational and monotone character of the Prandtl-Ishlinskii operator as linear combination of solution operators to simple variational inequalities. For the existence of solutions, the *convexity of the hysteresis loops* is, however, more substantial than monotonicity. We now show that the existence and regularity result is stable also with respect to *quasilinear perturbations*. In other words, no shocks occur provided the convex-concave hysteresis behaviour is preserved. Since monotonicity is lost, the question of uniqueness of solutions is open.

To be more specific, we consider the stress-strain relation in the form

$$\varepsilon \;=\; \mathcal{F}[\lambda, G(\sigma)]\,, \tag{1.4.1}$$

where $G : \mathbb{R} \to \mathbb{R}$ is a smooth "almost linear" increasing function, and $\mathcal{F}$ is the Prandtl-Ishlinskii operator given by (3.2.1). Note that the superposition $\mathcal{F} \circ G$ is the so-called *generalized Prandtl-Ishlinskii operator* as a special case of the Preisach operator (3.2.18), see [9] for the relationship between the functions h, g, and ψ. Similar operators play an important role in modelling of piezoelectricity, see [18]. Changing accordingly the notation, we rewrite system (1.0.1) with constitutive law of the type (1.4.1) in a more convenient form with a parameter $\delta > 0$ as

$$\left\{\begin{array}{rcl} \partial_t v &=& \partial_x(\sigma + \delta\, g(\sigma)) + f(\sigma, v, x, t)\,, \\ \partial_t \varepsilon &=& \partial_x v\,, \\ \varepsilon &=& \mathcal{F}[\lambda, \sigma]\,. \end{array}\right. \tag{1.4.2}$$

The existence part of Theorem 1.1.1 holds for δ sufficiently small in the following form.

Theorem 1.4.1. *Let Hypothesis 1.0.1 be fulfilled, let the function κ from (3.4.8) be positive on $[0, \infty[$, and let g be a non-decreasing function in $\mathbb{R}$ with locally Lipschitz continuous derivative. Then there exists $\delta_0 > 0$ such that System (1.4.2), (1.0.2)–(1.0.3) for $0 \le \delta \le \delta_0$ admits a solution $(v, \sigma, \varepsilon) \in C(\bar{Q}_T; \mathbb{R}^3)$ such that $\partial_t v, \partial_x v, \partial_t \sigma, \partial_x \sigma, \partial_t \varepsilon$ belong to $L^\infty(0, T;\, L^2(0, 1))$, and (1.4.1) holds almost everywhere in Q_T .*

Proof. We discretize the system (1.4.1) in space in the form similar to (1.1.4)–(1.1.6), more precisely

$$\dot{v}_j \;=\; n(\sigma_{j+1} - \sigma_j) + \delta n(g(\sigma_{j+1}) - g(\sigma_j)) + f_j(\sigma_j, v_j, t)\,, \quad j = 1, \ldots n-1\,, \tag{1.4.3}$$

$$\dot{\varepsilon}_j \;=\; n(v_j - v_{j-1})\,, \quad j = 1, \ldots n\,, \tag{1.4.4}$$

$$\varepsilon_j \;=\; \mathcal{F}[\lambda_j, \sigma_j]\,, \quad j = 1, \ldots n\,. \tag{1.4.5}$$

For each fixed n and δ, the existence of solutions to (1.4.3)–(1.4.5) together with conditions (1.1.7)–(1.1.10) is obtained in the same way as in the proof of Theorem 1.1.1. The estimates, however, have to be carried out in a more careful way.

We fix a bound $p_1 \geq \max_{j=1,\ldots n} |\sigma_j(0)|$ independent of n, and for each $p > p_1$ we solve (1.4.3)–(1.4.5) with $\delta = \delta(p) > 0$ such that

$$\delta(p) \max_{|s|\leq p} g'(s) \ \leq \ 1 \,, \quad \delta(p) \operatorname{sup\,ess}_{|s|\leq p} |g''(s)| \ \leq \ \frac{\kappa(p)}{4\,h(p)} \,. \tag{1.4.6}$$

We define the maximal time $T(n,p) > 0$ for which all $|\sigma_j(t)|$ remain bounded by p, that is,

$$T(n,p) \ = \ \max\{t \in [0,T]\,;\, \max_{j=1,\ldots n} |\sigma_j(t)| \leq p\} \,. \tag{1.4.7}$$

The counterpart of (1.1.18)–(1.1.19) reads

$$\begin{aligned} \ddot{v}_j \dot{v}_j \ &\leq \ n(\dot{\sigma}_{j+1} - \dot{\sigma}_j)\dot{v}_j + \delta(p) n \frac{d}{dt}(g(\sigma_{j+1}) - g(\sigma_j))\dot{v}_j && (1.4.8) \\ &\quad + \big(\beta_j(t) + \alpha_f(t)(|\dot{\sigma}_j| + |\dot{v}_j|)\big)|\dot{v}_j| \,, \\ \ddot{\varepsilon}_j \dot{\sigma}_j \ &= \ n(\dot{v}_j - \dot{v}_{j-1})\dot{\sigma}_j \,, && (1.4.9) \end{aligned}$$

hence

$$\frac{1}{n}\sum_{j=1}^{n-1} \ddot{v}_j\dot{v}_j + \frac{1}{n}\sum_{j=1}^{n}(1+\delta(p)g'(\sigma_j))\ddot{\varepsilon}_j\dot{\sigma}_j \ \leq \ \frac{1}{n}\sum_{j=1}^{n-1}\big(\beta_j(t)+\alpha_f(t)(|\dot{\sigma}_j|+|\dot{v}_j|)\big)|\dot{v}_j| \,. \tag{1.4.10}$$

We use Theorem 3.4.1 to obtain for all $0 \leq s < t \leq T(n,p)$ and all $j = 1, \ldots n$ that

$$\int_s^t \ddot{\varepsilon}_j \dot{\sigma}_j \, d\tau \ \geq \ \frac{\kappa(p)}{4} \int_s^t |\dot{\sigma}_j|^3 \, d\tau + \frac{1}{2}(\dot{\varepsilon}_j\dot{\sigma}_j)(t-) - \frac{1}{2}(\dot{\varepsilon}_j\dot{\sigma}_j)(s+) \,. \tag{1.4.11}$$

We see that the function

$$t \ \mapsto \ \frac{1}{2}(\dot{\varepsilon}_j\dot{\sigma}_j)(t) + \frac{\kappa(p)}{4}\int_0^t |\dot{\sigma}_j|^3 \, d\tau - \int_0^t \ddot{\varepsilon}_j\dot{\sigma}_j \, d\tau$$

in non-increasing in $[0, T(n,p)]$. For each absolutely continuous non-negative test function η and for all $t \in [0, T(n,p)]$ we thus have

$$\begin{aligned} &\int_0^t \left(\frac{1}{2}(\dot{\varepsilon}_j\dot{\sigma}_j)(\tau)\,\dot{\eta}(\tau) + \left((\ddot{\varepsilon}_j\dot{\sigma}_j)(\tau) - \frac{\kappa(p)}{4}|\dot{\sigma}_j(\tau)|^3\right)\eta(\tau)\right) d\tau && (1.4.12) \\ &\qquad \geq \ \frac{1}{2}(\dot{\varepsilon}_j\dot{\sigma}_j)(t)\,\dot{\eta}(t) - \frac{1}{2}(\dot{\varepsilon}_j\dot{\sigma}_j)(0+)\,\dot{\eta}(0) \,. \end{aligned}$$

We now set $\eta(t) = 1 + \delta(p)g'(\sigma_j(t))$ in (1.4.12). By hypothesis (1.4.6) we have $1 \leq \eta(t) \leq 2$ for $t \in [0, T(n,p)]$, hence

$$\begin{aligned} \frac{1}{2}(\dot{\varepsilon}_j\dot{\sigma}_j)(t) \ &\leq \ (\dot{\varepsilon}_j\dot{\sigma}_j)(0+) + \int_0^t (1 + \delta(p)g'(\sigma_j(\tau)))(\ddot{\varepsilon}_j\dot{\sigma}_j)(\tau)\, d\tau && (1.4.13) \\ &\quad + \int_0^t \left(\delta(p)g''(\sigma_j(\tau))(\dot{\varepsilon}_j\dot{\sigma}_j^2)(\tau) - \frac{\kappa(p)}{4}|\dot{\sigma}_j(\tau)|^3\right) d\tau \,. \end{aligned}$$

From (3.2.16) and hypothesis (1.4.6) it follows that for all $t \in [0, T(n,p)]$ we have

$$\delta(p)|g''(\sigma_j(t))\dot{\varepsilon}_j(t)| \;\leq\; \frac{\kappa(p)}{4}|\dot{\sigma}_j(t)|\,,$$

hence the last integral on the right-hand side of (1.4.13) is non-positive. We therefore have

$$\frac{1}{2n}\left(\sum_{j=1}^{n-1}\dot{v}_j^2(t) + \sum_{j=1}^{n}(\dot{\varepsilon}_j\dot{\sigma}_j)(t)\right) \;\leq\; \frac{1}{2n}\left(\sum_{j=1}^{n-1}\dot{v}_j^2(0) + 2\sum_{j=1}^{n}(\dot{\varepsilon}_j\dot{\sigma}_j)(0+)\right) \tag{1.4.14}$$
$$+\;\frac{1}{n}\int_0^t\left(\sum_{j=1}^{n-1}(\ddot{v}_j\dot{v}_j)(\tau) + \frac{1}{n}\sum_{j=1}^{n}((1+\delta(p)g'(\sigma_j(\tau)))(\ddot{\varepsilon}_j\dot{\sigma}_j)(\tau)\right)d\tau$$

for a. e. $t \in\,]0, T(n,p)[$. The initial conditions on the right-hand side of (1.4.14) can be estimated independently of n and p similarly as in the proof of Theorem 1.1.1, and as a consequence of (3.2.17) we have $\dot{\varepsilon}_j(t)\dot{\sigma}_j(t) \geq h(0)\dot{\sigma}_j^2(t)$ a. e. Combining (1.4.14) with (1.4.10) and using Gronwall's inequality we find a constant C_T independent of n and p (and possibly dependent on T) such that

$$\frac{1}{n}\left(\sum_{j=1}^{n-1}\dot{v}_j^2(t) + \sum_{j=1}^{n}\dot{\sigma}_j^2(t)\right) \;\leq\; C_T \quad \forall t \in [0, T(n,p)]\,. \tag{1.4.15}$$

Arguing as in the proof of Theorem 1.1.1, we find a constant $\bar{C}_T$ independent of n and p such that

$$n\left(\sum_{j=1}^{n}(v_j - v_{j-1})^2(t) + \sum_{j=1}^{n-1}(\sigma_{j+1}-\sigma_j)^2(t)\right) + \max_{j=1,\dots,n}|\sigma_j(t)| \;\leq\; \bar{C}_T \quad \forall t \in [0, T(n,p)]\,. \tag{1.4.16}$$

We now fix any $p_0 \geq \max\{p_1, \bar{C}_T\}$ and set $\delta_0 = \delta(p_0)$. Then we have $T(n,p) = T$ for all $n \in \mathbb{N}$, and the estimates (1.4.15)–(1.4.16) hold a. e. in $[0,T]$. This enables us to complete the proof passing to the limit as $n \to \infty$ similarly as in the proof of Theorem 1.1.1. ■

2 Periodic solutions

In this section, we consider the system (1.0.1) with boundary conditions (1.0.3) and with the time-periodicity condition

$$v(x, t+T) = v(x,t)\,,\;\; \sigma(x,t+T) = \sigma(x,t) \tag{2.0.17}$$

for all $(x,t) \in [0,1] \times [0,\infty[$ instead of (1.0.2), where $T > 0$ is a fixed period. Our analysis will be carried out in the spaces $L^p_T = L^p_{\mathrm{loc}}(]0,1[\times]0,\infty[)$ for $1 \leq p \leq \infty$ and in the space $\mathcal{C}^0_T$ of continuous functions, all satisfying the T-periodicity condition. Having in mind Corollary 3.1.3 which states that outputs of hysteresis operators with periodic inputs may possibly become periodic only after one period, we define the norms

$$\|w\|_p = \left(\int_T^{2T}\int_0^1 |w(x,t)|^p\,dx\,dt\right)^{1/p} \quad \text{for } w \in L^p_T\,,\ 1 \le p < \infty\,, \tag{2.0.18}$$

$$\|w\|_\infty = \operatorname{sup\,ess}\{|w(x,t)|\,;\ (x,t) \in]0,1[\times]T,2T[\} \quad \text{for } w \in L^\infty_T\,. \tag{2.0.19}$$

We endow the space $\mathcal{C}^0_T$ with the norm $\|\cdot\|_\infty$ as well. Recall that the compact embeddings

$$\begin{array}{l} H^{2,3}_T := \{w \in L^2_T\,;\ \partial_x w \in L^2_T\,, \partial_t w \in L^3_T\} \hookrightarrow\hookrightarrow \mathcal{C}^0_T \\ H^{3,2}_T := \{w \in L^2_T\,;\ \partial_x w \in L^3_T\,, \partial_t w \in L^2_T\} \hookrightarrow\hookrightarrow \mathcal{C}^0_T \end{array} \tag{2.0.20}$$

take place, and we fix constants $M, M' > 0$ such that

$$\begin{array}{lcl} \|w\|_\infty &\le& \left|\int_T^{2T}\int_0^1 w(x,t)\,dx\,dt\right| + M\,(\|\partial_x w\|_2 + \|\partial_t w\|_3) \quad \forall w \in H^{2,3}_T\,, \\ \|w\|_\infty &\le& \left|\int_T^{2T}\int_0^1 w(x,t)\,dx\,dt\right| + M'\,(\|\partial_x w\|_3 + \|\partial_t w\|_2) \quad \forall w \in H^{3,2}_T\,. \end{array} \tag{2.0.21}$$

An estimate for M, M' can be found in [14, Appendix 2].

We find sufficient conditions for the existence and uniqueness of solutions to the Dirichlet-periodic problem and prove its global asymptotic stability.

2.1 Statement of main results

In addition to Hypothesis 1.0.1, we impose the following more restrictive assumptions on f.

Hypothesis 2.1.1. *The following conditions hold for all admissible arguments.*

(i) $f(\sigma, v, x, t+T) = f(\sigma, v, x, t)$ *;*

(ii) $f^0, \beta_f \in L^2_T$ *;*

(iii) $|\partial_\sigma f(\sigma, v, x, t)| \le \gamma_f$ *;*

(iv) $-\gamma_f \le \partial_v f(\sigma, v, x, t) \le 0$*, where* $\gamma_f > 0$ *is a fixed constant.*

In this subsection we list the main results on existence (Theorem 2.1.2), uniqueness (Theorem 2.1.3), and asymptotic stability (Theorem 2.1.6) of periodic solutions to System (1.0.1), (1.0.3). Proofs are postponed to the next subsections.

Theorem 2.1.2. *Let Hypotheses 1.0.1 and 2.1.1 hold. Assume in addition to (3.2.15) that the functions* h *and* κ *in (3.2.1) and (3.4.8) satisfy the condition*

$$\limsup_{p\to\infty} \frac{h(p)}{p\,\kappa(p)} =: q < \infty \tag{2.1.1}$$

with

$$4\sqrt{T}\gamma_f M\, q\left(1+\frac{T}{2\pi}\gamma_f\left(1+\gamma_f \mathrm{e}^{\gamma_f}\right)\right) \;<\; 1\,. \tag{2.1.2}$$

Then Eqs. (1.0.1) with boundary conditions (1.0.3) and periodicity conditions (2.0.17) admit a solution $(v,\sigma) \in H_T^{3,2} \times H_T^{2,3}$ *, and Equations (1.0.1) are satisfied for a. e.* $(x,t) \in]0,1[\, \times\,]T,\infty[$ *.*

The situation is similar as in Subsections 1.2–1.3, cf. Remark 1.2.4. We are able to prove existence only if no negative friction is present. Moreover, uniqueness is obtained only if f is independent of σ, that is, if Problem (1.0.1) has the form (1.2.1). On the other hand, we can replace (2.1.1) by a weaker condition (2.1.3) below.

Theorem 2.1.3. *Let Hypotheses 1.0.1 and 2.1.1 hold with* f *independent of* σ*. Assume, instead of (2.1.1), that*

$$\limsup_{p\to\infty} \frac{h^{3/4}(p)}{p\,\kappa^{1/2}(p)} \;=:\; \tilde{q} \;<\; \infty \tag{2.1.3}$$

with

$$2\,T^{1/4}\sqrt{\|\beta_f\|_2}\, M\, \tilde{q}\left(1+\frac{T}{2\pi}\gamma_f\right) \;<\; 1\,. \tag{2.1.4}$$

Let further $\varrho \in \mathbb{R}$ *be given. Then Problem (1.2.1) with the periodicity conditions (2.0.17) admits a unique solution* $(v,\sigma) \in H_T^{3,2} \times H_T^{2,3}$ *, Eqs. (1.2.1) are satisfied for a. e.* $(x,t) \in]0,1[\, \times\,]T,\infty[$ *, and*

$$\frac{1}{T}\int_T^{2T} \sigma(0,t)\, dt = \varrho\,. \tag{2.1.5}$$

Remark 2.1.4. From (2.1.1) it follows that

$$\limsup_{p\to\infty} \frac{h(p)}{p\,(h(p)-h(p-1))} \;\le\; q\,, \tag{2.1.6}$$

hence

$$\frac{1}{(q+1)p} \;\le\; \frac{h(p)-h(p-1)}{h(p)} \;\le\; \log h(p) - \log h(p-1) \tag{2.1.7}$$

for p larger than some $p_1 > 0$. We thus obtain that

$$\lim_{p\to\infty} h(p) \;=\; \infty\,, \quad \lim_{p\to\infty}\frac{\sqrt{h(p)}}{p} \;=\; \lim_{p\to\infty}\sqrt{\frac{h(p)}{p\,\kappa(p)}}\sqrt{\frac{\kappa(p)}{p}} \;=\; 0\,. \tag{2.1.8}$$

In particular, if condition (2.1.1) is satisfied, then (2.1.3) holds with $\tilde{q} = 0$. The class of functions satisfying (2.1.1) is non-empty. For example, for every locally Lipschitz continuous function h such that

$$h_* \max\{r_0, r\}^{\alpha-1} \;\le\; h'(r) \;\le\; h^* \max\{r_0, r\}^{\alpha-1} \qquad \text{a. e.}\ \ , \tag{2.1.9}$$

where $0 < h_* \leq h^*$, $r_0 > 0$, $\alpha \in]0,1]$ are given numbers, we have for $p > r_0$ that $\kappa(p) \geq h_* p^{\alpha-1}$, $h(p) \leq h(0) + h^* p^\alpha/\alpha$, hence (2.1.1) holds. For the validity of (2.1.3), it suffices to have for instance

$$h_* \max\{r_0, r\}^{\alpha_1 - 1} \ \leq \ h'(r) \ \leq \ h^* \max\{r_0, r\}^{\alpha_2 - 1} \qquad \text{a.e.} \tag{2.1.10}$$

with some $\alpha_1 \in]-1,1]$ and $\alpha_1 \leq \alpha_2 \leq (2/3)(\alpha_1 + 1)$.

Remark 2.1.5. Below in Remark 2.3.1 at the end of Subsection 2.3 we comment on the non-uniqueness related to the fact that ϱ is arbitrary. Also the value of ϱ can be determined uniquely if we consider the Dirichlet boundary conditions in displacements instead of velocities.

To conclude this subsection, we state, as a complement to Theorem 2.1.3, a result on asymptotic stability of periodic solutions.

Theorem 2.1.6. *Let Hypotheses 1.0.1, 2.1.1 hold with f independent of σ, and let $\kappa(p) > 0$ for all $p > 0$. Let us define the set*

$$\begin{aligned} B \ = \ &\big\{(v,\sigma) \in L^\infty(\,]0,1[\times]0,\infty[\,)^2\,; \\ &\partial_t v, \partial_t \sigma, \partial_x v, \partial_x \sigma \in L^\infty(0,\infty\,;\, L^2(0,1))\,,\ v(0,t) = v(1,t) = 0\big\}\,, \end{aligned} \tag{2.1.11}$$

and assume that $(\underline{v}, \underline{\sigma}) \in B$ is a solution of the problem

$$\left\{ \begin{aligned} \partial_t \underline{v} \ &= \ \partial_x \underline{\sigma} + f(\underline{v}, x, t)\,, \\ \partial_t \underline{\varepsilon} \ &= \ \partial_x \underline{v}\,, \\ \underline{\varepsilon} \ &= \ \mathcal{F}[\underline{\lambda}, \underline{\sigma}]\,, \end{aligned} \right. \tag{2.1.12}$$

where $\underline{\lambda} \in C([0,1]\,;\,\Lambda_K)$ and $K > 0$ are fixed. Then there exists $\lambda \in C([0,1]\,;\,\Lambda_{\underline{K}})$ with $\underline{K} = \max\{K, \|\underline{\sigma}\|_\infty\}$ and a periodic solution $(v,\sigma) \in B$ of (1.2.1) such that

$$\lim_{t\to\infty} \big(|\underline{v}(\cdot,t) - v(\cdot,t)|_{L^\infty(0,1)} + |\underline{\sigma}(\cdot,t) - \sigma(\cdot,t)|_{L^\infty(0,1)}\big) \ = \ 0\,. \tag{2.1.13}$$

Remark 2.1.7. The assumptions of Theorem 2.1.6 are satisfied for instance under the hypotheses of Theorem 1.2.5. In such a case, condition (2.1.3) automatically holds.

2.2 Existence

This subsection is devoted to the proof of Theorem 2.1.2. We fix some $\varrho \in \mathbb{R}$ and consider the problem

$$\partial_t v \ = \ \partial_x \sigma^* + f(\sigma^* + \hat\sigma, v, x, t) - \frac{1}{T}\int_T^{2T} f(\sigma^* + \hat\sigma, v, x, \tau)\, d\tau\,, \tag{2.2.1}$$

$$\partial_t \varepsilon^* \ = \ \partial_x v\,, \tag{2.2.2}$$

$$\varepsilon^* \ = \ \mathcal{F}[\lambda, \sigma^*] \tag{2.2.3}$$

$$\frac{d}{dx}\hat\sigma(x) \ = \ -\frac{1}{T}\int_T^{2T} f(\sigma^* + \hat\sigma, v, x, \tau)\, d\tau\,, \quad \hat\sigma(0) = \varrho \tag{2.2.4}$$

together with the T-periodicity condition and

$$v(0,t) \;=\; v(1,t) \;=\; 0\,, \qquad \int_T^{2T} v(x,t)\,dt \;=\; \int_T^{2T} \sigma^*(x,t)\,dt \;=\; 0 \tag{2.2.5}$$

for all admissible arguments. We observe that if $(v, \sigma^*, \hat\sigma, \varepsilon^*)$ is a solution to (2.2.1)–(2.2.5), then (v, σ) with $\sigma = \sigma^* + \hat\sigma$ and $\varepsilon = \mathcal{F}[\lambda, \sigma]$ satisfy the conditions of Theorem 2.1.2, since by Proposition 3.3.1 we have $\partial_t \varepsilon = \partial_t \varepsilon^*$ for a. e. $t > T$.

The solution will be constructed by the Galerkin method. For $j \in \mathbb{Z}$, $k \in \mathbb{N} \cup \{0\}$, $t \ge 0$, $x \in [0,1]$ we define basis functions

$$e_j(t) \;=\; \begin{cases} \sin \frac{2\pi j}{T} t & \text{if } j > 0\,, \\ \cos \frac{2\pi j}{T} t & \text{if } j \le 0\,, \end{cases} \tag{2.2.6}$$

$$\varphi_k(x) \;=\; \sin k\pi x\,, \qquad \psi_k(x) \;=\; \cos k\pi x\,. \tag{2.2.7}$$

For all relevant values of j, k, x, t we have

$$\frac{d}{dt} e_j(t) \;=\; \frac{2\pi j}{T}\, e_{-j}(t)\,, \quad \frac{d}{dx}\varphi_k(x) \;=\; k\pi\, \psi_k(x)\,, \quad \frac{d}{dx}\psi_k(x) \;=\; -k\pi\, \varphi_k(x)\,. \tag{2.2.8}$$

For each fixed $n \in \mathbb{N}$ we set $J_n = \{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$, and define functions $v^{(n)}, \sigma^{(n)}, \varepsilon^{(n)}, \hat\sigma^{(n)}$ by the formulæ

$$v^{(n)}(x,t) \;=\; \sum_{j\in J_n} \sum_{k=1}^{n} v_{jk}\, e_j(t)\, \varphi_k(x)\,, \tag{2.2.9}$$

$$\sigma^{(n)}(x,t) \;=\; \sum_{j\in J_n} \sum_{k=1}^{n} \sigma_{jk}\, e_j(t)\, \psi_k(x)\,, \tag{2.2.10}$$

$$\varepsilon^{(n)}(x,t) \;=\; \mathcal{F}[\lambda, \sigma^{(n)}](x,t)\,, \tag{2.2.11}$$

$$\frac{d}{dx}\hat\sigma^{(n)}(x) \;=\; -\frac{1}{T}\int_T^{2T} f(\sigma^{(n)} + \hat\sigma^{(n)}, v^{(n)}, x, \tau)\,d\tau\,, \quad \hat\sigma^{(n)}(0) = \varrho\,, \tag{2.2.12}$$

where v_{jk}, σ_{jk} are solutions of the system

$$\int_T^{2T}\!\!\int_0^1 \left(\partial_t v^{(n)} - \partial_x \sigma^{(n)} - f(\sigma^{(n)} + \hat\sigma^{(n)}, v^{(n)}, x, t)\right)\, e_j(t)\, \varphi_k(x)\,dx\,dt \;=\; 0\,, \tag{2.2.13}$$

$$\int_T^{2T}\!\!\int_0^1 \left(\partial_t \varepsilon^{(n)} - \partial_x v^{(n)}\right)\, e_j(t)\, \psi_k(x)\,dx\,dt \;=\; 0 \tag{2.2.14}$$

for all $j \in J_n$, $k = 0, 1, \dots, n$. We first have to prove that (2.2.13)–(2.2.14) has a solution. The unknown in the problem is the vector $\mathbf{v} = (v_{jk}, \sigma_{jk})$, $j \in J_n$, $k = 0, 1, \dots, n$, with $v_{j0} = 0$ for all $j \in J_n$, hence $\mathbf{v}$ can be considered as an element of $\mathbf{V} := \mathbb{R}^{2n\times n} \times \mathbb{R}^{2n\times(n+1)}$. The mappings which with $\mathbf{v} \in \mathbf{V}$ associate the functions $\varepsilon^{(n)} \in \mathcal{C}_T^0$ and $\hat\sigma^{(n)} \in C[0,1]$ are continuous, hence the system (2.2.13)–(2.2.14) is of the form

$$\Phi(\mathbf{v}) \;=\; 0\,, \tag{2.2.15}$$

where Φ is a continuous mapping from $\mathbf{V}$ to $\mathbf{V}$. We define a homotopy $\Phi_s : \mathbf{V} \to \mathbf{V}$ with parameter $s \in [0,1]$ by the left-hand side of the system

$$\int_T^{2T} \int_0^1 \left(\partial_t v^{(n)} - \partial_x \sigma^{(n)} - s\, f(\sigma^{(n)} + \hat\sigma_s^{(n)}, v^{(n)}, x, t)\right) e_j(t)\, \varphi_k(x)\, dx\, dt \;=\; 0\,, \tag{2.2.16}$$

$$\int_T^{2T} \int_0^1 \left(\partial_t \varepsilon^{(n)} - \partial_x v^{(n)}\right) e_j(t)\, \psi_k(x)\, dx\, dt \;=\; 0\,, \tag{2.2.17}$$

where

$$\varepsilon^{(n)}(x,t) \;=\; \mathcal{F}[s\lambda, \sigma^{(n)}](x,t)\,, \tag{2.2.18}$$

$$\frac{d}{dx}\hat\sigma_s^{(n)}(x) \;=\; -\frac{s}{T}\int_T^{2T} f(\sigma^{(n)} + \hat\sigma_s^{(n)}, v^{(n)}, x, \tau)\, d\tau\,, \quad \hat\sigma_s^{(n)}(0) = \varrho\,. \tag{2.2.19}$$

We now check that (2.2.16)–(2.2.19) has no solution on the boundary of a sufficiently large ball independently of $s \in [0,1]$. The operator $\mathcal{F}[0,\cdot]$ corresponding to the initial configuration $\lambda \equiv 0$ is odd, hence also Φ_0 is odd in $\mathbf{V}$, so that its Brouwer degree with respect to this ball and to the point $0 \in \mathbf{V}$ is nonzero. By homotopy, also the degree of $\Phi_1 = \Phi$ is nonzero, hence a solution exists inside the ball. We thus establish the existence of a solution to (2.2.16)–(2.2.19) provided we prove the following statement.

$$\left.\begin{array}{l}\text{There exist } p_\infty, \tilde p_\infty > 0 \text{ independent of } n \in \mathbb{N} \text{ and } s \in [0,1] \text{ such} \\ \text{that if } \mathbf{v} \in \mathbf{V} \text{ is a solution of (2.2.16)–(2.2.19) with } \|\sigma^{(n)}\|_\infty = p\,, \\ \text{then } p \le p_\infty \text{ and } \|v^{(n)}\|_\infty \le \tilde p_\infty\,.\end{array}\right\} \tag{2.2.20}$$

To prove the conjecture (2.2.20), we consider some $p \ge K$, where K is as in Hypothesis 1.0.1, some $n \in \mathbb{N}$, $s \in [0,1]$, and a solution of (2.2.16)–(2.2.19) with $\|\sigma^{(n)}\|_\infty \le p$. We test (2.2.16) by $(2\pi j/T)^2 v_{jk}$, (2.2.17) by $(2\pi j/T)^2 \sigma_{jk}$, and sum them up. Integrating by parts and using the T-periodicity we obtain

$$\int_T^{2T} \int_0^1 \partial_{tt}\varepsilon^{(n)}\, \partial_t \sigma^{(n)}\, dx\, dt \;=\; \int_T^{2T} \int_0^1 \partial_t v^{(n)} \partial_t \left(s\, f(\sigma^{(n)} + \hat\sigma_s^{(n)}, v^{(n)}, x, t)\right) dx\, dt\,. \tag{2.2.21}$$

Similarly, testing (2.2.16) by $-(2\pi j/T) v_{-jk}$ and (2.2.17) by $-(2\pi j/T)\sigma_{-jk}$ yields

$$\begin{aligned} &\int_T^{2T} \int_0^1 |\partial_t v^{(n)}|^2\, dx\, dt \qquad\qquad (2.2.22) \\ &\quad = \int_T^{2T} \int_0^1 \left(\partial_t \varepsilon^{(n)}\, \partial_t \sigma^{(n)} + s\, \partial_t v^{(n)}\, f(\sigma^{(n)} + \hat\sigma_s^{(n)}, v^{(n)}, x, t)\right) dx\, dt\,. \end{aligned}$$

By Hypothesis 2.1.1, we have the pointwise relations

$$\partial_t v^{(n)}\, \partial_t \left(s\, f(\sigma^{(n)} + \hat\sigma_s^{(n)}, v^{(n)}, x, t)\right) \;\le\; |\partial_t v^{(n)}| \left(\gamma_f |\partial_t \sigma^{(n)}| + |\beta_f|\right) \tag{2.2.23}$$

$$\begin{aligned} s\, f(\sigma^{(n)} + \hat\sigma_s^{(n)}, v^{(n)}, x, t)\, \partial_t v^{(n)} \;&=\; \partial_t \left(s\, F(\hat\sigma_s^{(n)}, v^{(n)}, x, t)\right) \qquad (2.2.24) \\ - s\, (\partial_t F)(\hat\sigma_s^{(n)}, v^{(n)}, x, t) &+ s\, \left(f(\sigma^{(n)} + \hat\sigma_s^{(n)}, v^{(n)}, x, t) - f(\hat\sigma_s^{(n)}, v^{(n)}, x, t)\right) \partial_t v^{(n)}\,, \end{aligned}$$

where we set $F(\sigma, v, x, t) = \int_0^v f(\sigma, v', x, t)\, dv'$. We have $(\partial_t F)(\sigma, v, x, t) \le |v||\beta_f|$ a. e. Combining (2.2.21)–(2.2.24) and using (3.4.10) and (3.2.16) we obtain that

$$\begin{aligned} \frac{1}{4}\kappa(p)\, \|\partial_t \sigma^{(n)}\|_3^3 &\le \|\partial_t v^{(n)}\|_2 \left(\gamma_f \|\partial_t \sigma^{(n)}\|_2 + \|\beta_f\|_2\right) , && (2.2.25)\\ \|\partial_t v^{(n)}\|_2^2 &\le h(p)\|\partial_t \sigma^{(n)}\|_2^2 + \gamma_f \|\sigma^{(n)}\|_2 \|\partial_t v^{(n)}\|_2 + \|\beta_f\|_2 \|v^{(n)}\|_2 . && (2.2.26)\end{aligned}$$

We now use the embedding inequalities

$$\|\partial_t \sigma^{(n)}\|_2 \le T^{1/6}\|\partial_t \sigma^{(n)}\|_3 , \quad \|\sigma^{(n)}\|_2 \le \frac{T}{2\pi}\|\partial_t \sigma^{(n)}\|_2 , \quad \|v^{(n)}\|_2 \le \frac{T}{2\pi}\|\partial_t v^{(n)}\|_2 \tag{2.2.27}$$

(note that both $\sigma^{(n)}(x,\cdot)$ and $v^{(n)}(x,\cdot)$ have zero average on $[0,T]$) and set

$$\left.\begin{array}{l} x(p) = \frac{1}{p}\sup\{\|\partial_t \sigma^{(n)}\|_3 \,;\, \|\sigma^{(n)}\|_\infty \le p\} \\ y(p) = \frac{1}{p}\sup\{\|\partial_t v^{(n)}\|_2 \,;\, \|\sigma^{(n)}\|_\infty \le p\} \\ z(p) = \frac{1}{p}\sup\{\|\partial_x \sigma^{(n)}\|_2 \,;\, \|\sigma^{(n)}\|_\infty \le p\} \end{array}\right\} , \tag{2.2.28}$$

where the supremum is taken over all possible solutions of (2.2.16)–(2.2.19) and over all $n \in \mathbb{N}$. From (2.2.25)–(2.2.28) we obtain

$$\begin{aligned} \frac{1}{4} p\, \kappa(p)\, x^3(p) &\le y(p)\left(T^{1/6}\gamma_f\, x(p) + \frac{\|\beta_f\|_2}{p}\right) , && (2.2.29)\\ y^2(p) &\le T^{1/3} h(p)\, x^2(p) + \frac{T}{2\pi} y(p)\left(T^{1/6}\gamma_f\, x(p) + \frac{\|\beta_f\|_2}{p}\right) , && (2.2.30)\end{aligned}$$

and an elementary computation based on hypothesis (2.1.1) and (2.1.8) yields that

$$\limsup_{p\to\infty} \sqrt{h(p)}\, x(p) \le 4T^{1/3}\gamma_f q , \quad \limsup_{p\to\infty} y(p) \le 4T^{1/2}\gamma_f q . \tag{2.2.31}$$

We estimate $z(p)$ using Eq. (2.2.16), which yields

$$\begin{aligned} \|\partial_x \sigma^{(n)}\|_2 &\le \|\partial_t v^{(n)}\|_2 + \|f^0\|_2 + \gamma_f \left(\|\sigma^{(n)}\|_2 + \|v^{(n)}\|_2 + \|\hat{\sigma}_s^{(n)}\|_2\right) && (2.2.32)\\ &\le \left(1 + \frac{T}{2\pi}\gamma_f\right)\|\partial_t v^{(n)}\|_2 + \frac{T}{2\pi}\gamma_f \|\partial_t \sigma^{(n)}\|_2 + \|f^0\|_2 \\ &\quad + T^{1/2}\gamma_f \left(\int_0^1 |\hat{\sigma}_s^{(n)}(x)|^2\, dx\right)^{1/2} . \end{aligned}$$

To estimate the last term on the right-hand side of (2.2.32), we use Eq. (2.2.19) and obtain

$$\begin{aligned} \frac{d}{dx}\left|\hat{\sigma}_s^{(n)}(x)\right| &\le \frac{s}{T}\int_T^{2T} \left|f(\sigma^{(n)} + \hat{\sigma}_s^{(n)}, v^{(n)}, x, t)\right|\, dt && (2.2.33)\\ &\le \gamma_f \left|\hat{\sigma}_s^{(n)}(x)\right| + \frac{1}{T}\int_T^{2T} |f^0(x,t)|\, dt + \frac{\gamma_f}{T}\int_T^{2T} \left(|\sigma^{(n)}| + |v^{(n)}|\right)(x,t)\, dt , \end{aligned}$$

hence

$$\max_{x\in[0,1]} \left|\hat{\sigma}_s^{(n)}(x)\right| \;\leq\; \mathrm{e}^{\gamma_f}\left(|\varrho| + T^{-1/2}\left(\|f^0\|_2 + \gamma_f\left(\|\sigma^{(n)}\|_2 + \|v^{(n)}\|_2\right)\right)\right) . \tag{2.2.34}$$

Combining (2.2.32) with (2.2.34) yields

$$\begin{aligned}\|\partial_x\sigma^{(n)}\|_2 \;\leq\;& T^{1/2}\gamma_f \mathrm{e}^{\gamma_f}|\varrho| + (1+\gamma_f\mathrm{e}^{\gamma_f})\,\|f^0\|_2 \\ &+ \left(1+\frac{T}{2\pi}\gamma_f\,(1+\gamma_f\mathrm{e}^{\gamma_f})\right)\|\partial_t v^{(n)}\|_2 + \gamma_f\frac{T^{7/6}}{2\pi}\,(\mathrm{e}^{\gamma_f}+1)\,\|\partial_t\sigma^{(n)}\|_3\,,\end{aligned} \tag{2.2.35}$$

hence, in view of (2.2.31),

$$\limsup_{p\to\infty} z(p) \;\leq\; 4T^{1/2}\gamma_f q\left(1+\frac{T}{2\pi}\gamma_f\,(1+\gamma_f\mathrm{e}^{\gamma_f})\right) . \tag{2.2.36}$$

By virtue of (2.1.2), (2.1.8), (2.2.31), and (2.2.36), we may choose $p_\infty > 0$ such that

$$M(x(p)+z(p)) \;<\; 1 \qquad \text{for } p \geq p_\infty\,. \tag{2.2.37}$$

In other words, from (2.0.21), (2.2.28) and (2.2.37) it follows that whenever we have a solution of (2.2.16)–(2.2.19), then the implication

$$\left(p \geq p_\infty\,,\ \|\sigma^{(n)}\|_\infty \leq p\right) \implies \|\sigma^{(n)}\|_\infty < p \tag{2.2.38}$$

holds, hence $\|\sigma^{(n)}\|_\infty < p_\infty$ independently of $n \in \mathbb{N}$. From (2.2.17), (3.2.16), and (2.2.31) we further obtain that $\|\partial_x v^{(n)}\|_3 \leq h(p_\infty)\|\partial_t\sigma^{(n)}\|_3 \leq p_\infty\, h(p_\infty)\, x(p_\infty)$, $\|\partial_t v^{(n)}\|_2 \leq p_\infty\, y(p_\infty)$, hence also $\|v^{(n)}\|_\infty < M' p_\infty(h(p_\infty)\, x(p_\infty) + y(p_\infty))$ as a consequence of (2.0.21). We thus proved the conjecture (2.2.20) which implies that (2.2.16)–(2.2.19) has a solution for every $n \in \mathbb{N}$. Moreover, we have found a bound independent of n for $\sigma^{(n)}$ in $H_T^{2,3}$ and for $v^{(n)}$ in $H_T^{3,2}$. Using the compact embedding (2.0.20), we may find a subsequence (still indexed by n) and some elements $\sigma^* \in H_T^{2,3}$ and $v \in H_T^{3,2}$ such that $\int_T^{2T} v(x,t)\,dt = \int_T^{2T} \sigma^*(x,t)\,dt = 0$ a. e., and

$$v^{(n)} \to v\,,\quad \sigma^{(n)} \to \sigma^* \qquad \text{uniformly,} \tag{2.2.39}$$

$$\partial_t v^{(n)} \to \partial_t v\,,\quad \partial_x\sigma^{(n)} \to \partial_x\sigma^* \qquad \text{weakly in } L_T^2\,, \tag{2.2.40}$$

$$\partial_x v^{(n)} \to \partial_x v\,,\quad \partial_t\sigma^{(n)} \to \partial_t\sigma^* \qquad \text{weakly in } L_T^3\,. \tag{2.2.41}$$

We can pass to the limit as $n \to \infty$ in (2.2.11)–(2.2.14) and find $\varepsilon^* \in \mathcal{C}_T^0$, $\hat{\sigma} \in W^{1,2}(0,1)$ such that

$$\varepsilon^*(x,t) \;=\; \mathcal{F}[\lambda,\sigma^*](x,t)\,, \tag{2.2.42}$$

$$\frac{d}{dx}\hat{\sigma}(x) \;=\; -\frac{1}{T}\int_T^{2T} f(\sigma^*+\hat{\sigma},v,x,\tau)\,d\tau\,,\quad \hat{\sigma}(0)=\varrho\,, \tag{2.2.43}$$

$$\int_T^{2T}\int_0^1 \left(\partial_t v - \partial_x\sigma^* - f(\sigma^*+\hat{\sigma},v,x,t)\right)\,\vartheta(x,t)\,dx\,dt \;=\; 0\,, \tag{2.2.44}$$

$$\int_T^{2T}\int_0^1 \left(\partial_t\varepsilon^* - \partial_x v\right)\,\vartheta(x,t)\,dx\,dt \;=\; 0 \tag{2.2.45}$$

for every test function $\vartheta \in L^2_T$ such that $\int_T^{2T} \vartheta(x,t)\,dt = 0$ a. e. We now obtain (2.2.1)–(2.2.2) (and thus complete the proof of Theorem 2.1.2) by putting $\vartheta = \partial_t v - \partial_x \sigma^* - f(\sigma^* + \hat\sigma, v, x, t) + (1/T)\int_T^{2T} f(\sigma^* + \hat\sigma, v, x, \tau)\,d\tau$ in (2.2.44), and $\vartheta = \partial_t \varepsilon^* - \partial_x v$ in (2.2.45).

2.3 Uniqueness

In this section, we prove Theorem 2.1.3. Since the nonlinearity f is now independent of σ, the counterpart of (2.2.1)–(2.2.5) reads

$$
\begin{aligned}
\partial_t v &= \partial_x \sigma^* + f(v,x,t) - \frac{1}{T}\int_T^{2T} f(v,x,\tau)\,d\tau\,, && (2.3.1)\\
\partial_t \varepsilon^* &= \partial_x v\,, && (2.3.2)\\
\varepsilon^* &= \mathcal{F}[\lambda, \sigma^*] && (2.3.3)\\
\frac{d}{dx}\hat\sigma(x) &= -\frac{1}{T}\int_T^{2T} f(v,x,\tau)\,d\tau\,, \quad \hat\sigma(0) = \varrho && (2.3.4)
\end{aligned}
$$

together with the T-periodicity condition and

$$
v(0,t) \;=\; v(1,t) \;=\; 0\,, \quad \int_T^{2T} v(x,t)\,dt \;=\; \int_T^{2T} \sigma^*(x,t)\,dt \;=\; 0\,. \tag{2.3.5}
$$

We will not repeat all details of the existence proof which exactly follows the lines of the proof of Theorem 2.1.2. Estimates analogous to (2.2.25)–(2.2.26) for the system (2.3.1)–(2.3.5) have the form

$$
\begin{aligned}
\frac{1}{4}\kappa(p)\,\|\partial_t \sigma^{(n)}\|_3^3 &\le \|\beta_f\|_2\,\|\partial_t v^{(n)}\|_2\,, && (2.3.6)\\
\|\partial_t v^{(n)}\|_2^2 &\le h(p)\|\partial_t \sigma^{(n)}\|_2^2 + \|\beta_f\|_2\|v^{(n)}\|_2 && (2.3.7)
\end{aligned}
$$

which, with the notation of (2.2.28), yields similarly as in (2.2.31) that

$$
\limsup_{p\to\infty} \sqrt{h(p)}\,x(p) \le 2T^{1/12}\,\tilde q\sqrt{\|\beta_f\|_2}\,, \quad \limsup_{p\to\infty} y(p) \le 2T^{1/4}\,\tilde q\sqrt{\|\beta_f\|_2}\,. \tag{2.3.8}
$$

Instead of (2.2.32) we directly have

$$
\|\partial_x \sigma^{(n)}\|_2 \;\le\; \left(1 + \frac{T}{2\pi}\gamma_f\right)\|\partial_t v^{(n)}\|_2 + \|f^0\|_2\,, \tag{2.3.9}
$$

and the rest of the existence argument is identical to the one in the previous subsection.

To prove the uniqueness, we consider two solutions $(v_1, \sigma_1^*), (v_2, \sigma_2^*)$ of (2.3.1)–(2.3.5) (with $\varepsilon_i^*, \hat\sigma_i\,,\ i = 1,2$ having the corresponding meaning) associated with two different values ϱ_1, ϱ_2 of ϱ in (2.3.4). Set $\bar v = v_1 - v_2\,,\ \bar\sigma = \sigma_1^* - \sigma_2^*\,,\ \bar\varepsilon = \varepsilon_1^* - \varepsilon_2^*$. As f is non-increasing in v, we obtain from (2.3.1)–(2.3.2) that

$$
\int_T^{2T}\int_0^1 \partial_t\bar\varepsilon\,\bar\sigma\,dx\,dt \;=\; \int_T^{2T}\int_0^1 \bar v(f(v_1,x,t) - f(v_2,x,t))\,dx\,dt \;\le\; 0\,. \tag{2.3.10}
$$

By Proposition 3.3.2 there exists $\sigma^0 \in W^{1,2}(0,1)$ such that $\bar{\sigma}(x,t) = \sigma^0(x)$ for $t \geq T$. In view of (2.3.5), we have $\sigma^0 \equiv 0$, hence $\sigma_1^* = \sigma_2^*$, consequently also $\varepsilon_1^* = \varepsilon_2^*$ and $v_1 = v_2$. We thus have

$$\hat{\sigma}_1(x) = \hat{\sigma}_2(x) \;=\; \varrho_1 - \varrho_2 \tag{2.3.11}$$

for all $x \in [0,1]$, and the uniqueness follows.

Remark 2.3.1. The ambiguity due to the arbitrary choice of ϱ in (2.3.4) can be removed by considering the Dirichlet boundary conditions in displacements instead of velocities. More specifically, we denote by $(v, \sigma^*, \hat{\sigma}_0)$ the solution of (2.3.1)–(2.3.5) corresponding to $\varrho = 0$. We know from (2.3.11) that $(v, \sigma^*, \hat{\sigma}_0 + \varrho)$ is then the solution to (2.3.1)–(2.3.5) for any ϱ. For $(x,t) \in [0,1] \times [T, \infty[$ and $\varrho \in \mathbb{R}$ set

$$\varepsilon^{(\varrho)}(x,t) \;=\; \mathcal{F}[\lambda, \sigma^* + \hat{\sigma}_0 + \varrho](x,t)\,, \quad u^{(\varrho)}(x,t) \;=\; \int_T^t v(x,t')\,dt' + \int_0^x \varepsilon^{(\varrho)}(x',T)\,dx'\,. \tag{2.3.12}$$

We then have $\partial_t u^{(\varrho)} = v$, $\partial_x u^{(\varrho)} = \varepsilon^{(\varrho)}$, $u^{(\varrho)}(x, t+T) = u^{(\varrho)}(x,t)$ for all $(x,t) \in [0,1] \times [T, \infty[$, and $u^{(\varrho)}(1,t) = \int_0^1 \varepsilon^{(\varrho)}(x,T)\,dx$. We claim that

$$\exists!\, \varrho \in \mathbb{R}: \quad u^{(\varrho)}(1,t) = 0 \quad \forall t \geq T\,. \tag{2.3.13}$$

This conjecture follows from the fact that for $\varrho_1 > \varrho_2$ we have by (3.3.6) and (2.3.11) that $\varepsilon^{(\varrho_1)}(x,t) - \varepsilon^{(\varrho_2)}(x,t) \geq h(0)(\varrho_1 - \varrho_2)$, and that $\varepsilon^{(\varrho)}$ depends continuously on ϱ.

2.4 Asymptotic stability

This subsection is devoted to the proof of Theorem 2.1.6. For $\lambda_1, \lambda_2 \in C([0,1]\,;\,\Lambda_{\underline{K}})$, $(v_1, \sigma_1), (v_2, \sigma_2) \in B$, we define the functional

$$\begin{aligned} V(\lambda_1, \lambda_2, v_1, v_2, \sigma_1, \sigma_2)(t) \;&=\; \int_0^1 \left(h(0)\,(\sigma_1 - \sigma_2)^2 + (v_1 - v_2)^2\right)\,dx \\ &\quad + \int_0^1 \int_0^\infty \left(\mathfrak{p}_r[\lambda_1, \sigma_1] - \mathfrak{p}_r[\lambda_2, \sigma_2]\right)^2\,dh(r)\,dx\,. \end{aligned} \tag{2.4.1}$$

Using (3.3.2) we check that whenever (v_i, σ_i) for $i = 1, 2$ are solutions of (1.2.1) with the respective choice of $\lambda = \lambda_i$, then

$$\frac{d}{dt} V(\lambda_1, \lambda_2, v_1, v_2, \sigma_1, \sigma_2)(t) \;\leq\; 0 \qquad \text{a. e.} \tag{2.4.2}$$

For $n \in \mathbb{N}$ and $x \in [0,1]$, $t \geq 0$, $r \geq 0$ we define the sequences

$$v_n(x,t) = \underline{v}(x, t+nT)\,,\ \sigma_n(x,t) = \underline{\sigma}(x, t+nT)\,,\ \lambda_n(x,r) = \mathfrak{p}_r[\underline{\lambda}(,\cdot), \underline{\sigma}(,\cdot)](nT)\,. \tag{2.4.3}$$

By Lemma 3.1.2 and Proposition 3.1.1 we have $\lambda_n \in C([0,1]\,;\,\Lambda_{\underline{K}})$ for all n, and putting $\varepsilon_n(x,t) = \mathcal{F}[\underline{\lambda}, \underline{\sigma}](x, t+nT)$ we obtain for all $x \in [0,1]$ and $t \geq 0$ that

$$\varepsilon_n(x,t) = \mathcal{F}[\lambda_n, \sigma_n](x,t)\,. \tag{2.4.4}$$

The sequence $\{(v_n, \sigma_n)\,;\, n \in \mathbb{N}\}$ is equibounded in B; there exists therefore a subsequence $\{n_k\}$ in $\mathbb{N}$ and an element $(v, \sigma) \in B$ such that

$$\left.\begin{array}{rcl} (\partial_t v_{n_k}, \partial_x v_{n_k}, \partial_t \sigma_{n_k}, \partial_x \sigma_{n_k}) & \to & (\partial_t v, \partial_x v, \partial_t \sigma, \partial_x \sigma) \\ & & \text{weakly-star in } \; L^\infty(0,\infty\,;\, L^2(0,1))\,, \\ (v_{n_k}, \sigma_{n_k}) & \to & (v, \sigma) \\ & & \text{locally uniformly in } \; [0,1]\times[0,\infty[\;. \end{array}\right\} \tag{2.4.5}$$

From (3.1.14) it follows that $\{\lambda_n\}$ is an equibounded and equicontinuous sequence in $C([0,1]\,;\,\Lambda_{\underline{K}})$. Since $\Lambda_{\underline{K}})$ is a compact subset of $C[0,\underline{K}]$, we may use the Arzelà-Ascoli Theorem and assume that the subsequence $\{n_k\}$ is such that

$$\lambda_{n_k} \to \lambda \in C([0,1]\,;\,\Lambda_{\underline{K}}) \quad \text{uniformly in } [0,1]\times[0,\underline{K}]\,. \tag{2.4.6}$$

All elements (v_n, σ_n) are solutions to (1.2.1) with ε_n given by (2.4.4). Passing to the limit as $n_k \to \infty$ we conclude that (v, σ) is a solution to (1.2.1). For all $k \in \mathbb{N}$ we have by (2.4.2) that

$$\frac{d}{dt} V(\lambda_{n_k}, \lambda, v_{n_k}, v, \sigma_{n_k}, \sigma)(t) \;\le\; 0 \qquad \text{a. e.} \quad , \tag{2.4.7}$$

hence

$$\sup_{t\ge 0} V(\lambda_{n_k}, \lambda, v_{n_k}, v, \sigma_{n_k}, \sigma)(t) \;\le\; V(\lambda_{n_k}, \lambda, v_{n_k}, v, \sigma_{n_k}, \sigma)(0)\,. \tag{2.4.8}$$

The right-hand side of (2.4.8) tends to 0 as $k \to \infty$, and we conclude that

$$\lim_{k\to\infty} \sup_{t\ge 0} \int_0^1 \left(|v_{n_k} - v|^2 + |\sigma_{n_k} - \sigma|^2\right)(x,t)\,dx \;=\; 0\,. \tag{2.4.9}$$

We now prove that both v and σ are T-periodic. Put $v_+(x,t) = v(x, t+T)$, $\sigma_+(x,t) = \sigma(x,t+T)$, $\lambda_+(x,r) = \mathfrak{p}_r[\lambda(,\cdot), \sigma(,\cdot)](T)$, and

$$\beta \;=\; \lim_{t\to\infty} V(\lambda_+, \lambda, v_+, v, \sigma_+, \sigma)(t) \;\ge\; 0\,. \tag{2.4.10}$$

For all $t \ge 0$ we have

$$\beta \;=\; \lim_{k\to\infty} V(\lambda_{n_k+1}, \lambda_{n_k}, v_{n_k+1}, v_{n_k}, \sigma_{n_k+1}, \sigma_{n_k})(t) \;=\; V(\lambda_+, \lambda, v_+, v, \sigma_+, \sigma)(t)\,, \tag{2.4.11}$$

hence $(d/dt)V(\lambda_+, \lambda, v_+, v, \sigma_+, \sigma) = 0$ a. e. in $[0,\infty[$. By construction we have $\lambda_+(x,0) = \sigma_+(x,0)$, $\lambda(x,0) = \sigma(x,0)$ for all $x \in [0,1]$. From Proposition 3.3.2 it follows that there exists a function $R(x,t)$ such that $R(x,\cdot)$ is non-decreasing for every x and $\sigma_+(x,t) - \sigma(x,t) = \lambda_+(x, R(x,t)) - \lambda(x, R(x,t))$. For every x there exists therefore the limit $\sigma_\infty(x) = \lim_{t\to\infty}(\sigma_+(x,t) - \sigma(x,t)) = \lim_{t\to\infty}(\sigma(x,t+T) - \sigma(x,t))$. Since σ is bounded, we have

$$\lim_{t\to\infty} (\sigma_+(x,t) - \sigma(x,t)) \;=\; 0 \quad \forall x \in [0,1]\,. \tag{2.4.12}$$

Using again Proposition 3.3.2, we similarly obtain

$$\lim_{t\to\infty}(\mathfrak{p}_r[\lambda_+,\sigma_+](x,t)-\mathfrak{p}_r[\lambda,\sigma](x,t)) \;=\; 0 \quad \forall x\in[0,1] \;\; \forall r>0\,. \tag{2.4.13}$$

Let $\delta>0$ be arbitrarily given. We fix some $m\in\mathbb{N}$ and $t_m>0$ such that for all $t\ge t_m$ and all $j=1,\dots,m$ we have

$$|\sigma(j/m,t+T)-\sigma(j/m,t)| \;<\; \delta/2\,. \tag{2.4.14}$$

For each $y\in[(j-1)/m,j/m]$ and $t\ge 0$ we have

$$|\sigma(y,t)-\sigma(j/m,t)| \;\le\; \frac{1}{\sqrt{m}}\left(\int_{(j-1)/m}^{j/m}|\partial_x\sigma(x,t)|^2\,dx\right)^{1/2} \;\le\; \frac{C}{\sqrt{m}} \tag{2.4.15}$$

with some constant C independent of t and m. We thus can find $t^*>0$ such that

$$|\sigma(\cdot,t+T)-\sigma(\cdot,t)|_{L^\infty(0,1)} \;<\; \delta \quad \text{for } t\ge t^*\,. \tag{2.4.16}$$

Let $\ell\in\mathbb{N}$ be such that, by virtue of (2.4.9), we have

$$\left|\int_0^1|\sigma(x,\cdot)-\sigma_{n_k}(x,\cdot)|^2\,dx\right|_{L^\infty(0,\infty)} \;<\; \delta^2 \quad \text{for } k\ge\ell\,. \tag{2.4.17}$$

Put $t^{**}=t^*+n_\ell T$. For $s\ge T^{**}$, we have $s-n_\ell T\ge t^*$, hence

$$\begin{aligned}
|\underline{\sigma}(\cdot,s+T)-\underline{\sigma}(\cdot,s)|_{L^2(0,1)} \;&\le\; |\underline{\sigma}(\cdot,s+T)-\sigma(\cdot,s-n_\ell T+T)|_{L^2(0,1)} \\
&\quad+\; |\sigma(\cdot,s-n_\ell T+T)-\sigma(\cdot,s-n_\ell T)|_{L^2(0,1)} \\
&\quad+\; |\sigma(\cdot,s-n_\ell T)-\underline{\sigma}(\cdot,s)|_{L^2(0,1)} \\
&\le\; 3\delta\,.
\end{aligned} \tag{2.4.18}$$

Let now $t\ge 0$ be arbitrary. We fix $k\ge\ell$ such that $t+n_kT\ge t^{**}$. Then

$$\begin{aligned}
|\sigma(\cdot,t+T)-\sigma(\cdot,t))|_{L^2(0,1)} \;&\le\; |\sigma(\cdot,t+T)-\sigma_{n_k}(\cdot,t+T)|_{L^2(0,1)} \\
&\quad+\; |\underline{\sigma}(\cdot,t+n_kT+T)-\underline{\sigma}(\cdot,t+n_kT)|_{L^2(0,1)} \\
&\quad+\; |\sigma(\cdot,t)-\sigma_{n_k}(\cdot,t)|_{L^2(0,1)} \\
&\le\; 5\delta\,.
\end{aligned} \tag{2.4.19}$$

Since $\delta>0$ was arbitrary, we obtain $\sigma(x,t+T)=\sigma(x,t))$ for all x and t, and from the fact that (v,σ) is a solution to (1.2.1) we obtain also $v(x,t+T)=v(x,t))$. Using (2.4.11) and (2.4.13) we obtain $\beta=0$ and $\lambda_+=\lambda$.

To conclude the proof, consider the sequence

$$d_n \;:=\; V(\lambda_n,\lambda,v_n,v,\sigma_n,\sigma)(0)\,. \tag{2.4.20}$$

By (2.4.2), we have $d_{n+1} \le d_n$ for all $n \in \mathbb{N}$. As $d_{n_k} \to 0$, the whole sequence $\{d_n\}$ converges to 0 and using (2.4.2) again, we obtain

$$\lim_{n\to\infty} \sup_{t\ge 0} \int_0^1 \left(|v_n - v|^2 + |\sigma_n - \sigma|^2\right)(x,t)\, dx \;=\; 0 . \tag{2.4.21}$$

Combining (2.4.21) with the elementary interpolation inequality

$$|w|_{L^\infty(0,1)} \;\le\; |w|_{L^2(0,1)} + 2|w|^{1/2}_{L^2(0,1)} |\partial_x w|^{1/2}_{L^2(0,1)} ,$$

we see that the whole sequence $\{(v_n, \sigma_n)\}$ converges uniformly to (v, σ) in $[0,1] \times [0, \infty[$. It remains to prove that (2.1.13) holds. To this end, we consider again any $\delta > 0$ and find n_0 such that $|v_n(x,t) - v(x,t)| + |\sigma_n(x,t) - \sigma(x,t)| < \delta$ for all (x,t) and all $n \ge n_0$. For $t \ge n_0 T$ we find $n \ge n_0$ such that $t - nT \in [0, T]$. Then

$$\begin{aligned} &|\underline{v}(\cdot,t) - v(\cdot,t))|_{L^\infty(0,1)} + |\underline{\sigma}(\cdot,t) - \sigma(\cdot,t))|_{L^\infty(0,1)} \\ &\quad = \; |v_n(\cdot, t-nT) - v(\cdot, t-nT))|_{L^\infty(0,1)} + |\sigma_n(\cdot, t-nT) - \sigma(\cdot, t-nT))|_{L^\infty(0,1)} \;<\; \delta , \end{aligned} \tag{2.4.22}$$

and Theorem 2.1.6 is proved.

3 Hysteresis operators

The first axiomatic approach to hysteresis was proposed by Madelung in [19], and a basic mathematical theory of hysteresis operators has been developed by M. Krasnosel'skii and his collaborators. The results of this group are summarized in the monograph [10] which constitutes until now the main source of reference on hysteresis. Our presentation here is based on more recent results from [14] which are needed here, in particular the energy inequalities in Subsection 3.4. The so-called *play operator* introduced in [10] is the main building block of the theory.

3.1 The play operator

For our purposes, it is convenient to work in the space $G_R(\mathbb{R}_+)$ of right-continuous regulated functions of time $t \in \mathbb{R}_+$, that is, functions $w : \mathbb{R}_+ \to \mathbb{R}$ which admit the left limit $w(t-)$ at each point $t > 0$, and the right limit $w(t+)$ exists and coincides with $w(t)$ for each $t \ge 0$. More information about regulated functions can be found e. g. in [1, 2, 7, 16, 24].

We endow the space $G_R(\mathbb{R}_+)$ with the system of seminorms

$$\|w\|_{[0,t]} \;=\; \sup\{|w(\tau)|\, ;\, \tau \in [0,t]\} \qquad \text{for } w \in G_R(\mathbb{R}_+) \text{ and } t \in \mathbb{R}_+ . \tag{3.1.1}$$

With the metric

$$\Delta(u,v) \;=\; \sup_{T>0} \frac{\|u-v\|_{[0,T]}}{1 + \|u-v\|_{[0,T]}} \qquad \text{for } u, v \in G_R(\mathbb{R}_+) , \tag{3.1.2}$$

the set $G_R(\mathbb{R}_+)$ becomes a Fréchet space. Similarly, $BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ will denote the space of right-continuous functions of bounded variation on each interval $[0,T]$ for any $T > 0$, and $C(\mathbb{R}_+)$ is the space of continuous functions on $\mathbb{R}_+$. We have $BV_R^{\mathrm{loc}}(\mathbb{R}_+) \subset G_R(\mathbb{R}_+)$ and the embedding is dense, while $C(\mathbb{R}_+)$ is a closed subspace of $G_R(\mathbb{R}_+)$.

The uniform approximation problem for real-valued regulated functions by functions of bounded variation has actually an interesting solution. For each $w \in G_R(\mathbb{R}_+)$, a parameter $r > 0$, and an initial condition $\xi_r^0 \in [w(0) - r, w(0) + r]$, there exists a unique $\xi_r \in BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ in the r-neighborhood of w with minimal total variation, that is (see Figure 1 for $\xi_r^0 = w(0)$),

$$
\begin{aligned}
&|w(t) - \xi_r(t)| \le r \qquad \forall t \ge 0\,, && (3.1.3)\\
&\xi_r(0) \;=\; \xi_r^0\,, && (3.1.4)\\
&\operatorname*{Var}_{[0,t]} \xi_r \;=\; \min\{\operatorname*{Var}_{[0,t]} \eta\,;\, \eta \in BV_R^{\mathrm{loc}}(\mathbb{R}_+),\ \eta(0) = \xi_r^0,\ \|w - \eta\|_{[0,t]} \le r\} \quad \forall t > 0\,. && (3.1.5)
\end{aligned}
$$

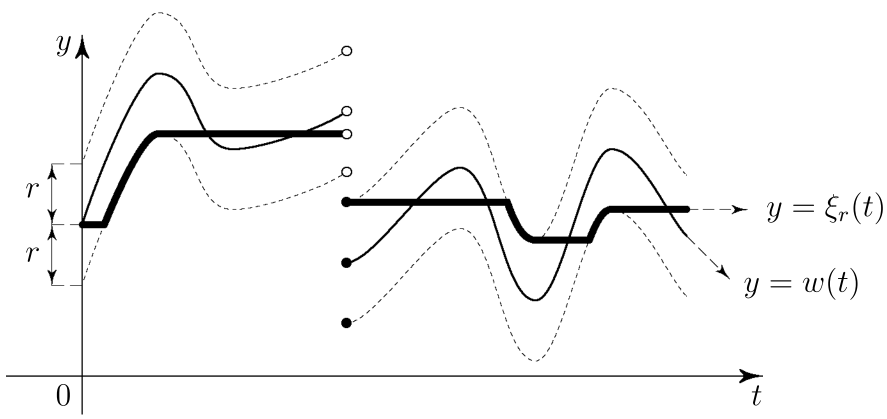

Figure 1: Optimal BV-approximation.

This result goes back to A. Vladimirov and V. Chernorutskii for the case of continuous functions w; for a proof see [23]. An extension to $L^\infty(\mathbb{R}_+)$ has been done in [17]. The function ξ_r can also be characterized as the unique solution of the variational inequality

$$
\begin{aligned}
&|w(t) - \xi_r(t)| \;\le\; r \qquad \forall t \ge 0\,, && (3.1.6)\\
&\xi_r(0) \;=\; \xi_r^0\,, && (3.1.7)\\
&\int_0^t (w(\tau) - \xi_r(\tau) - y(\tau))\, d\xi_r(\tau) \;\ge\; 0 && (3.1.8)\\
&\qquad\qquad \forall t \ge 0 \;\; \forall y \in G_R(\mathbb{R}_+)\,,\ \|y\|_{[0,t]} \le r\,,
\end{aligned}
$$

where the integration in (3.1.8) is understood in the Young or Kurzweil sense, see [16, 17]. If moreover w is continuous, then ξ_r is continuous, we can restrict ourselves to continuous test functions y, and (3.1.8) can be interpreted as the usual Stieltjes integral.

Let $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ denote the space of absolutely continuous functions on $\mathbb{R}_+$. It is an easy exercise to show that if $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, then the solution ξ_r to (3.1.6)–(3.1.8) belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ and fulfils the variational inequality

$$\dot{\xi}_r(t)\,(w(t) - \xi_r(t) - y) \;\geq\; 0 \;\;\text{a.e.} \quad \forall y \in [-r, r]\,. \tag{3.1.9}$$

Let us consider the mapping $\hat{\mathfrak{p}}_r : \mathbb{R} \times G_R(\mathbb{R}_+) \to BV^{\mathrm{loc}}_R(\mathbb{R}_+)$ which with each $\hat{\xi}^0_r \in \mathbb{R}$ and $w \in G_R(\mathbb{R}_+)$ associates the solution ξ_r of (3.1.6) – (3.1.8) with

$$\xi^0_r \;=\; \max\{w(0) - r, \min\{w(0) + r, \hat{\xi}^0_r\}\}\,. \tag{3.1.10}$$

Then $\hat{\mathfrak{p}}_r$ is a hysteresis operator called the *play*, and alternative equivalent definitions of the play can be found in [3, 10, 25]. Figure 2 shows a typical $w - \xi_r$ diagram. The horizontal parts of the graph are reversible, motions along the lines $\xi_r = w \pm r$ are irreversible.

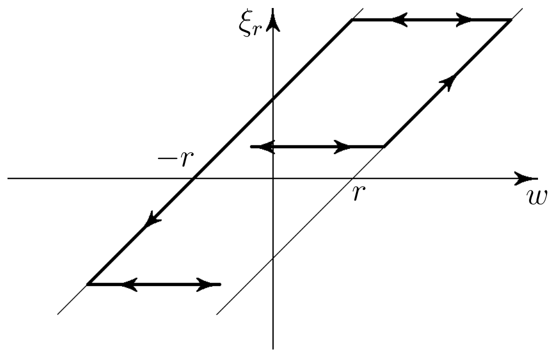

Figure 2: A diagram of the play.

More complex hysteresis behavior can be modelled by considering the whole family $\{\xi_r\}_{r>0}$ corresponding to a given $w \in G_R(\mathbb{R}_+)$. In fact, [3, Theorem 2.7.7] shows that a very large class of hysteresis operators admits a representation by means the one-parametric play system which accounts for the *hysteresis memory* and the parameter r plays the role of *memory variable*. We introduce the *hysteresis state space*

$$\Lambda \;=\; \{\lambda : \mathbb{R}_+ \to \mathbb{R}\,;\, |\lambda(r) - \lambda(s)| \leq |r - s| \;\; \forall r, s \in \mathbb{R}_+\,,\, \lim_{r \to +\infty} \lambda(r) = 0\}\,, \tag{3.1.11}$$

and choose the initial condition $\{\hat{\xi}^0_r\}_{r>0}$ in the form

$$\hat{\xi}^0_r = \lambda(r) \qquad \text{for } r > 0\,, \tag{3.1.12}$$

where $\lambda \in \Lambda$ is given. We define the operators $\mathfrak{p}_r : \Lambda \times G_R(\mathbb{R}_+) \to BV^{\mathrm{loc}}_R(\mathbb{R}_+)$ for $r > 0$ by the formula

$$\mathfrak{p}_r[\lambda, w] \;=\; \hat{\mathfrak{p}}_r[\lambda(r), w] \tag{3.1.13}$$

for $\lambda \in \Lambda$ and $w \in G_R(\mathbb{R}_+)$. Consistently with the definition we set $\mathfrak{p}_0[\lambda, w](t) = w(t)$ for all $t \geq 0$.

The following result was proved in [14, 17].

Proposition 3.1.1. *For every $\lambda \in \Lambda$, $w \in G_R(\mathbb{R}_+)$, and $t \geq 0$, the mapping $r \mapsto \lambda_t(r) = \mathfrak{p}_r[\lambda, w](t)$ belongs to Λ, and for all $\lambda_1, \lambda_2 \in \Lambda$, $w_1, w_2 \in G_R(\mathbb{R}_+)$ and $t \geq 0$ we have*

$$|\mathfrak{p}_r[\lambda_1, w_1](t) - \mathfrak{p}_r[\lambda_2, w_2](t)| \ \leq \ \max\{|\lambda_1(r) - \lambda_2(r)|, \|w_1 - w_2\|_{[0,t]}\} \ \ \forall r \geq 0\,, \tag{3.1.14}$$

The play operator thus generates for every $t \geq 0$ a continuous *state mapping* $\Pi_t : \Lambda \times G_R(\mathbb{R}_+) \to \Lambda$ which with each $(\lambda, w) \in \Lambda \times G_R(\mathbb{R}_+)$ associates the state $\lambda_t \in \Lambda$ at time t.

In order to study further properties of the play, we first derive an explicit formula for $\mathfrak{p}_r[\lambda, w]$ if w is a step function of the form

$$w(t) = \sum_{k=1}^{m} w_{k-1} \chi_{[t_{k-1}, t_k[}(t) \qquad \text{for} \quad t \geq 0 \tag{3.1.15}$$

with some given $w_i \in \mathbb{R}$, $i = 0, 1, \dots, m-1$, where $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = +\infty$ is a given sequence and χ_A for $A \subset \mathbb{R}$ is the characteristic function of the set A, that is, $\chi_A(t) = 1$ for $t \in A$, $\chi_A(t) = 0$ otherwise. We define analogously to (3.1.10) for $\lambda \in \Lambda$ and $v \in \mathbb{R}$ the function $P[\lambda, v] : \mathbb{R}_+ \to \mathbb{R}$ by the formula

$$P[\lambda, v](r) \ = \ \max\{v - r, \min\{v + r, \lambda(r)\}\}\,, \tag{3.1.16}$$

see Fig. 3. In particular, P can be considered as a mapping from $\Lambda \times \mathbb{R}$ to Λ. One can directly check as a one-dimensional counterpart of [16, Proposition 4.3] using the Young or Kurzweil integral calculus and the inequality

$$(P[\lambda, v](r) - \lambda(r))\ (v - P[\lambda, v](r) - z) \ \geq \ 0 \quad \forall |z| \leq r \tag{3.1.17}$$

that we have

$$\xi_r(t) = \sum_{k=1}^{m} \xi_{k-1}^{(r)} \chi_{[t_{k-1}, t_k[}(t) \qquad \text{for} \quad t \geq 0\,, \tag{3.1.18}$$

with

$$\xi_k^{(r)} = \lambda_k(r)\,, \quad \lambda_k = P[\lambda_{k-1}, w_k]\,, \quad \lambda_{-1} = \lambda\,. \tag{3.1.19}$$

for $k = 0, \dots m-1$, see Figure 3.

Every function $w \in G_R(\mathbb{R}_+)$ can be approximated uniformly on every compact interval by step functions of the form (3.1.15). Proposition 3.1.1 enables us to extend Eq. (3.1.19) to the whole space $G_R(\mathbb{R}_+)$ and obtain for a function $w \in G_R(\mathbb{R}_+)$ which is monotone (non-decreasing or non-increasing) in an interval $[t_0, t_1]$ the representation formula

$$\mathfrak{p}_r[\lambda, w](t) \ = \ P[\lambda_{t_0}, w(t)](r) \ = \ \max\{w(t) - r, \min\{w(t) + r, \lambda_{t_0}(r)\}\} \tag{3.1.20}$$

for $t \in [t_0, t_1]$, see Figure 2. It is perhaps interesting to note that (3.1.20) has originally been used in [10] as alternative definition of the play on continuous piecewise monotone inputs, extended afterwards by density and continuity to the whole space of continuous functions.

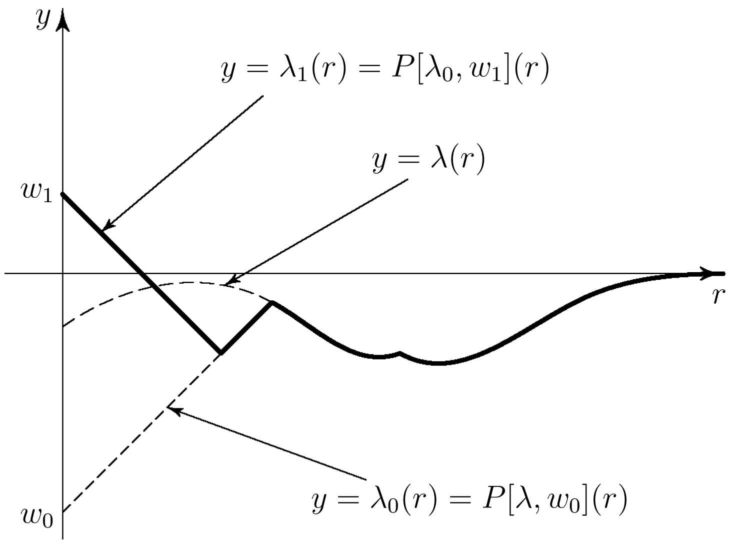

Figure 3: Distribution of play operators in consecutive times.

More generally, the play possesses the *semigroup property* as a time-continuous version of (3.1.19), namely

$$\mathfrak{p}_r[\lambda, w](t+s) = \mathfrak{p}_r[\lambda_s, w(s+\cdot)](t) \tag{3.1.21}$$

for all $w \in G_R(\mathbb{R}_+)$, $\lambda \in \Lambda$ and $s, t \geq 0$.

The choice (3.1.11) of the state space is justified by the fact that it consists of elements which are *asymptotically reachable from the reference initial state* $\lambda \equiv 0$, that is,

$$\forall \lambda \in \Lambda \;\; \exists w \in G_R(\mathbb{R}_+) \;\; \forall \varepsilon > 0 \;\; \exists T > 0 \; : \; \sup_{r>0} |\lambda(r) - \mathfrak{p}_r[0, w](T)| \; < \; \varepsilon\,. \tag{3.1.22}$$

Instead of a formal proof of this statement, we rather illustrate the construction of w on Figure 4. We set for instance $T_k = 2k$ for $k = 0, 1, 2, \ldots$ and fix a sequence $\varepsilon_k \to 0$ as $k \to \infty$. The function w will be defined as a step function successively in $[T_k, T_{k+1}]$ with a maximum absolute value at $T_k + 1$ and with jumps of decreasing amplitude at points $T_k + 1 < t_1 < t_2 < \cdots < T_{m_k} < T_{k+1}$. The graph of the function $\lambda_k(r) = \mathfrak{p}_r[0, w](T_{k+1})$ is piecewise affine with alternating slopes $+1$ and -1 for $0 \leq r \leq |w(T_k + 1)|$, and is chosen so as $|\lambda(r) - \lambda_k(r)| \; < \; \varepsilon_k$ for $k = 0, 1, 2, \ldots$.

Consider now a subset Λ_K of the state space Λ defined as

$$\Lambda_K \;=\; \{\lambda \in \Lambda\,;\, \lambda(r) = 0 \;\text{ for }\; r \geq K\} \tag{3.1.23}$$

for any $K > 0$. We now prove another property of the play which is used several times throughout the text.

Lemma 3.1.2. *Let $w \in G_R(\mathbb{R}_+)$ and $t \geq 0$ be given. Set*

$$w_{\max}(t) = \sup_{\tau \in [0,t]} w(\tau)\,, \qquad w_{\min}(t) = \inf_{\tau \in [0,t]} w(\tau)\,. \tag{3.1.24}$$

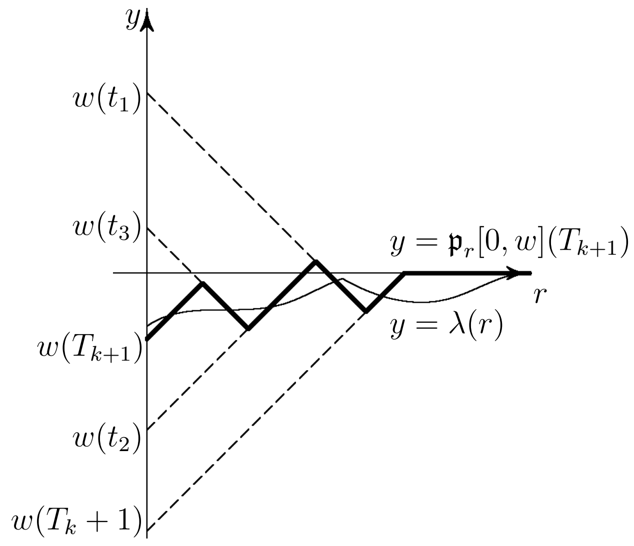

Figure 4: Construction of w in (3.1.22) at times $T_k + 1 < t_1 < t_2 < t_3 < T_{k+1}$.

Then for all $\lambda \in \Lambda$ *and* $r > 0$ *we have*

$$\mathfrak{p}_r[\lambda, w](\tau) \;\leq\; \max\{\lambda(r),\, w_{\max}(t) - r\} \quad \forall\, \tau \in [0, t]\,, \tag{3.1.25}$$

$$\mathfrak{p}_r[\lambda, w](\tau) \;\geq\; \min\{\lambda(r),\, w_{\min}(t) + r\} \quad \forall\, \tau \in [0, t]\,, \tag{3.1.26}$$

$$\mathfrak{p}_r[\lambda, w](t) \;=\; \lambda(r) \qquad \textit{for } r > \|m_\lambda(w(\cdot))\|_{[0,t]}\,, \tag{3.1.27}$$

where for $v \in \mathbb{R}$ *we put* $m_\lambda(v) = \inf\{r \geq 0\,;\, |\lambda(r) - v| = r\}$. *In particular, for* $K > 0$, $\lambda \in \Lambda_K$ *we have* $\lambda_t \in \Lambda_{K_t}$ *for all* $t \geq 0$, *where* $K_t = \max\{K, \|w\|_{[0,t]}\}$.

Proof. By density and continuity, it suffices to prove the assertion for step functions w of the form (3.1.15) using the recurrent formula (3.1.19). We show by induction that

$$\lambda_{k-1}(r) \;\leq\; \max\{\lambda(r),\, w_{\max}(t) - r\} \quad \forall\, r \geq 0 \tag{3.1.28}$$

for every $t_{k-1} \leq t$. Indeed, (3.1.28) holds for $k = 0$. Assume now that for some $k > 0$, $t_{k-1} \leq t$, and $r > 0$, we have $\lambda_{k-1}(r) > w_{\max}(t) - r$. By virtue of (3.1.16), (3.1.19) and of the induction hypothesis, we have $\lambda(r) \geq \lambda_{k-2}(r) > w_{\max}(t) - r$, hence $\lambda_{k-1}(r) \leq \lambda_{k-2}(r) \leq \lambda(r)$, and (3.1.25) follows. The proof of (3.1.26) is similar. To check Eq. (3.1.27), we notice that the function $r \mapsto r - |v - \lambda(r)|$ is non-decreasing for every $v \in \mathbb{R}$, hence for $r > \|m_\lambda(w(\cdot))\|_{[0,t]}$ and for all $\tau \in [0, t]$ we have $|\lambda(r) - w(\tau)| \leq r$, that is, $\lambda(r) - r \leq w(\tau) \leq \lambda(r) + r$. Then (3.1.16), (3.1.19) yield immediately that $\lambda_{k-1}(r) = \lambda(r)$. For $\lambda \in \Lambda_K$ and $v \in \mathbb{R}$ we have $\max\{K, |v|\} \geq m_\lambda(v)$, and using (3.1.27) we easily complete the proof. ∎

Let us derive some consequences from Lemma 3.1.2. Assume that $m_\lambda(w(\cdot))$ attains at a point $\bar{t} \geq 0$ its maximum over $[0, \bar{t}]$, that is,

$$\bar{r} \;:=\; m_\lambda(w(\bar{t})) \;=\; \|m_\lambda(w(\cdot))\|_{[0,\bar{t}]}\,. \tag{3.1.29}$$

The case $\bar{r} = 0$ is trivial, as it implies $w(t) = \lambda(0)$ for all $t \in [0, \bar{t}]$. For $\bar{r} > 0$ we distinguish the cases

(i) $w(\bar{t}) = \lambda(\bar{r}) + \bar{r}$,

(ii) $w(\bar{t}) = \lambda(\bar{r}) - \bar{r}$.

If (i) holds and $w(t) > w(\bar{t})$ for some $t \in [0, \bar{t}]$, then $\lambda(\bar{r}) + \bar{r} < w(t)$, hence $m_\lambda(w(t)) > \bar{r}$ which contradicts (3.1.29). We thus have $w(\bar{t}) = w_{\max}(\bar{t})$, and Lemma 3.1.2 together with (3.1.6) yield

$$\mathfrak{p}_r[\lambda, w](\bar{t}) \; = \; \max\{\lambda(r), w(\bar{t}) - r\}\,. \tag{3.1.30}$$

Similarly, in the case (ii) we have $w(\bar{t}) = w_{\min}(\bar{t})$ and

$$\mathfrak{p}_r[\lambda, w](\bar{t}) \; = \; \min\{\lambda(r), w(\bar{t}) + r\}\,. \tag{3.1.31}$$

From the above considerations we conclude

Corollary 3.1.3. *Let $w \in G_R(\mathbb{R}_+)$ be T-periodic that is $w(t+T) = w(t)$ for all $t \geq 0$, with a fixed period $T > 0$. Then $\mathfrak{p}_r[\lambda, w]$ is T-periodic for $t \geq T$ for all $\lambda \in \Lambda$.*

Proof. We may again consider only step functions w and then pass to the uniform limit, if necessary. The function $m_\lambda(w(\cdot))$ is T-periodic and attains its maximum at some point $\bar{t} \in [0, T]$, hence also at all points $\bar{t} + kT$, $k \in \mathbb{N}$. From (3.1.30)–(3.1.31) and the semigroup property (3.1.21) we obtain the assertion. ■

3.2 Prandtl-Ishlinskii operator

We describe here a construction which has been suggested in [8, 21] as a model for elastoplastic hysteresis. Each individual play represents a rigid-plastic element with kinematic hardening, and their linear superposition corresponds to a combination in series of such elements. A passage to the whole one-parametric continuum of plays can be done by homogenization, see e. g. [6].

Given a *distribution function* $h \in BV_R^{\mathrm{loc}}(\mathbb{R}_+)$, we define the value of the *Prandtl-Ishlinskii operator* $\mathcal{F} : \Lambda \times G_R(\mathbb{R}_+) \to G_R(\mathbb{R}_+)$ *generated by* h for an initial state $\lambda \in \Lambda$ an an input $w \in G_R(\mathbb{R}_+)$ by the formula

$$\mathcal{F}[\lambda, w](t) \; = \; h(0)\, w(t) + \int_0^\infty \mathfrak{p}_r[\lambda, w](t)\, dh(r)\,. \tag{3.2.1}$$

By (3.1.27), the definition is meaningful if and only if

$$\int_0^\infty \lambda(r)\, dh(r) \; < \; \infty\,. \tag{3.2.2}$$

This is always true if for instance $\lambda \in \Lambda_K$ for some $K > 0$. The function

$$H(s) \;=\; \int_0^s h(r)\,dr \tag{3.2.3}$$

is the so-called *initial loading curve* which depicts the reaction of a hysteresis system with no previous memory to an input which monotonically increasing from zero. Indeed, assuming $\lambda \equiv 0$, $w(0) = 0$, and w increasing in $[0, T]$, we obtain from (3.1.20) that $\mathfrak{p}_r[\lambda, w](t) = P[0, w(t)](r) = \max\{w(t) - r, 0\}$, hence

$$\mathcal{F}[\lambda, w](t) \;=\; h(0)\,w(t) + \int_0^{w(t)} (w(t) - r)\,dh(r) \;=\; H(w(t))\,. \tag{3.2.4}$$

Let us have a short look at the hysteresis branches starting from the initial loading curve at time t_0. Assume that $w(t_0) > 0$, $\lambda_{t_0} = \max\{w(t_0) - r, 0\}$, and that w decreases in $[t_0, t_1]$, $t_1 > t_0$, $w(t_1) > -w(t_0)$. By (3.1.20) we have

$$\mathfrak{p}_r[\lambda, w](t) \;=\; \begin{cases} w(t) + r & \text{for } \; 0 < r < \frac{1}{2}(w(t_0) - w(t))\,, \\ w(t_0) - r & \text{for } \; \frac{1}{2}(w(t_0) - w(t)) \le r < w(t_0)\,, \\ 0 & \text{for } \; w(t_0) \le r\,, \end{cases} \tag{3.2.5}$$

hence

$$\mathcal{F}[\lambda, w](t) \;=\; H\left(w(t_0)\right) - 2H\left(\frac{1}{2}(w(t_0) - w(t))\right)\,. \tag{3.2.6}$$

A similar computation in the case $w(t_0) < 0$, $\lambda_{t_0} = \min\{w(t_0) + r, 0\}$, w increases in $[t_0, t_1]$, $t_1 > t_0$, $w(t_1) < -w(t_0)$, yields

$$\mathcal{F}[\lambda, w](t) \;=\; H\left(w(t_0)\right) + 2H\left(\frac{1}{2}(w(t) - w(t_0))\right)\,. \tag{3.2.7}$$

We see that the hysteresis branches are homothetic copies with factor 2 of the initial loading curve, reversed if w decreases. This phenomenon is known in plasticity as the “Masing law”. Figure 5 shows two typical situations, where $h(r) \ge 0$ and

- either h is non-decreasing and the loops are oriented counterclockwise,
- or h is non-increasing and the loops are oriented clockwise.

We will see below that the orientation of the loops is important for the energy dissipation properties of the model. Furthermore, it was shown in [11] that the two cases correspond to mutually inverse operators associated with mutually inverse initial loading curves (note that if H is convex, then H^{-1} is concave and vice versa). The following result is a variant of [14, Corollary II.3.4].

Proposition 3.2.1. *Let $h \in BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ be such that $h(r) > 0$ for all $r > 0$, and let H given by (3.2.3) be unbounded. Let H^{-1} be the inverse function to H, let $\hat{\mathcal{F}}$ be the Prandtl-Ishlinskii operator of the form (3.2.1) generated by $\hat{h} = dH^{-1}/dr$. Then for all $w \in G_R(\mathbb{R}_+)$, $K > 0$, $\lambda \in \Lambda_K$, and $t \ge 0$ we have*

$$\hat{\mathcal{F}}[\mu, \mathcal{F}[\lambda, w]](t) \;=\; w(t)\,, \tag{3.2.8}$$

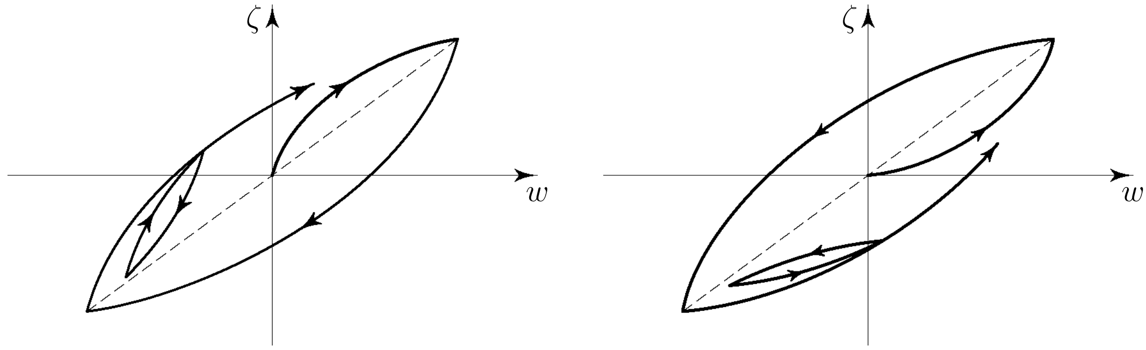

Figure 5: Clockwise and counterclockwise hysteresis in $\zeta(t) = \mathcal{F}[0, w](t)$.

where $\mu \in \Lambda_{H(K)}$ *is given for* $s \geq 0$ *by the formula*

$$\mu(s) \;=\; -\int_{H^{-1}(s)}^{\infty} \lambda'(r)\, h(r)\, dr\,. \tag{3.2.9}$$

The local Lipschitz continuity of $\mathcal{F}$ follows immediately from Proposition 3.1.1 and Lemma 3.1.2, and we state the result explicitly as follows.

Proposition 3.2.2. *Let* $h \in BV_R^{\mathrm{loc}}(\mathbb{R}_+)$ *and* $K > 0$ *be given, and let* $\mathcal{F}$ *be the operator (3.2.1). Then for all* $w_1, w_2 \in G_R(\mathbb{R}_+)$,, $\lambda_1, \lambda_2 \in \Lambda_K$, *and* $t \geq 0$ *we have*

$$\begin{aligned} |\mathcal{F}[\lambda_1, w_1](t) - \mathcal{F}[\lambda_2, w_2](t)| \;&\leq\; |h(0)|\, |w_1(t) - w_2(t)| \\ &+ \left(\operatorname*{Var}_{[0,R(t)]} h\right) \max\{\|\lambda_1 - \lambda_2\|_{[0,K]}\,, \|w_1 - w_2\|_{[0,t]}\}\,, \end{aligned} \tag{3.2.10}$$

where $R(t) = \max\{K, \|w_1\|_{[0,t]}\,, \|w_2\|_{[0,t]}\}$, *and* Var *denotes the total variation.*

We will not consider here the question of continuous dependence of $\mathcal{F}$ on the distribution function h, and an interested reader may find more information on this subject in [6].

Let $w \in G_R(\mathbb{R}_+)$ and $0 \leq t_1 < t_2$ be arbitrarily chosen. Putting in (3.2.16) $\lambda_1 = \lambda_2 =: \lambda$ and $w_1 = w$, $w_2(t) = w(t)$ for $t \in [0, t_1[$, $w_2(t) = w(t_1)$ for $t \in [t_1, t_2]$, $\zeta = \mathcal{F}[\lambda, w]$, we obtain that

$$|\zeta(t_2) - \zeta(t_1)| \;\leq\; \left(|h(0)| + \operatorname*{Var}_{[0,R(t_2)]} h\right) \|w - w(t_1)\|_{[t_1,t_2]}\,. \tag{3.2.11}$$

In particular, if $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, then $\zeta \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, and we have

$$|\dot\zeta(t)| \;\leq\; \left(|h(0)| + \operatorname*{Var}_{[0,R(t)]} h\right) |\dot w(t)| \qquad \text{a.e.}\;, \quad \text{with}\;\; R(t) = \max\{K, \|w\|_{[0,t]}\}\,. \tag{3.2.12}$$

Moreover, if $\dot w(t) = 0$, then $\dot\zeta(t)$ exists and equals 0. Assume now that w increases in an interval $[t_0, t_1]$. From Eq. (3.1.20) it follows for $t \in [t_0, t_1]$ that

$$\begin{aligned} \mathcal{F}[\lambda, w](t) - \mathcal{F}[\lambda, w](t_0) &= h(0)\,(w(t) - w(t_0)) \\ &\quad + \int_0^{m_{\lambda_{t_0}}(w(t))} (w(t) - r - \lambda_{t_0}(r))\, dh(r) \\ &= \int_0^{m_{\lambda_{t_0}}(w(t))} h(r)\,(1 + \lambda'_{t_0}(r))\, dr \;=\; \int_{w(t_0)}^{w(t)} h(m_{\lambda_{t_0}}(u))\, du\,. \end{aligned} \tag{3.2.13}$$

Similarly, if w decreases in $[t_0, t_1]$, then

$$\mathcal{F}[\lambda, w](t) - \mathcal{F}[\lambda, w](t_0) \;=\; -\int_{w(t)}^{w(t_0)} h(m_{\lambda_{t_0}}(u))\, du \qquad \text{for} \quad t \in [t_0, t_1]\,. \tag{3.2.14}$$

From now on, we restrict ourselves to counterclockwise Prandtl-Ishlinskii operators and assume that

$$\text{The function } h \text{ is positive and non-decreasing in } [0, \infty[\,. \tag{3.2.15}$$

Then Eq. (3.2.12) reads

$$|\dot\zeta(t)| \;\le\; h(R(t))\,|\dot w(t)| \qquad \text{a.e.} \tag{3.2.16}$$

If w is monotone in a neighborhood of t, $\dot w(t) \ne 0$, and $\dot\zeta(t)$ exist at some point t, then and we may conclude using (3.2.13) that $\dot\zeta(t)$ and $\dot w(t)$ have the same sign, and

$$\dot\zeta(t)\,\dot w(t) \;\ge\; h(0)\,\dot w^2(t)\,. \tag{3.2.17}$$

Inequality (3.2.17) therefore holds a. e. if w is continuously differentiable. By [14, Proposition II.4.2], the Prandtl-Ishlinskii operator is locally Lipschitz continuous in $W^{1,1}(0,T)$ for every $T > 0$, hence (3.2.17) can be a. e. extended to any $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$.

Remark 3.2.3. The Prandtl-Ishlinskii operator (3.2.1) can be considered as a special case of the *Preisach operator*

$$\mathcal{P}[\lambda, w](t) \;=\; a\,w(t) + \int_0^\infty \psi(r, \mathfrak{p}_r[\lambda, w](t))\, dr\,, \tag{3.2.18}$$

where $a \in \mathbb{R}$ is a constant and ψ is a given function of two variables. The original construction based on the concept proposed in [22] and based on the concept of *two-parametric relays*, used systematically in [20, 25], is shown in [12] to be equivalent to (3.2.18). More about the relationship between the operators (3.2.1) and (3.2.18) can be found in [13, 14].

3.3 Monotonicity

The variational character of the Prandtl-Ishlinskii operator induces natural monotonicity for absolutely continuous inputs. Assume that h satisfies (3.2.15), $w_1, w_2 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, and $\lambda_1, \lambda_2 \in \Lambda$ are given, and set $\xi_r^{(i)} = \mathfrak{p}_r[\lambda_i, w_i]$, $\zeta_i = \mathcal{F}[\lambda_i, w_i]$ for $i = 1, 2$, where $\mathcal{F}$ is given by (3.2.1). From (3.1.9) it follows that $(\dot{\xi}_r^{(1)} - \dot{\xi}_r^{(2)})(w_1 - w_2 - \xi_r^{(1)} + \xi_r^{(2)}) \geq 0$ a. e., hence

$$\frac{1}{2}\frac{d}{dt}(\xi_r^{(1)} - \xi_r^{(2)})^2 \;\leq\; (\dot{\xi}_r^{(1)} - \dot{\xi}_r^{(2)})(w_1 - w_2) \;\text{ a. e.,} \tag{3.3.1}$$

$$\frac{1}{2}\frac{d}{dt}\left(h(0)(w_1 - w_2)^2 + \int_0^\infty (\xi_r^{(1)} - \xi_r^{(2)})^2 \, dh(r)\right) \;\leq\; (\dot{\zeta}_1 - \dot{\zeta}_2)(w_1 - w_2) \;\text{ a. e.} \tag{3.3.2}$$

Let $W^{1,1}_T(\mathbb{R}_+)$ denote the space of T-periodic absolutely continuous functions defined on $\mathbb{R}_+$. In view of Corollary 3.1.3, we obtain for all $w_1, w_2 \in W^{1,1}_T(\mathbb{R}_+)$ and ζ_i, λ_i as above that ζ_1, ζ_2 are T-periodic for $t \geq T$ and

$$\int_T^{2T} (\dot{\zeta}_1(t) - \dot{\zeta}_2(t))(w_1(t) - w_2(t)) \, dt \;\geq\; 0\,. \tag{3.3.3}$$

We obviously have equality in (3.3.3) provided $w_1 - w_2 =$ const., but in this case we actually can easily prove more, namely

Proposition 3.3.1. *Let $\lambda_1, \lambda_2 \in \Lambda$, $w_1 \in W^{1,1}_T(\mathbb{R}_+)$ and $c \in \mathbb{R}$ be given, and put $w_2(t) = w_1(t) + c$, $\zeta_i = \mathcal{F}[\lambda_i, w_i]$ for $i = 1, 2$, with $\mathcal{F}$ given by (3.2.1). Then there exists $\tilde{c} \in \mathbb{R}$ such that $\zeta_2(t) = \zeta_1(t) + \tilde{c}$ for $t \geq T$.*

Proof. For $r > 0$ and $i = 1, 2$ set $\xi_r^{(i)} = \mathfrak{p}_r[\lambda_i, w_i]$. By (3.3.1) we have

$$\frac{d}{dt}\left(\xi_r^{(1)} - \xi_r^{(2)} + c\right)^2(t) \;\leq\; 0 \quad \text{a. e.}$$

From Corollary 3.1.3 we obtain that $\xi_r^{(1)}(t) - \xi_r^{(2)}(t) = c_r =$ const. for $t \geq T$, and the assertion follows. ■

The converse of Proposition 3.3.1 holds if h in (3.2.1) is strictly monotone, so that inequalities (3.3.2) and (3.3.3) are in fact "almost" strict. This fact is less obvious and we state it in the form given in [14, Theorem II.4.10, Corollary II.4.11, and Proposition II.4.12].

Proposition 3.3.2. *Let the function h in (3.2.15) be increasing and let $w_1, w_2 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, $\lambda_1, \lambda_2 \in \Lambda$ be given, $\zeta_i = \mathcal{F}[\lambda_i, w_i]$ for $i = 1, 2$, with $\mathcal{F}$ given by (3.2.1). Assume that (3.3.2) holds with equality sign a. e. Then there exists a non-decreasing function $R : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\mathfrak{p}_r[\lambda_1, w_1](t) - \mathfrak{p}_r[\lambda_2, w_2](t) \;=\; \begin{cases} \lambda_1^0(r) - \lambda_2^0(r) & \text{for } r \geq R(t)\,, \\ \lambda_1^0(R(t)) - \lambda_2^0(R(t)) & \text{for } 0 \leq r < R(t)\,, \end{cases} \tag{3.3.4}$$

where $\lambda_i^0(r) = \mathfrak{p}_r[\lambda_i, w_i](0)$ *for* $i = 1, 2$. *In particular,* $w_1(t) - w_2(t) = \lambda_1^0(R(t)) - \lambda_2^0(R(t))$ *for all* $t \geq 0$. *If* $w_1, w_2 \in W_T^{1,1}(\mathbb{R}_+)$ *and*

$$\int_T^{2T} (\dot{\zeta}_1(t) - \dot{\zeta}_2(t))(w_1(t) - w_2(t))\, dt \;\leq\; 0\,, \tag{3.3.5}$$

then $w_1(t) - w_2(t) \equiv$ *const. If moreover* $\lambda_1 = \lambda_2 = \lambda \in \Lambda_K$, *then*

$$h(0)(w_1(t) - w_2(t))^2 \leq (\zeta_1(t) - \zeta_2(t))(w_1(t) - w_2(t)) \leq h(k(t))(w_1(t) - w_2(t))^2 \tag{3.3.6}$$

with $k(t) = \max\{K, \|w_1\|_{[0,t]}, \|w_2\|_{[0,t]}\}$.

We see that Prandtl-Ishlinskii operators possess some sort of "two-level monotonicity" which may be used in a Minty-type argument, see [14, Section III.3].

3.4 Energy dissipation

We still assume that (3.2.15) holds. With the operator $\mathcal{F}$, we associate the *potential energy operator* $\mathcal{U}$ of the form

$$\mathcal{U}[\lambda, w](t) \;=\; \frac{1}{2} h(0)\, |w(t)|^2 + \frac{1}{2} \int_0^\infty |\mathfrak{p}_r[\lambda, w](t)|^2\, dh(r)\,. \tag{3.4.1}$$

If we interpret w as stress, $\zeta = \mathcal{F}[\lambda, w]$ as strain, and $U = \mathcal{U}[\lambda, w]$ as potential energy, then, putting $\xi_r(t) = \mathfrak{p}_r[\lambda, w](t)$ and assuming that the input w is absolutely continuous, we obtain for the dissipation rate $d(t)$ the integral expression

$$d(t) \;:=\; \dot{\zeta}(t)\, w(t) - \dot{U}(t) \;=\; \int_0^\infty \dot{\xi}_r(t)(w(t) - \xi_r(t))\, dh(r) \;\; \text{a.e.} \tag{3.4.2}$$

Let us examine this formula in more detail. By (3.1.9), we can have $\dot{\xi}_r(t) > 0$ only if $w(t) - \xi_r(t) = r$ and $\dot{\xi}_r(t) < 0$ only if $w(t) - \xi_r(t) = -r$. Consequently, we have

$$d(t) \;=\; \int_0^\infty \left|\dot{\xi}_r(t)\right| r\, dh(r) \;\geq\; 0 \;\; \text{a.e.} \tag{3.4.3}$$

in agreement with the Second Principle of Thermodynamics. The integral of $d(t)$ over a closed cycle yields the area of the corresponding hysteresis loop, indeed.

For the sake of completeness, we derive a formula for the potential energy operator associated with the inverse operator

$$\begin{aligned} w(t) \;&=\; \hat{\mathcal{F}}[\mu, \zeta](t) \;=\; \hat{h}(0)\, \zeta(t) + \int_0^\infty \mathfrak{p}_r[\mu, \zeta](t)\, d\hat{h}(r) \\ &=\; \hat{h}(\infty)\, \zeta(t) - \int_0^\infty (\zeta(t) - \mathfrak{p}_r[\mu, \zeta](t))\, d\hat{h}(r)\,. \end{aligned} \tag{3.4.4}$$

generated by $\hat{h}$ as in Proposition 3.2.1. The function $\hat{h}$ is non-increasing and positive, hence $\hat{h}(\infty) = \lim_{r\to\infty} \hat{h}(r)$ is well defined. Putting

$$e(t) \;=\; \hat{\mathcal{U}}[\mu,\zeta](t) \;=\; \frac{1}{2}\hat{h}(\infty)\,|\zeta(t)|^2 - \frac{1}{2}\int_0^\infty |\zeta(t) - \mathfrak{p}_r[\mu,\zeta](t)|^2\, d\hat{h}(r)\,, \tag{3.4.5}$$

we easily check that the positive sign in (3.4.3) is preserved.

Besides the "physical energy inequality" (3.4.2), the Prandtl-Ishlinskii operator (3.2.1), (3.2.15) (as well as other hysteresis operators with convex/concave branches, for a detailed discussion on this subject see [14]) admits a "higher order energy inequality"

$$\ddot{\zeta}(t)\,\dot{w}(t) - \dot{V}(t) \;\ge\; 0 \tag{3.4.6}$$

in the sense of distributions (see (3.4.9) below), where we set

$$V(t) \;=\; \frac{1}{2}\dot{\zeta}(t)\,\dot{w}(t) \qquad \text{for} \quad \text{a.e. } t > 0\,. \tag{3.4.7}$$

This observation has been made for the first time in [11] in the context of periodic functions, and later on several different proofs have been published. Since this result plays a central role in our analysis, we state it precisely and give a sketch of the proof. As time differentiation is involved, we restrict ourselves to regular inputs and outputs.

Theorem 3.4.1. *Let hypothesis (3.2.15) hold, and for $p > 0$ set*

$$\kappa(p) \;=\; \inf\left\{\frac{h(r)-h(s)}{r-s}\,;\, 0 \le s < r \le p\right\}. \tag{3.4.8}$$

Then for every $K > 0$, $\lambda \in \Lambda_K$, and $w \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+)$ such that $\zeta = \mathcal{F}[\lambda, w]$ with $\mathcal{F}$ given by (3.2.1) belongs to $W^{2,1}_{\mathrm{loc}}(\mathbb{R}_+)$, the function $V(t)$ given by (3.4.7) equals almost everywhere to a function of bounded variation. Moreover, for every $T > 0$, $p \ge \max\{K, \|w\|_{[0,T]}\}$, and every $0 \le t_0 < t_1 < T$ we have

$$\int_{t_0}^{t_1} \ddot{\zeta}(t)\,\dot{w}(t)\,dt - V(t_1-) + V(t_0+) \;\ge\; \frac{1}{4}\kappa(p)\int_{t_0}^{t_1} |\dot{w}(t)|^3\,dt\,. \tag{3.4.9}$$

In particular, if w is T-periodic, then

$$\int_T^{2T} \ddot{\zeta}(t)\,\dot{w}(t)\,dt \;\ge\; \frac{1}{4}\kappa(p)\int_T^{2T} |\dot{w}(t)|^3\,dt\,. \tag{3.4.10}$$

Remark 3.4.2. As noticed in Remark 1.2.4, the function κ is a measure for the curvature of the initial loading curve H given by (3.2.3); in particular, H is strictly convex if κ is positive. The "dissipation term" on the right-hand side of (3.4.9) is thus proportional to the minimal curvature of H. Inequality (3.4.9) would be in fact a trivial application of the integration by parts formula if the hysteresis branches $\eta(t) = g(w(t))$ were smooth enough. Indeed, in this case we would have

$$\ddot{\zeta}(t)\,\dot{w}(t) - \dot{V}(t) \;=\; \frac{1}{2}\,g''(w(t))\,\dot{w}^3(t)\,. \tag{3.4.11}$$

The right-hand side of (3.4.11) is formally positive because g is convex if w increases and concave if w decreases, cf. the counterclockwise case on Figure 5. However, the "second order potential energy" $V(t)$ (which is indeed positive by virtue of (3.2.17)) is typically discontinuous in time *even if* h *is smooth*, e. g. on the transition from a minor loop to the major loop, and this fact makes the rigorous argument technically complicated.

The proof of Theorem 3.4.1 is based on a series of lemmas below.

Lemma 3.4.3. *Let* $\varphi :]a, b[\to \mathbb{R}$ *and* $c \geq 0$ *be such that* $\varphi(a+) > 0$ *and the function* $v \mapsto \varphi(v) - cv$ *is non-decreasing in* $]a, b[$. *Then the function*

$$\psi(v) \ := \ \frac{1}{\varphi(v)} + c \int_a^v \frac{ds}{\varphi^2(s)}$$

is non-increasing in $]a, b[$.

Proof. The assertion is obvious if φ is absolutely continuous; otherwise we approximate φ by piecewise linear interpolates and pass to the limit in continuity points of φ. Discontinuity points can be handled directly. ∎

Lemma 3.4.4. *Let* $w \in W^{1,\infty}(T_0, T_1)$ *be an increasing function, and let* $c \geq 0$ *and* $g : [w(T_0), w(T_1)] \to \mathbb{R}$ *be such that the function* $v \mapsto g(v) - \frac{c}{2} v^2$ *is convex in* $[w(T_0), w(T_1)]$, $g'(w(T_0)+) > 0$. *Assume that* $\zeta = g(w) \in W^{2,1}(T_0, T_1)$, *and for* $t \in]T_0, T_1[$ *put* $V(t) = \frac{1}{2}\dot{\zeta}(t)\, \dot{w}(t)$. *Then* V *coincides a. e. with a function of bounded variation in* $[T_0, T_1]$, *and for every* $T_0 \leq t_0 < t_1 \leq T_1$ *we have*

$$\int_{t_0}^{t_1} \ddot{\zeta}(t)\, \dot{w}(t)\, dt - V(t_1-) + V(t_0+) \ \geq \ \frac{c}{2} \int_{t_0}^{t_1} |\dot{w}(t)|^3 \, dt \,. \tag{3.4.12}$$

Proof. We first choose $t_0 < t_1$ such that $w(t_0), w(t_1)$ are continuity points of g'. By Lemma 3.4.3, the function

$$\eta(t) \ = \ \frac{1}{g'(w(t))} + c \int_{t_0}^t \frac{\dot{w}(\tau)}{(g'(w(\tau)))^2} \, d\tau \tag{3.4.13}$$

is non-increasing in $[t_0, t_1]$. Integrating by parts we obtain

$$\begin{aligned} \int_{t_0}^{t_1} \ddot{\zeta}(t)\, \dot{w}(t)\, dt \ &= \ \int_{t_0}^{t_1} \frac{1}{g'(w(t))} \frac{1}{2} \frac{d}{dt} \left(\dot{\zeta}^2(t) \right) dt \\ &= \ V(t_1) - V(t_0) - \int_{t_0}^{t_1} \frac{1}{2} \dot{\zeta}^2(t)\, d\left(\frac{1}{g'(w)} \right)(t) \\ &= \ V(t_1) - V(t_0) - \int_{t_0}^{t_1} \frac{1}{2} \dot{\zeta}^2(t)\, d\eta(t) + \frac{c}{2} \int_{t_0}^{t_1} \frac{\dot{\zeta}^2(t)\, \dot{w}(t)}{(g'(w(t)))^2} \, dt \\ &\geq \ V(t_1) - V(t_0) + \frac{c}{2} \int_{t_0}^{t_1} |\dot{w}(t)|^3 \, dt \,. \end{aligned} \tag{3.4.14}$$

Consequently, the function

$$t \ \mapsto \ \int_{t_0}^{t} \ddot{\zeta}(\tau)\, \dot{w}(\tau)\, d\tau - \frac{c}{2} \int_{t_0}^{t} |\dot{w}(\tau)|^3\, d\tau - V(t)$$

is a. e. non-decreasing, hence V has (up to a set of measure zero) bounded variation, and (3.4.12) is obtained by passing to the limit. ■

We do not repeat the same proof for the following "decreasing" counterpart to Lemma 3.4.4.

Lemma 3.4.5. *Let $w \in W^{1,\infty}(T_0, T_1)$ be a decreasing function, and let $c \geq 0$ and $g : [w(T_1), w(T_0)] \to \mathbb{R}$ be such that the function $v \mapsto g(v) + \frac{c}{2}v^2$ is concave in $[w(T_1), w(T_0)]$, $g'(w(T_0)-) > 0$. Let $\zeta(t)$ and $V(t)$ be as in Lemma 3.4.4. Then (3.4.12) holds for all $T_0 \leq t_0 < t_1 \leq T_1$.*

We are now ready to pass to the proof of Theorem 3.4.1.

Proof of Theorem 3.4.1. Let $t_0 < t_1$ be fixed, and set

$$N \ = \ \{t \in [t_0, t_1] \ ; \ \dot{\zeta}(t) = 0\}\,.$$

The function ζ is continuously differentiable, hence N is closed, and there exist pairwise disjoint intervals $]\tau_j, \tau^j[$, j belonging to an at most countable index set J, such that

$$]t_0, t_1[\,\backslash N \ = \ \bigcup_{j \in J}]\tau_j, \tau^j[\ . \tag{3.4.15}$$

Let us now fix some $j \in J$. The function ζ (and also w by virtue of (3.2.16)) are strictly monotone in $]\tau_j, \tau^j[$, hence we are in the situation of either Lemma 3.4.4 or Lemma 3.4.5 with g defined by (3.2.13) or (3.2.14). To be more precise, we distinguish between the two cases in order to determine the constant c.

(i) Let $\dot{\zeta} > 0$ in $]\tau_j, \tau^j[$, and put

$$g_j(v) \ = \ \mathcal{F}[\lambda, w](\tau_j) + \int_{w(\tau_j)}^{v} h(m_{\lambda_{\tau_j}}(u))\, du\,. \tag{3.4.16}$$

For a. e. $w(\tau_j) < v_1 < v_2 < w(\tau^j)$ we have

$$\frac{g_j'(v_2) - g_j'(v_1)}{v_2 - v_1} \ = \ \frac{h(m_{\lambda_{\tau_j}}(v_2)) - h(m_{\lambda_{\tau_j}}(v_1))}{m_{\lambda_{\tau_j}}(v_2) - m_{\lambda_{\tau_j}}(v_1)} \cdot \frac{m_{\lambda_{\tau_j}}(v_2) - m_{\lambda_{\tau_j}}(v_1)}{v_2 - v_1}\,. \tag{3.4.17}$$

Set $r_i = m_{\lambda_{\tau_j}}(v_i)$ for $i = 1, 2$. Then

$$v_1 - v_2 = (r_1 + \lambda_{\tau_j}(r_1)) - (r_2 + \lambda_{\tau_j}(r_2)) \leq 2(r_1 - r_2)\,.$$

From (3.4.17), (3.4.8), and Lemma 3.1.2 it follows that

$$\frac{g_j'(v_2) - g_j'(v_1)}{v_2 - v_1} \ \geq \ \frac{\kappa(p)}{2}\,. \tag{3.4.18}$$

The function $v \mapsto g_j'(v) - \frac{\kappa(p)}{2}\, v$ is non-increasing, hence $v \mapsto g_j(v) - \frac{\kappa(p)}{4}\, v^2$ is convex, and we may use Lemma 3.4.4 to obtain that

$$\int_{\tau_j}^{\tau^j} \ddot{\zeta}(t)\, \dot{w}(t)\, dt - V(\tau^j-) + V(\tau_j+) \;\geq\; \frac{1}{4}\kappa(p) \int_{\tau_j}^{\tau^j} |\dot{w}(t)|^3\, dt\,. \tag{3.4.19}$$

(ii) Let $\dot{\zeta} < 0$ in $]\tau_j, \tau^j[$, and put

$$g_j(v) \;=\; \mathcal{F}[\lambda, w](\tau_j) - \int_v^{w(\tau_j)} h(m_{\lambda_{\tau_j}}(u))\, du\,. \tag{3.4.20}$$

Repeating the above procedure we show that the function $v \mapsto g_j(v) + \frac{\kappa(p)}{4}\, v^2$ is concave, and Lemma 3.4.4 yields again that (3.4.19) holds.

At all points τ_j, τ^j except possibly the cases $\tau_j = t_0$ or $\tau^j = t_1$, we have $\dot{\zeta}(\tau_j) = \dot{\zeta}(\tau^j) = 0$, hence $V(\tau^j-) = V(\tau_j+) = 0$. Furthermore, almost everywhere in N we have $\dot{w}(t) = 0$, hence we may sum all inequalities (3.4.19) over $j \in J$ and obtain the assertion. The periodic case follows from Corollary 3.1.3 which enables us to consider in (3.4.9) any integration domain of length T in $[T, \infty[$.

■

3.5 Parameter dependent hysteresis

We now extend the Prandtl-Ishlinskii construction to functions depending also on a spatial variable x by assuming that each point x has its own memory. In our situation, we only consider the one-dimensional case $x \in [0,1]$ and input functions continuous in t.

Let an initial memory distribution $\lambda \in L^1(0,1;\Lambda_K)$ be given for some $K > 0$. For inputs w defined in $[0,1] \times \mathbb{R}_+$ and such that $w \in L^1(0,1;C[0,T]) \cap L^\infty(]0,1[\times]0,T[)$ we define similarly as in (3.1.13) and (3.2.1)

$$\mathfrak{p}_r[\lambda, w](x,t) \;=\; \hat{\mathfrak{p}}_r[\lambda(x,r), w(x,\cdot)](t)\,, \tag{3.5.1}$$

$$\mathcal{F}[\lambda, w](x,t) \;=\; h(0)w(x,t) + \int_0^\infty \mathfrak{p}_r[\lambda, w](x,t)\, dh(r) \tag{3.5.2}$$

for $(x,t) \in [0,1] \times \mathbb{R}_+$, where h is a function satisfying (3.2.15). In fact, we may have considered h which depends also on x, and a detailed discussion on this subject can be found in [6]. Here, for the sake of simplicity, we restrict ourselves to the *spatially homogeneous case*.

Assume first that both λ and w are continuous in x. Then for all $x, y \in [0,1]$ and $t \in [0,T]$ we have by virtue of (3.2.10) that

$$\begin{aligned} &|\mathcal{F}[\lambda, w](x,t) - \mathcal{F}[\lambda, w](y,t)| \\ &\quad\leq\; h(R(T)) \max\{\|\lambda(x,\cdot) - \lambda(y,\cdot)\|_{[0,K]}\,, \|w(x,\cdot) - w(y,\cdot)\|_{[0,t]}\}\,, \end{aligned} \tag{3.5.3}$$

where $R(T) = \max\{K, \sup\{|w(z,t)|\,(z,t) \in [0,1]\times[0,T]\}\}$, hence $\mathcal{F}[\lambda, w]$ is continuous on $[0,1]\times[0,T]$. Using (3.2.10) again for sequences $\lambda^{(n)}$ and $w^{(n)}$, we derive the implications

$$\left.\begin{array}{ll} \lambda^{(n)} \to \lambda & \text{uniformly} \\ w^{(n)} \to w & \text{uniformly} \end{array}\right\} \Longrightarrow \mathcal{F}[\lambda^{(n)}, w^{(n)}] \to \mathcal{F}[\lambda, w] \quad \text{uniformly}\,, \tag{3.5.4}$$

$$\left.\begin{array}{lll} \lambda^{(n)} \to \lambda & \text{strongly in} & L^1(0,1;\Lambda_K)\,, \\ w^{(n)} \to w & \text{strongly in} & L^1(0,1;C[0,T])\,, \\ w^{(n)} & \text{bounded in} & L^\infty(]0,1[\times]0,T[) \end{array}\right\} \tag{3.5.5}$$

$$\Longrightarrow \left\{\begin{array}{lll} \mathcal{F}[\lambda^{(n)}, w^{(n)}] \to \mathcal{F}[\lambda, w] & \text{strongly in} & L^1(0,1;C[0,T])\,, \\ \mathcal{F}[\lambda^{(n)}, w^{(n)}] & \text{bounded in} & L^\infty(]0,1[\times]0,T[)\,. \end{array}\right.$$

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