Robust and efficient guaranteed error bounds for reaction-diffusion problems

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Reaction-diffusion problem

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^d, \qquad \kappa > 0$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak formulation:

$$u \in V : \quad (\nabla u, \nabla v) + \kappa^2 (u, v) = (f, v) \quad \forall v \in V$$

Linear FEM:

$$u_h \in V_h : \quad (\nabla u_h, \nabla v_h) + \kappa^2 (u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Notation:

- $V = H_0^1(\Omega)$
- $(u, v) = \int_\Omega uv \, \mathrm{d}x$
- $V_h = \{v_h \in V : v_h|_K \in P^1(K), K \in \mathcal{T}_h\}$
- $|||v|||^2 = \|\nabla v\|^2 + \kappa^2 \|v\|^2$

Reaction-diffusion problem

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Goals:

- Guaranteed upper bound: $|||u - u_h||| \leq \eta$
- Local efficiency: $\eta_K \leq C |||u - u_h|||_K$
- Robustness: C is independent of h and κ
- Fast algorithm

Guaranteed error bound

Theorem

$$|||u - u_h|||^2 \leq \sum_{K \in \mathcal{T}_h} \left[\eta_K(\boldsymbol{\tau}) + \mathrm{osc}_K(f)\right]^2 \quad \forall \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}, \Omega)$$

where

- $\eta_K^2(\boldsymbol{\tau}) = \|\boldsymbol{\tau} - \nabla u_h\|_K^2 + \frac{1}{\kappa^2} \left\| f_K - \kappa^2 u_h + \mathrm{div}\, \boldsymbol{\tau} \right\|_K^2$
- $\mathrm{osc}_K(f) = \min\left\{ \frac{h_K}{\pi}, \frac{1}{\kappa} \right\} \|f - f_K\|_K$
- $f_K = \Pi_K f$, i.e., $f_K \in P^1(K) : \ (f - f_K, \varphi)_K = 0 \quad \forall \varphi \in P^1(K)$

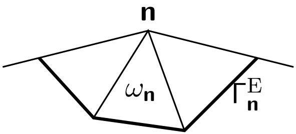
Goal: Construct $\boldsymbol{\tau}$ such that we have (robust) local efficiency:

$$\eta_K(\boldsymbol{\tau}) \leq C\left(|||u - u_h|||_{\widetilde{K}} + \text{h.o.t.} \right)$$

Flux reconstruction

Definition

$$\boldsymbol{\tau} = \sum_{\mathbf{n}\in\mathcal{N}_h} \boldsymbol{\tau}_{\mathbf{n}}$$



where $\boldsymbol{\tau}_{\mathbf{n}} \in \mathbf{W}(\omega_{\mathbf{n}})$ minimizes

$$\|\boldsymbol{\tau}_{\mathbf{n}} - \theta_{\mathbf{n}}\nabla u_h\|^2_{\omega_{\mathbf{n}}} + \frac{1}{\kappa^2}\left\|\Pi_K\left(\theta_{\mathbf{n}}(f_K - \kappa^2 u_h)\right) - \nabla\theta_{\mathbf{n}}\cdot\nabla u_h + \operatorname{div}\boldsymbol{\tau}_{\mathbf{n}}\right\|^2_{\omega_{\mathbf{n}}}$$

Notation

- $\mathbf{W}(\omega_{\mathbf{n}}) = \{\boldsymbol{\tau} \in \mathbf{H}(\operatorname{div},\omega_{\mathbf{n}}) : \boldsymbol{\tau}|_K \in \mathbf{RT}_1(K),\ \boldsymbol{\tau}\cdot\boldsymbol{\nu}_{\omega_{\mathbf{n}}} = 0 \text{ on } \Gamma^{\mathrm{E}}_{\mathbf{n}}\}$
- $\theta_{\mathbf{n}}$ ... hat functions (form partition of unity: $\sum_{\mathbf{n}\in\mathcal{N}_h}\theta_{\mathbf{n}} = 1$)
- $\mathcal{N}_h$ ... nodes (vertices) in triangulation $\mathcal{T}_h$

[Braess, Schöberl 2008], [Ern, Vohralík 2009–]

Flux reconstruction

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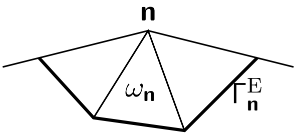


where $\boldsymbol{\tau}_{\mathbf{n}} \in \mathbf{W}(\omega_{\mathbf{n}})$ minimizes

$$\|\boldsymbol{\tau}_{\mathbf{n}} - \theta_{\mathbf{n}} \nabla u_h\|_{\omega_{\mathbf{n}}}^2 + \frac{1}{\kappa^2} \left\| \Pi_K \left(\theta_{\mathbf{n}} (f_K - \kappa^2 u_h) \right) - \nabla \theta_{\mathbf{n}} \cdot \nabla u_h + \operatorname{div} \boldsymbol{\tau}_{\mathbf{n}} \right\|_{\omega_{\mathbf{n}}}^2$$

Euler–Lagrange equations

Minimizer $\boldsymbol{\tau}_{\mathbf{n}} \in \mathbf{W}(\omega_{\mathbf{n}})$ satisfies

$$\frac{1}{\kappa^2} (\operatorname{div} \boldsymbol{\tau}_{\mathbf{n}}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{n}}} + (\boldsymbol{\tau}_{\mathbf{n}}, \mathbf{w}_h)_{\omega_{\mathbf{n}}}$$
$$= (\theta_{\mathbf{n}} \nabla u_h, \mathbf{w}_h)_{\omega_{\mathbf{n}}} - \frac{1}{\kappa^2} \left(\theta_{\mathbf{n}} (f_K - \kappa^2 u_h) - \nabla \theta_{\mathbf{n}} \cdot \nabla u_h, \operatorname{div} \mathbf{w}_h \right)_{\omega_{\mathbf{n}}}$$

for all $\mathbf{w}_h \in \mathbf{W}(\omega_{\mathbf{n}})$.

Local efficiency

Theorem. Assume regular family of triangulations. Then there exists $C > 0$ such that

$$\eta_K^2(\boldsymbol{\tau}) \leq C \left(|\!|\!| u - u_h |\!|\!|_{\widetilde{K}}^2 + \mathrm{h.o.t.}^2 \right)$$

where

$$\mathrm{h.o.t.}^2 = \min\left\{ h_K, \frac{1}{\kappa} \right\}^2 \left(\| f - f_K \|_{\widetilde{K}}^2 + \sum_{\mathbf{n} \in \mathcal{N}_K} \| f_{\mathbf{n}} - f_K \|_{\omega_{\mathbf{n}}}^2 \right)$$

and $f_{\mathbf{n}} \in P^1(\omega_{\mathbf{n}}) : \quad (f - f_{\mathbf{n}}, \varphi)_{\omega_{\mathbf{n}}} = 0 \quad \forall \varphi \in P^1(\omega_{\mathbf{n}}).$

Idea of proof

Lemma

$$\eta_K^2(\boldsymbol{\tau}) \leq C \sum_{\mathbf{n}\in\mathcal{N}_K}$$

$$\left[\|\boldsymbol{\tau}_\mathbf{n} - \theta_\mathbf{n}\nabla u_h\|_{\omega_\mathbf{n}}^2 + \frac{1}{\kappa^2}\left\|\Pi_K(\theta_\mathbf{n}(f_K - \kappa^2 u_h)) - \nabla\theta_\mathbf{n}\cdot\nabla u_h + \operatorname{div}\boldsymbol{\tau}_\mathbf{n}\right\|_{\omega_\mathbf{n}}^2\right]$$

Proof

$$\begin{aligned}
\eta_K^2(\boldsymbol{\tau}) &= \|\boldsymbol{\tau} - \nabla u_h\|_K^2 + \frac{1}{\kappa^2}\left\|f_K - \kappa^2 u_h + \operatorname{div}\boldsymbol{\tau}\right\|_K^2 \\
&= \left\|\sum_{\mathbf{n}\in\mathcal{N}_K}(\boldsymbol{\tau}_\mathbf{n} - \theta_\mathbf{n}\nabla u_h)\right\|_K^2 \\
&+ \frac{1}{\kappa^2}\left\|\sum_{\mathbf{n}\in\mathcal{N}_K}\left[\Pi_K\left(\theta_\mathbf{n}(f_K - \kappa^2 u_h)\right) - \nabla\theta_\mathbf{n}\cdot\nabla u_h + \operatorname{div}\boldsymbol{\tau}_\mathbf{n}\right]\right\|_K^2
\end{aligned}$$

Idea of proof

Lemma

$$\eta_K^2(\boldsymbol{\tau}) \leq C \sum_{\mathbf{n}\in\mathcal{N}_K}$$

$$\left[\|\boldsymbol{\tau}_{\mathbf{n}} - \theta_{\mathbf{n}} \nabla u_h\|_{\omega_{\mathbf{n}}}^2 + \frac{1}{\kappa^2} \left\| \Pi_K(\theta_{\mathbf{n}}(f_K - \kappa^2 u_h)) - \nabla\theta_{\mathbf{n}} \cdot \nabla u_h + \operatorname{div} \boldsymbol{\tau}_{\mathbf{n}} \right\|_{\omega_{\mathbf{n}}}^2 \right]$$

Case 1. $\kappa h_K \leq 1$ [Ainsworth, V. 2011, 2014]

- $\boldsymbol{\sigma}_{\mathbf{n}}^{(1)} \in \mathbf{W}(\omega_{\mathbf{n}})$
- $\Pi_K(\theta_{\mathbf{n}}(f_K - \kappa^2 u_h)) - \nabla\theta_{\mathbf{n}} \cdot \nabla u_h + \operatorname{div} \boldsymbol{\sigma}_{\mathbf{n}}^{(1)} = 0$
- $\|\boldsymbol{\sigma}_{\mathbf{n}}^{(1)} - \theta_{\mathbf{n}} \nabla u_h\|_{\omega_{\mathbf{n}}}^2 \leq C \left(\|\!|u - u_h\|\!|_{\widetilde{K}}^2 + h_K^2 \|f - f_K\|_{\widetilde{K}} \right)$

[Ainsworth, Babuška 1999], [Verfürth 1998]

Idea of proof

Lemma

$$\eta_K^2(\boldsymbol{\tau}) \leq C \sum_{\mathbf{n}\in\mathcal{N}_K}$$

$$\left[\|\boldsymbol{\tau}_{\mathbf{n}} - \theta_{\mathbf{n}}\nabla u_h\|_{\omega_{\mathbf{n}}}^2 + \frac{1}{\kappa^2}\left\|\Pi_K(\theta_{\mathbf{n}}(f_K - \kappa^2 u_h)) - \nabla\theta_{\mathbf{n}}\cdot\nabla u_h + \operatorname{div}\boldsymbol{\tau}_{\mathbf{n}}\right\|_{\omega_{\mathbf{n}}}^2\right]$$

Case 2. $\kappa h_K > 1$

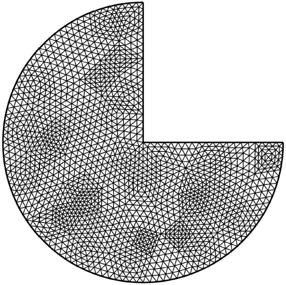
- $\boldsymbol{\sigma}_{\mathbf{n}}^{(2)} = \dfrac{1}{\kappa^2}\theta_{\mathbf{n}}\nabla f_{\mathbf{n}}$
- $\boldsymbol{\sigma}_{\mathbf{n}}^{(2)} \in \mathbf{W}(\omega_{\mathbf{n}})$
-

$$\left\|\boldsymbol{\sigma}_{\mathbf{n}}^{(2)} - \theta_{\mathbf{n}}\nabla u_h\right\|_{\omega_{\mathbf{n}}}^2 + \frac{1}{\kappa^2}\left\|\Pi_K\left(\theta_{\mathbf{n}}(f_K - \kappa^2 u_h)\right) - \nabla\theta_{\mathbf{n}}\cdot\nabla u_h + \operatorname{div}\boldsymbol{\sigma}_{\mathbf{n}}^{(2)}\right\|_{\omega_{\mathbf{n}}}^2$$

$$\leq C\left(|\!|\!| u - u_h |\!|\!|_{\omega_{\mathbf{n}}}^2 + \frac{1}{\kappa^2}\|f - f_K\|_{\omega_{\mathbf{n}}}^2 + \frac{1}{\kappa^2}\|f_{\mathbf{n}} - f_K\|_{\omega_{\mathbf{n}}}^2\right)$$

Numerical example

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

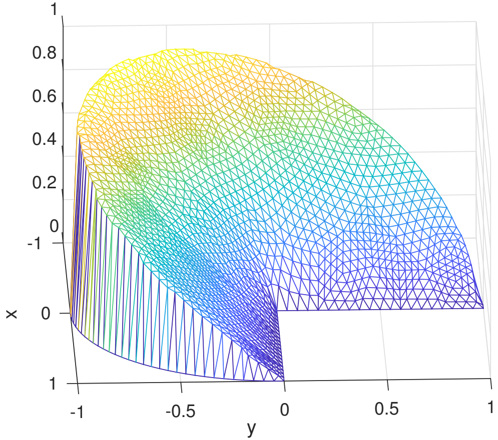


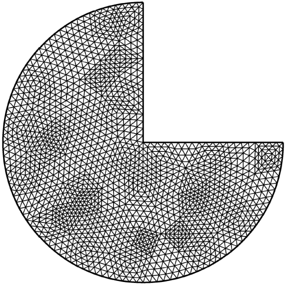
- $f = \kappa^2 r^{2/3} \sin \frac{2\phi - \pi}{3}$
- $u = \left(r^{2/3} - \frac{I_{2/3}(\kappa r)}{I_{2/3}(\kappa)} \right) \sin \frac{2\phi - \pi}{3}$
- $I_\alpha(x)$ ... modified Bessel function of the first kind
- $\kappa = 10^{-3}, \ldots, 10^6$

Numerical example

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

$\kappa = 100$





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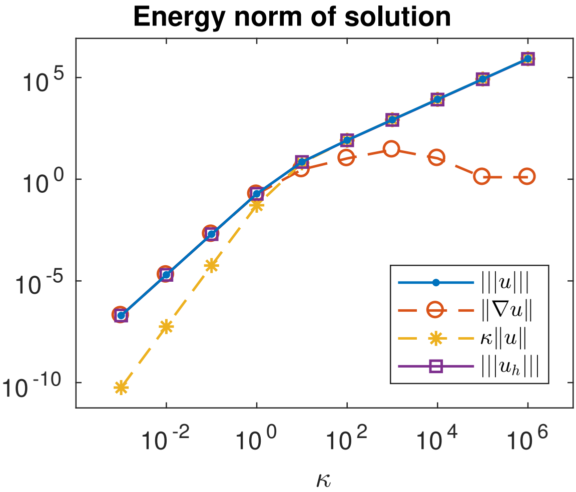
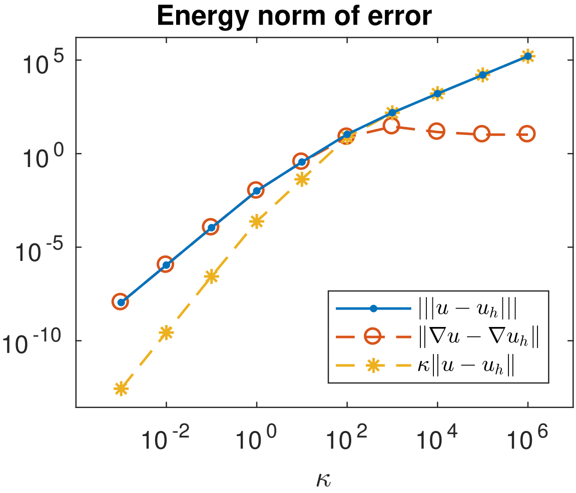


Numerical example

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$e = u - u_h$$



Numerical example

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
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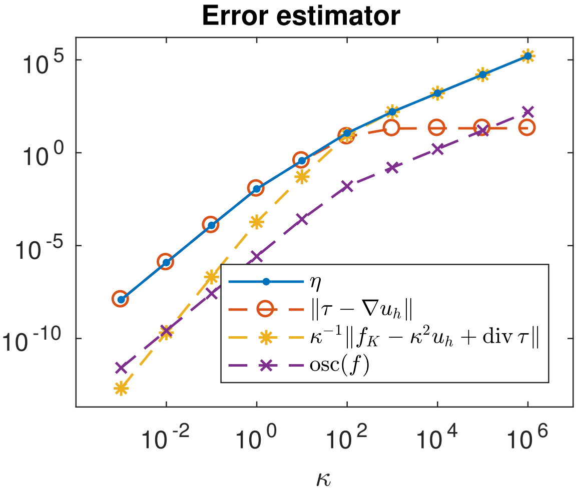
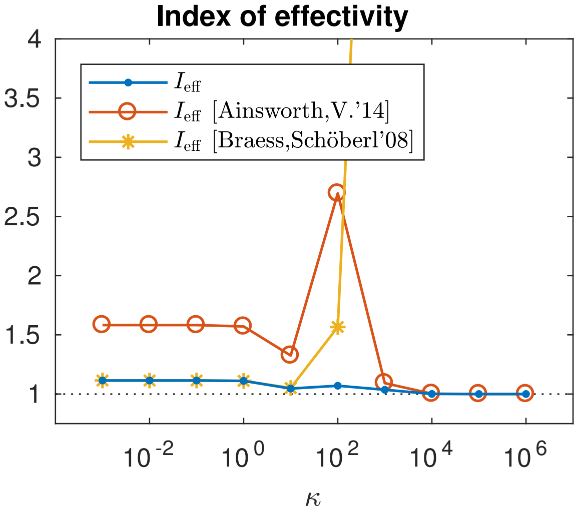
$$\eta^2 = \sum_{K \in \mathcal{T}_h} [\eta_K(\boldsymbol{\tau}) + \mathrm{osc}_K(f)]^2$$



Numerical example

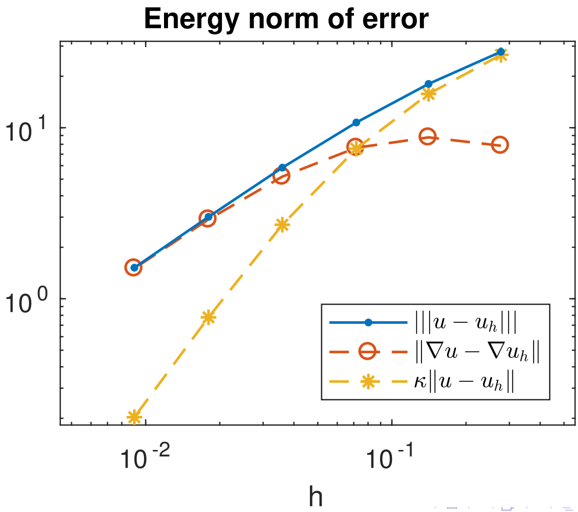
$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$I_{\text{eff}} = \frac{\eta}{\|u - u_h\|}$$



Numerical example

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega \qquad \kappa = 100$$



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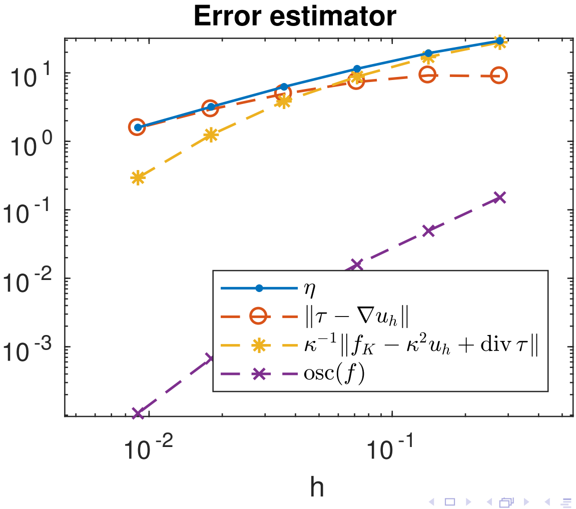

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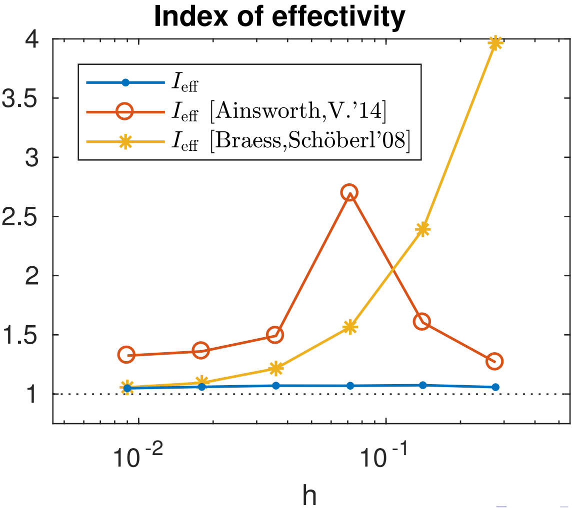
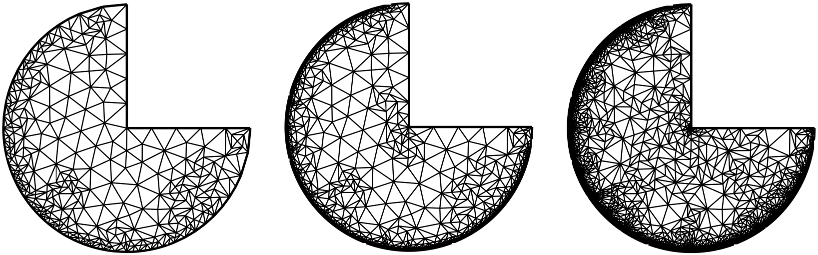
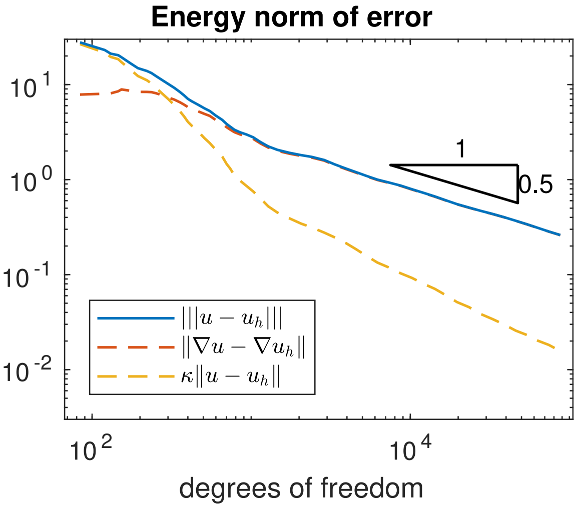


Adaptivity

- Solve. FEM
- Estimate. Error indicators $\eta_K(\boldsymbol{\tau})$
  Error estimator $\eta^2 = \sum_{K \in \mathcal{T}_h} [\eta_K(\boldsymbol{\tau}) + \mathrm{osc}_K(f)]^2$
- Stop. If $\|u - u_h\| \leq \eta \leq \mathrm{TOL}$
- Mark. Dörfler strategy
- Refine. Longest edge bisection

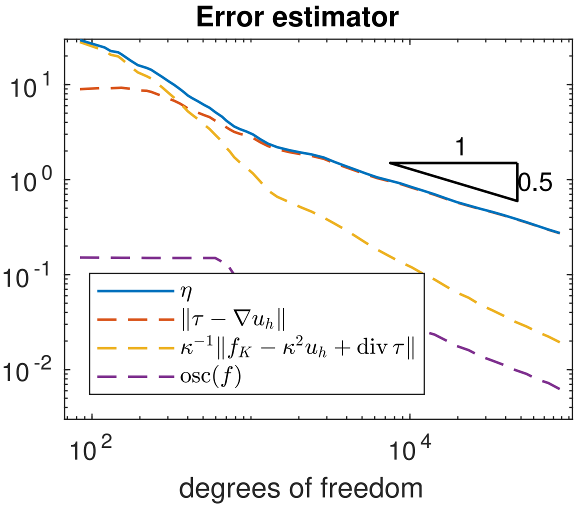


Adaptivity

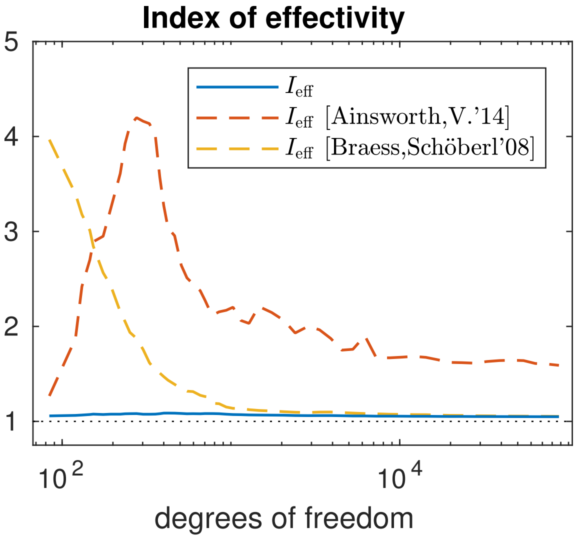


Energy norm of error
10^1
10^0
10^-1
10^-2
10^2
10^4
degrees of freedom
1
0.5
$|||u - u_h|||$
$\|\nabla u - \nabla u_h\|$
$\kappa\|u - u_h\|$

Adaptivity


Error estimator
10^1
10^0
10^-1
10^-2
10^2
10^4
degrees of freedom
1
0.5
η
$\|\tau - \nabla u_h\|$
$\kappa^{-1}\|f_K - \kappa^2 u_h + \operatorname{div}\tau\|$
$\operatorname{osc}(f)$

Adaptivity



Index of effectivity
5
4
3
2
1
I_{eff}
I_{eff} [Ainsworth,V.'14]
I_{eff} [Braess,Schöberl'08]
10^2
10^4
degrees of freedom

Conclusions

Highlights

- Guaranteed upper bound on error (reliability)
- Local lower bound on error (efficiency)
- Robust with respect to both h and κ
- Flux reconstruction based on small local problems

Open problems

- Variable κ
- Neumann boundary conditions
- Anisotropic meshes – talk of Natalia Kopteva

Thank you for your attention

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